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z-Classes in finite groups of conjugate type $(n, 1)$

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Abstract. Two elements in a group G are said to be *z-equivalent* or to be in the same *z-class* if their centralizers are conjugate in G. In a recent work, Kulkarni *et al.* (*J. Algebra Appl.*, **15** (2016) 1650131) proved that a non-abelian p-group G can have at most $\frac{p^k-1}{p-1}+1$ number of z-classes, where $|G/Z(G)| = p^k$. Here, we characterize the p-groups of conjugate type $(n, 1)$ attaining this maximal number. As a corollary, we characterize p-groups having prime order commutator subgroup and maximal number of z-classes.

Keywords. Conjugacy classes of centralizers; z-classes; p-groups; extraspecial groups.

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1. Introduction

Two elements in a group G are said to be *z-equivalent* or to be in the same *z-class* if their centralizers are conjugate in G. Since z-equivalence is an equivalence relation on G, it yields a partition of G into disjoint equivalence classes. It is coarser than the conjugacy relation. An infinite group may have infinitely many conjugacy classes, but in many groups the number of z-classes is finite. This finiteness of the z-classes gives a rough idea about 'dynamical types' in a homogeneous space on which G acts [12]. Following this motivation to use the z-classes to classify 'dynamical types' of transformations, the z-classes of real hyperbolic isometries have been classified and counted by Gongopadhyay and Kulkarni [4].

Apart from geometric motivations, the z-classes are important objects in their own right. Characterizations of the z-classes in groups and explicit computations of their numbers is an interesting problem to study. A related problem is to compute the 'genus number' in a linear algebraic group, see [1]. The z-classes play an important role in the study of characters of finite groups of Lie type [2]. Steinberg [15] proved that the number of z-classes is finite in a reductive algebraic group over a field of good characteristic. The z-classes in general linear groups and affine groups over arbitrary fields have been classified by Kulkarni [13]. Gouraige [7] obtained characterization of the z-classes in general linear groups over division rings. Singh [14] computed the z-classes of semisimple elements in the compact real form G_2. Gongopadhyay and Kulkarni [5] investigated the z-classes in orthogonal and symplectic groups over fields of characteristic different from two. Gongopadhyay [6] classified the z-classes in the isometry groups of the hyperbolic spaces and counted them for the quaternionic hyperbolic space.

In a recent work, Kulkarni *et al.* [11] investigated z-classes in finite p-groups. They proved that a non-abelian p-group has at least $p+2$ z-classes and characterized the groups that attain this lower bound. They also proved that a non-abelian p-group can have at most $\frac{p^k-1}{p-1}+1$ number of z-classes, where $|G/Z(G)| = p^k$. They proved the following theorem that gives necessary conditions for a p-group G to attain this maximal number of z-classes.

Theorem KKJ [11, Theorem A3]. *Let G be a non-abelian group with $|G/Z(G)| = p^k$. If the number of z-classes in G is $\frac{p^k-1}{p-1}+1$, then either $G/Z(G) \cong C_p \times C_p$ or the following holds*:

(1) *G has no abelian subgroup of index p.*
(2) *$G/Z(G)$ is an elementary abelian p-group.*

Further, a counter-example was given in [11] to show that the above conditions (1) and (2) are not sufficient to attain this bound. It would be interesting to obtain sufficient conditions for a group G to attain the maximal number of z-classes and to characterize such groups.

In this note, we characterize the groups of conjugate type $(n, 1)$ attaining this bound. To state our result, we recall the following definition by Ito [9].

DEFINITION [9]

Let G be a finite group. Let $n_1, n_2, \ldots, n_r$ be integers, where $n_1 > n_2 > \ldots > n_r = 1$, be all the numbers which are the indices of the centralizers of elements of G in G. The vector $(n_1, n_2, \ldots, n_r)$ is called conjugate type vector of G. A group with the conjugate type vector $(n_1, n_2, \ldots, n_r)$ is called a group of conjugate type $(n_1, n_2, \ldots, n_r)$, or simply, a group of type $(n_1, n_2, \ldots, n_r)$.

In [9], Ito investigated the groups of type $(n, 1)$. Any such group is nilpotent and $n = p^a$ for some prime p. A group of type $(p^a, 1)$ is the direct product of a p-group of the same type and of an abelian group. Thus, the study of groups of type $(n, 1)$ can be reduced to that of p-groups of type $(p^a, 1)$. He showed that such a group G contains an abelian normal subgroup A such that G/A is of exponent p.

Here we prove the following.

Theorem 1.1. *Let G be a non-abelian group of type $(n, 1)$ and $[G : Z(G)] = p^k$. Then G has $\frac{p^k-1}{p-1}+1$ z-classes if and only if*

(1) *$G/Z(G)$ is elementary abelian, and*
(2) *for all $x \in G \setminus Z(G)$, $Z(C_G(x)) = \langle x, Z(G)\rangle$.*

As an application of the above result, we have the following.

COROLLARY 1.2

Let G be a non-abelian group with $|G/Z(G)| = p^k$, $(k \geq 2)$ and $|G'| = p$. Then the number of z-classes in G is $\frac{p^k-1}{p-1} + 1$ if and only if G is isoclinic to an extraspecial p-group.

We prove these results in section 3.

We note that Corollary 1.2 may be compared with Theorem 5.3.1 of [10, section 5.3].

2. Preliminaries

We recall a few basic facts that will be used in the proof of the above results.

Let G be a group. We denote $C_G(x)$ to be the centralizer of an element x in G.

DEFINITION 2.1

Let G be a group. Two elements $x, y \in G$ are said to be z-equivalent if their centralizers are conjugate, i.e. there exist $g \in G$ such that

$$g^{-1}C_G(x)g = C_G(y).$$

Let G be a group acting on a set X. For $x \in X$, let $G(x)$ denote the orbit of x and G_x be the stabilizer subgroup of x:

$$G(x) = \{y \in X : \text{there exists } g \in G \text{ and } y = gx\}.$$
$$G_x = \{g \in G : gx = x\}.$$

We have a partition of X as

$$X = \bigcup_{x \in X} G(x).$$

Let $y \in G(x)$ be some element. Then for some $g \in G$, we have $y = gx$ and $G_y = gG_xg^{-1}$.

For subgroups A and B in G, write $A \sim_g B$ if they are conjugate. This gives us the following.

DEFINITION 2.2

Let $x, y \in X$ be any two element, then they are said to be in the same orbit class if $G_x \sim_g G_y$. Denote this equivalence relation as $x \sim_0 y$ and denote $R(x)$ as the equivalence class of x.

This gives another partition of X:

$$X = \bigcup_{x \nsim_0 y} R(x).$$

Let F_x denote the points fixed by G_x, i.e.,

$$F_x = \{y \in X : G_y \supset G_x\}.$$

Let $F'_x = \{y \in X : G_y = G_x\}$. Define $W_x = N_x/G_x$, where N_x is the normalizer of the G_x in G. Then we have the following theorem (cf. [12]).

Theorem 2.3 **[12, Theorem 2.1].** *Let G act on a set X. Then the map $\phi : G/G_x \times F'_x \to R(x)$, $\phi(gG_x, y) = gy$ is well-defined and induces a bijection $\bar{\phi} : (G/G_x \times F'_x)/W_x \to R(x)$.*

When we take our space $X = G$, the group itself, then $R(x)$ in the above notation becomes the z-classes and the above theorem gives us the following equation, which is used to calculate the size of a z-class: for any $x \in G$, we have

$$|z\text{-class of } x| = [G : N_G(C_G(x))] \cdot |F'_x|,$$

where $C_G(x)$ is the centralizer of x and $F'_x = \{y \in G : C_G(y) = C_G(x)\}$.

The following theorem states that that number of z-classes is invariant under the isoclinism. For basic notions on isoclinism, see [8,11].

Theorem 2.4 **[11, Theorem 2.2].** *Let G_1 and G_2 be isoclinic groups, with an isoclinism (ϕ, ψ). Then the isoclinism (ϕ, ψ) induces a bijection between the z-classes in G_1 and G_2.*

Finally we note the following theorem by Hall [8] that is used in the proof of Theorem 1.2.

Theorem 2.5. *Every group is isoclinic to a group whose center is contained in the commutator subgroup.*

3. Proof of the main results

It was proved in [11, Lemma 3.1] that if G is a finite group with $|G/Z(G)| = p^k$, then G is isoclinic to a finite p-group. Since number of z-classes is invariant under isoclinism, hence it is enough to prove our results assuming that G is a p-group.

Lemma 3.1. *Let G be a non-abelian p-group with $[G : Z(G)] = p^k$. If*

(1) *$G/Z(G)$ is elementary abelian, and*
(2) *for all $z \in G \setminus Z(G)$, $Z(C_G(x)) = \langle x, Z(G)\rangle$,*

then G has $\frac{p^k-1}{p-1} + 1$ z-classes.

Proof. Let $x \in G \setminus Z(G)$. Since $G/Z(G)$ is abelian, hence $C_G(x) \trianglelefteq G$, i.e., $[G : N(C_G(x))] = 1$. Consider

$$F'_x = \{y \in G : C_G(y) = C_G(x)\}.$$

By definition, $F'_x \subseteq Z(C_G(x)) \setminus Z(G) = \langle x, Z(G)\rangle \setminus Z(G)$. Obviously $C_G(x) = C_G(xt)$ for $t \in Z(G)$. Also since the exponent of $G/Z(G)$ is p, thus $C_G(x) = C_G(x^i)$ for any $i \in \{1, 2, \ldots, p-1\}$. Hence $\langle x, Z(G)\rangle \setminus Z(G) \subseteq F'_x$ and

$$F'_x = Z(C_G(x)) \setminus Z(G) = \langle x, Z(G)\rangle \setminus Z(G) = xZ(G) \cup \cdots \cup x^{p-1}Z(G).$$

This implies that $|F'_x| = (p-1)|Z(G)|$. Therefore,

$$|z\text{-class of } x| = [G : N_G(C_G(x))] \cdot |F'_x| = 1 \cdot (p-1)|Z(G)|.$$

Thus $G \setminus Z(G)$ contributes to a total $\frac{p^k-1}{p-1}$ number of z-classes of G. Clearly, $Z(G)$ is also a z-class. Hence, G has $\frac{p^k-1}{p-1} + 1$ z-classes. □

COROLLARY 3.2

Let G be a non-abelian p-group with $|G/Z(G)| = p^k$. If

(1) *G has no abelian subgroup of order exceeding $p|Z(G)|$,*
(2) *and $G/Z(G)$ is an elementary abelian p-group,*

then G has $\frac{p^k-1}{p-1} + 1$ number of z-classes.

Proof. We claim $C_G(x) = \langle x, Z(G)\rangle$, and the rest follows from Lemma 3.1.

Clearly $\langle x, Z(G)\rangle \subseteq C_G(x)$. Let us assume that $y \notin \langle x, Z(G)\rangle$ and $y \in C_G(x)$. But then $\langle x, y, Z(G)\rangle$ is an abelian subgroup and $|\langle x, y, Z(G)\rangle| > p|Z(G)|$. Thus $C_G(x) = \langle x, Z(G)\rangle$. □

3.1 *Proof of Theorem 1.1*

Proof. Let G be an non-Abelian group of type $(n, 1)$ and G has $\frac{p^k-1}{p-1} + 1$ z-classes, where $[G : Z(G)] = p^k$. (1) follows from Theorem KKJ and we prove (2).

Let $x \in G \setminus Z(G)$. Then for any $t \in Z(G)$, $C_G(x) = C_G(xt)$. Also for any m relatively prime to order of x, we have

$$C_G(x) = C_G(\langle x\rangle) = C_G(\langle x^m\rangle)$$

Thus z-class of x contains

$$xZ(G) \cup x^2 Z(G) \cup \cdots \cup x^{p-1} Z(G),$$

and hence, for all $x \in G \setminus Z(G)$, the size of a z-class is at least $(p-1)|Z(G)|$. Since $Z(G)$ is also a z-class, hence it follows from the hypothesis that $G \setminus Z(G)$ constitute exactly $\frac{p^k-1}{p-1}$ z-classes. Now, $|G \setminus Z(G)| = |G| - |Z(G)|$. Hence, we must have for $x \in G \setminus Z(G)$,

$$|z\text{-class of } x| = (p-1)|Z(G)|.$$

Now, it is clear that for $x \in G \setminus Z(G)$, $\langle x, Z(G)\rangle \subseteq Z(C_G(x))$. Let $y \in Z(C_G(x))$ such that $y \notin \langle x, Z(G)\rangle$. Then a simple observation shows that $C_G(y) \supseteq C_G(x)$. But since G is of type $(n, 1)$, we have $|C_G(x)| = |C_G(y)|$, and thus we must have $C_G(x) = C_G(y)$. This implies that y belongs to the z-class of x. Since $y \notin \langle x, Z(G)\rangle$, $|z\text{-class of } x| > (p-1)|Z(G)|$, which is a contradiction. Thus our assumption is incorrect and $Z(C_G(x)) = \langle x, Z(G)\rangle$ for all $x \in G \setminus Z(G)$.

The converse part follows from Lemma 3.1. This completes the proof. □

3.2 *Proof of Corollary 1.2*

Proof. Without loss of generality, using Theorem 2.5, we assume up to isoclinism that G is a group such that $Z(G) \subseteq G'$. Let G has $\frac{p^k-1}{p-1} + 1$ z-classes. Since $|G'| = p$ and in a non-Abelian p-group $|Z(G)| > 1$, it follows that $G' = Z(G)$; hence, $|G'| = |Z(G)| = p$. Therefore G and $G/Z(G)$ have the same exponent. Thus G is an extra-special p-group.

Conversely, let G be an extraspecial p-group. So, $|Z(G)| = |G'| = p$. For $x \in G \backslash Z(G)$, define the map

$$\phi_x : G \to G', \\ g \mapsto [x, g].$$

As $G/Z(G)$ is elementary abelian, ϕ_x is a homomorphism with $\ker(\phi_x) = C_G(x)$ and since $|G'| = p$, the map is also surjective. Hence we get

$$G/C_G(x) \cong G'$$

Thus each $C_G(x)$ is a maximal subgroup of index p in G. This shows that extraspecial p-groups are of type $(n, 1)$.

Now, we claim that $Z(C_G(x)) = \langle x, Z(G) \rangle$. This follows from the fact that $G/Z(G)$ can be equipped with an alternating non-degenerate form induced by the commutator, see [3, Theorem 3.14] or [16, p. 83]. The claim follows from [3, Lemma 3.11] by noting that $Z(C_G(x))/Z(G)$ is the radical of co-dimension one subspace $C_G(x)/Z(G)$.

The result now follows from Theorem 1.1. □

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