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Subspace Lang conjecture and some remarks on a transcendental criterion

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Abstract. Let $b \geq 2$ be an integer and α is a non-zero real number written in b-ary expansion. Adamczewski *et al.* (*C. R. Acad. Sci. Paris* **339** (2004) 11–14) provided a criterion for an irrational number to be a transcendental number using b-ary expansion. In this paper, we make some remarks on this criterion and, under the assumption of Subspace Lang Conjecture, we extend this criterion for a much wider class of irrational numbers.

Keywords. b-Ary expansion; transcendental numbers; subspace theorem; Lang's conjecture.

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1. Introduction

Let $b \geq 2$ be an integer. A *b-ary expansion* is denoted by $a_0 \cdot a_1 a_2 \cdots a_n \cdots$ and is defined by a series of the form

$$a_0 + \frac{a_1}{b} + \frac{a_2}{b^2} + \cdots + \frac{a_n}{b^n} + \cdots, \tag{1.1}$$

where $a_0 \in \mathbb{Z}$, $a_n \in \{0, 1, \ldots, b-1\}$ for each $n \geq 1$.

A basic result states that every real number has a b-ary expansion and this way of representing real numbers provides a necessary and sufficient condition for it to be a rational number. More precisely, we know that a real number α has an eventually periodic b-ary expansion (means, $a_{n+k} = a_n$ for all $n \geq N_0$ and for some integer $k \geq 1$) if and only if it is a rational number.

A real number α is said to be an *algebraic number* if there exists a non-zero polynomial $P(X) \in \mathbb{Z}[X]$ such that $P(\alpha) = 0$. Otherwise, the real number is called *a transcendental number*.

We ask the basic question as to whether the b-ary expansion can help us to decide whether a given real number is transcendental or not? In this article, we shall deal with this question. Though the answer is not as satisfactory as in the case of rational numbers, we do have some positive answer to this question.

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In 2004, Adamczewski *et al.* [2] (see also in [1]) provided the following criterion for transcendental numbers.

Theorem A [**2**]. *Let $b \geq 2$ be an integer. Let $\alpha \in [0, 1)$ be a non-zero real number satisfying* (1.1). *Suppose there exists $\epsilon > 0$ and there exist infinitely many 3-tuples (j_n, k_n, ℓ_n) of natural numbers satisfying*

$$a_{j_n+i} = a_{j_n+k_n+i} \quad \text{for all } i = 1, 2, \ldots, \ell_n \quad \text{and for all } n = 1, 2, \ldots \tag{1.2}$$

and

$$\epsilon(j_n + k_n) \leq \ell_n \leq k_n \quad \text{for all } n = 1, 2, \ldots. \tag{1.3}$$

Then α is either a rational number or a transcendental number.

We first remark about the parameters, mainly, k_n and ℓ_n in Theorem A as follows.

Remark 1. In Theorem A, we claim that the sequence $\{k_n\}$ is unbounded. If possible, the sequence $\{k_n\}$ is bounded. That is, there exists a constant $K > 0$ such that $k_n \leq K$ for all $n = 1, 2, \ldots$. Therefore, by (1.3), it is clear that $\ell_n \leq K$ for all $n = 1, 2, \ldots$ and hence

$$\epsilon(j_n + k_n) \leq k_n \leq K \implies j_n \leq j_n + k_n \leq K\epsilon^{-1}$$

for all integers $n \geq 1$. This means that k_n, ℓ_n and j_n are bounded. Therefore, the number of tuples (j_n, k_n, ℓ_n) is finite, which is a contradiction. In [2], they consider two cases, namely, the sequence $\{k_n\}$ is bounded or otherwise. In the first case, they use Ridout's theorem [4] to prove Theorem A.

Remark 2. By Remark 1 and (1.3), we see that the sequence $\{\ell_n\}$ is also unbounded.

By (1.3), we see that ℓ_n varies between $\epsilon(j_n + k_n)$ and k_n. In this article, we look at the extreme cases, namely, $\ell_n = k_n$ for infinitely many n and the lower bound for ℓ_n. More precisely, we prove the following theorems.

Theorem 1. *Let $b \geq 2$ be an integer. Let $\alpha \in [0, 1)$ be a non-zero real number satisfying* (1.1). *Suppose there exists $\epsilon > 0$ and there exist infinitely many 3-tuples (j_n, k_n, ℓ_n) of natural numbers satisfying* (1.2), (1.3) *together with $\ell_n = k_n$ and the sequence $\{j_n\}$ is bounded. Then α is a transcendental number.*

Theorem 2. *Let $b \geq 2$ be an integer. Let $\alpha \in [0, 1)$ be a non-zero real number satisfying* (1.1) *and $\epsilon > 0$ be given. Suppose there exist infinitely many 3-tuples (j_n, k_n, ℓ_n) of natural numbers satisfying* (1.2) *and*

$$\frac{(2 + \epsilon)(\log(j_n + k_n) + \log\log b)}{\log b} \leq \ell_n \leq k_n. \tag{1.4}$$

If the Subspace Lang Conjecture is true (*see Conjecture* 1 *in* § 2), *then α is either rational or transcendental.*

Note that the lower bound $\frac{(2+\epsilon)(\log(j_n+k_n)+\log\log b)}{\log b}$ is much smaller than that of $\epsilon(j_n+k_n)$ and hence allowing a wider class of real numbers to satisfy the hypothesis of Theorem 1 compared to Theorem A at the expense of an unproven hypothesis. We prove Theorem 2 along similar lines as that of the proof of Theorem A.

To illustrate Theorem 2, we take $u_n = n[\log n]^2$ for all integers $n \geq 1$ and let

$$\alpha = \sum_{n\geq 1} \frac{1}{10^{u_n}}$$

written in base 10. By Theorem 2, it follows that α is a transcendental number. To see this, first we observe that the block of zeroes in between $\frac{1}{10^{u_n}}$ and $\frac{1}{10^{u_{n+1}}}$ is of length $u_{n+1} - u_n - 1 = (n+1)[\log(n+1)]^2 - n[\log n]^2 - 1 \geq 100[\log n]$ for all sufficiently large integer n and hence α cannot be a rational number. Now, for all large enough integers n, we let $\ell_n = 60[\log n]$, $j_n = n[\log n]^2$ and $k_n = (n+1)[\log(n+1)]^2 - n[\log n]^2$. Hence, we get, $j_n + k_n = (n+1)[\log(n+1)]^2$. Therefore, there does not exist any $\epsilon > 0$ such that $\epsilon(j_n + k_n) \leq \ell_n$ for infinitely many values of n. However, we note that

$$\log(j_n + k_n) = \log(n+1) + 2\log([\log(n+1)]) \leq 16[\log n]$$

for all sufficiently large value of n. By taking $\epsilon = 1$ in the statement of Theorem 2, we see that ℓ_n satisfies the required lower bound and hence by Theorem 2, α must be a transcendental number, provided the Subspace Lang Conjecture is true.

2. Preliminaries

In order to state Subspace Lang Conjecture, we need to introduce p-adic absolute values on a finite extension K over $\mathbb{Q}$. First, we shall define the p-adic absolute value on $\mathbb{Q}$ and then we shall extend this absolute value for a finite extension K over $\mathbb{Q}$. Any finite extension $K/\mathbb{Q}$ is referred as a *a number field*.

Let p be a prime number in $\mathbb{Z}$. Let x/y be any rational number, where $x \in \mathbb{Z}\backslash\{0\}$, $y \geq 1$ integer and $(x, y) = 1$. We define

$$\mathrm{ord}_p(x/y) = \begin{cases} n & \text{if } p^n \| x, \\ -n & \text{if } p^n \| y. \end{cases}$$

Then, the p-adic absolute value on $\mathbb{Q}$, denoted by $|\cdot|_p$ and is defined as

$$\left|\frac{x}{y}\right|_p = \left(\frac{1}{p}\right)^{\mathrm{ord}_p(x/y)} \quad \text{and} \quad |0|_p = 0.$$

In this set up, the usual absolute value $|\cdot|$ on $\mathbb{Q}$ will be denoted by $|\cdot|_\infty$.

Now, let $K/\mathbb{Q}$ be a number field and $\mathcal{O}_K$ be its ring of integers. Then, for any prime number $p \in \mathbb{Z}$, the ideal $p\mathcal{O}_K$ in $\mathcal{O}_K$ can be factored into product of prime ideals as

$$p\mathcal{O}_K = \mathfrak{p}_1^{e_1} \ldots \mathfrak{p}_g^{e_g}$$

with $e_i \geq 1$ integers and $\mathfrak{p}_i$ are prime ideals in $\mathcal{O}_K$.

Hence $p\mathcal{O}_K \subset \mathfrak{p}_i$ for all $i = 1, 2, \ldots, g$. In this situation, we say $\mathfrak{p}_i | p$ ($\mathfrak{p}_i$ divides p) for all $i = 1, 2, \ldots, g$.

Since K is the quotient field of $\mathcal{O}_K$, any $\alpha \in K$ can be written as $\alpha = x/y$ where $x, y \in \mathcal{O}_K$ with $\gcd(x\mathcal{O}_K, y\mathcal{O}_K) = \mathcal{O}_K$. Therefore, for any $\alpha \in K$ and for a given prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$, we can define

$$\mathrm{ord}_{\mathfrak{p}}(\alpha) = \begin{cases} n & \text{if } \mathfrak{p}^n \| x\mathcal{O}_K, \\ -n & \text{if } \mathfrak{p}^n \| y\mathcal{O}_K. \end{cases}$$

Also, for any non-zero prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$, the norm of $\mathfrak{p}$ is denoted by $N\mathfrak{p}$ and is defined by $N\mathfrak{p} = |\mathcal{O}_K/\mathfrak{p}|$, cardinality of the quotient ring (which is known to be finite). Now, we can extend the p-adic absolute value for any $\alpha \in K\backslash\{0\}$ as

$$|\alpha|_p = \left|\frac{x}{y}\right|_p = \prod_{\mathfrak{p}|p} \left(\frac{1}{N\mathfrak{p}}\right)^{\mathrm{ord}_{\mathfrak{p}}(\alpha)}.$$

If $p = \infty$, then we define

$$|\alpha|_\infty = |N_{K/\mathbb{Q}}(\alpha)|,$$

where $N_{K/\mathbb{Q}}(\alpha)$ is the norm of α (which is nothing but the product of all the Galois conjugates of α) in $K/\mathbb{Q}$. With these definitions, one can check that the product formula

$$|\alpha|_\infty \prod_p |\alpha|_p = 1 \tag{2.1}$$

holds for all $\alpha \in K\backslash\{0\}$. For any vector $(y_1, \ldots, y_n) \in K^n\backslash(0, 0, \ldots, 0)$, we define the *height*

$$H((y_1, \ldots, y_n)) = \prod_p \max_{1\le i\le n} \{|y_1|_p, \ldots, |y_n|_p\}$$

and the *norm*

$$\|(y_1, \ldots, y_n)\|_p = \max\{|y_1|_p, \ldots, |y_n|_p\}$$

for all primes $p \in \mathbb{Z} \cup \{\infty\}$.

We are now ready to state the p-adic version of the subspace theorem which was proved by Schlickewei [6]. This is a generalization of Roth [5], Ridout [4] and Schmidt [7] results.

Theorem 2.1 **[6]**. *Let S_f be a finite subset of prime numbers and let $S = S_f \cup \{\infty\}$. Let $n > 1$ be an integer. For every prime $p \in S$, let $L_{1,p}, \ldots, L_{n,p}$ be the given linearly independent linear forms in n-variables whose coefficients are algebraic numbers. Let $\epsilon > 0$ be given. Then any non-zero vector $(y_1, \ldots, y_n) \in \mathbb{Q}^n$ satisfying the inequality*

$$\prod_{p\in S} \prod_{i=1}^{n} \frac{|L_{i,p}(y_1, \ldots, y_n)|_p}{\|(y_1, \ldots, y_n)\|_p} \le \frac{1}{H^{n+\epsilon}((y_1, \ldots, y_n))}$$

lies only in finitely many proper subspaces of $\mathbb{Q}^n$.

It is worth mentioning that Adamczewski *et al.* [2] proved Theorem A as an application of Theorem 2.1.

Extension of Theorem 2.1 is an analogue conjecture of Lang stated by Dixit *et al.* [3] over any number fields. For our purposes, we state the conjecture over $\mathbb{Q}$.

Conjecture 1. Let S_f be a finite subset of prime numbers and let $S = S_f \cup \{\infty\}$. Let $n > 1$ be an integer. For every prime $p \in S$, let the given linear forms $L_{1,p}, \ldots, L_{n,p}$ in n-variables, whose coefficients are algebraic numbers, be linearly independent. Let $\epsilon > 0$. Then for any non-zero vector $(y_1, \ldots, y_n) \in \mathbb{Q}^n$ satisfying the inequality

$$\prod_{p\in S}\prod_{i=1}^{n}\frac{\left|L_{i,p}(y_1,\ldots,y_n)\right|_p}{\|(y_1,\ldots,y_n)\|_p} \le \frac{1}{H^n((y_1,\ldots,y_n))\log^{n-1+\epsilon}(H((y_1,\ldots,y_n)))} \tag{2.2}$$

lies only in finitely many proper subspaces of $\mathbb{Q}^n$.

3. Proofs of Theorems 1 and 2

We first prove Theorem 2 and then prove Theorem 1. Our proofs closely follows that of [2].

Proof of Theorem 2. Given that there exists $\epsilon > 0$ and a sequence $(j_n, k_n, \ell_n)_n$ of 3-tuples of natural numbers satisfying (1.2) and (1.4).

Using these conditions, we shall construct a sequence $\{\alpha_n\}_n$ of rational numbers as follows. For each $n \ge 1$, we let

$$\alpha_n = 0.a_1a_2\ldots a_{j_n}\overline{a_{j_n+1}\ldots a_{j_n+k_n}},$$

where $\overline{a_{j_n+1}\ldots a_{j_n+k_n}}$ means this block of digits are repeating. Hence, α_n is a rational number for all integers $n \ge 1$. Thus, we get

$$\begin{aligned}\alpha_n &= \frac{a_1}{b}+\cdots+\frac{a_{j_n}}{b^{j_n}}+\left(\frac{a_{j_n+1}}{b^{j_n+1}}+\cdots+\frac{a_{j_n+k_n}}{b^{j_n+k_n}}\right)\left(1+\frac{1}{b^{k_n}}+\frac{1}{b^{2k_n}}+\cdots\right)\\ &= \frac{a_1}{b}+\cdots+\frac{a_{j_n}}{b^{j_n}}+\left(\frac{a_{j_n+1}}{b^{j_n+1}}+\cdots+\frac{a_{j_n+k_n}}{b^{j_n+k_n}}\right)\frac{b^{k_n}}{(b^{k_n}-1)} = \frac{p_n}{b^{j_n}(b^{k_n}-1)}\end{aligned}$$

for some integer p_n for all integers $n \ge 1$. Therefore,

$$b^{j_n}(b^{k_n}-1)\alpha_n = p_n \quad \text{for all integers } n \ge 1. \tag{3.1}$$

Since $a_{j_n+1} = a_{j_n+k_n+1}, \ldots, a_{j_n+\ell_n} = a_{j_n+k_n+\ell_n}$, we get

$$\begin{aligned}|\alpha-\alpha_n| &= \left|\alpha-\frac{p_n}{(b^{k_n}-1)b^{j_n}}\right| \le \frac{|a_{j_n+k_n+\ell_n+1}-a_{j_n+1}|}{b^{j_n+k_n+\ell_n+1}}\\ &\quad+\cdots < \frac{b-1}{b^{j_n+k_n+\ell_n+1}}\left(1+\frac{1}{b}+\cdots\right)\\ &\le \left(\frac{b-1}{b^{j_n+k_n+\ell_n+1}}\right)\left(\frac{b}{b-1}\right) = \frac{1}{b^{j_n+k_n+\ell_n}}.\end{aligned}$$

Thus

$$|\alpha-\alpha_n| < \frac{1}{b^{j_n+k_n+\ell_n}} \quad \text{for all integers } n \ge 1. \tag{3.2}$$

Consider

$$\begin{aligned}|b^{j_n+k_n}\alpha - b^{j_n}\alpha - p_n| &= |b^{j_n+k_n}\alpha - b^{j_n}\alpha - (b^{j_n}(b^{k_n}-1))\alpha_n|\\ &= b^{j_n}(b^{k_n}-1)|\alpha-\alpha_n|\\ &< \frac{b^{j_n}(b^{k_n}-1)}{b^{j_n+k_n+\ell_n}} < \frac{1}{b^{\ell_n}}.\end{aligned}$$

Thus, we get

$$|b^{j_n+k_n}\alpha - b^{j_n}\alpha - p_n| < \frac{1}{b^{\ell_n}}. \tag{3.3}$$

By Remark 1 in the Introduction, we have seen that the sequence $\{k_n\}$ is unbounded. Without loss of generality, we shall assume that

$$k_1 < k_2 < \cdots < k_n < \cdots \text{ such that } k_n \to \infty \text{ as } n \to \infty. \tag{3.4}$$

Since we want to prove α is either rational or transcendental, we shall assume that α is not a transcendental number (and hence it is an algebraic number). Now to complete the proof, we need to prove that α is a rational number.

In order to prove that α is a rational number, we shall apply the Subspace Lang Conjecture (Conjecture 1). Let $S = \{\infty\} \cup \{p : p \text{ is a prime and } p|b\}$ be a finite subset of prime numbers which includes the infinite prime. For each prime $q \in S$, we need to define linearly independent linear forms with algebraic coefficients. Consider

$$\begin{aligned}&L_{1,\infty}(X_1, X_2, X_3) = X_1,\ L_{2,\infty}(X_1, X_2, X_3) = X_2 \quad \text{and}\\ &L_{3,\infty}(X_1, X_2, X_3) = \alpha X_1 - \alpha X_2 - X_3.\end{aligned} \tag{3.5}$$

Clearly, as α is algebraic, the linear form $L_{i,\infty}$ is with algebraic coefficients for all $i = 1, 2, 3$. Since the determinant of the coefficient matrix

$$\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ \alpha & -\alpha & -1\end{pmatrix}$$

is non-zero, we conclude that $L_{1,\infty}$, $L_{2,\infty}$ and $L_{3,\infty}$ are linearly independent linear forms. Now, for any finite prime $p \in S$, we define

$$\begin{aligned}&L_{1,p}(X_1, X_2, X_3) = X_1,\ L_{2,p}(X_1, X_2, X_3) = X_2 \quad \text{and}\\ &L_{3,p}(X_1, X_2, X_3) = X_3.\end{aligned} \tag{3.6}$$

Clearly, the linear forms $L_{1,p}$, $L_{2,p}$ and $L_{3,p}$ are linearly independent.

For any integer $n \geq 1$, we let

$$\mathbf{x}^{(n)} = (b^{j_n+k_n}, b^{j_n}, p_n) \in \mathbb{Z}^3. \tag{3.7}$$

Since $\alpha_n \in (0, 1)$ for all $n = 1, 2, \ldots$, we see that $p_n < b^{j_n+k_n}$ and hence, we get

$$\begin{aligned}H((b^{j_n+k_n}, b^{j_n}, p_n)) &= \max\{|b^{j_n+k_n}|, |b^{j_n}|, |p_n|\}\\ &\leq b^{j_n+k_n} \quad \text{for all } n = 1, 2, \ldots,\end{aligned} \tag{3.8}$$

since $|x_i|_p \le 1$ for all $x_i \in \mathbb{Z}\backslash\{0\}$ and for all finite primes p.

Note that by the definition of norm and height, we see that

$$\prod_{i=1}^{3}\prod_{p\in S}\|\mathbf{x}^{(n)}\|_p = \prod_{i=1}^{3}\prod_{p\in S}\|(b^{j_n+k_n}, b^{j_n}, p_n)\|_p \\ \ge H^3((b^{j_n+k_n}, b^{j_n}, p_n)). \tag{3.9}$$

In order to apply the Subspace Lang Conjecture, we need to compute the quantity

$$\prod_{q\in S}\left(\frac{\left|L_{1,q}\left(\mathbf{x}^{(n)}\right)\right|_q}{\|\mathbf{x}^{(n)}\|_q}\frac{\left|L_{2,q}\left(\mathbf{x}^{(n)}\right)\right|_q}{\|\mathbf{x}^{(n)}\|_q}\frac{\left|L_{3,q}\left(\mathbf{x}^{(n)}\right)\right|_q}{\|\mathbf{x}^{(n)}\|_q}\right) \tag{3.10}$$

for every integer $n = 1, 2, \ldots$.

First note that

$$\prod_{i=1}^{3}|L_{i,\infty}((b^{j_n+k_n}, b_n^j, p_n))|_\infty = b^{j_n+k_n}b^{j_n}|\alpha b^{j_n+k_n} - \alpha b^{j_n} - p_n|_\infty. \tag{3.11}$$

For any prime $q \in S\backslash\{\infty\}$, we have

$$\prod_{i=1}^{3}|L_{i,q}((b^{j_n+k_n}, b_n^j, p_n))|_q = |b^{j_n+k_n}|_q|b^{j_n}|_q|p_n|_q. \tag{3.12}$$

Therefore, by (3.11), (3.12) and the product formula (2.1),

$$\prod_{i=1}^{3}\prod_{p\in S}|L_{i,p}(\mathbf{x}^{(n)})|_p \\ = \prod_{p\in S}|b^{j_n+k_n}|_p\prod_{p\in S}|b^{j_n}|_p|\alpha b^{j_n+k_n} - \alpha b^{j_n} - p_n|\prod_{p\in S\backslash\{\infty\}}|p_n|_p \\ \le |\alpha b^{j_n+k_n} - \alpha b^{j_n} - p_n|.$$

Thus, by (3.3), we get

$$\prod_{i=1}^{3}\prod_{p\in S}|L_{i,p}(\mathbf{x}^{(n)})|_p < \frac{1}{b^{\ell_n}}. \tag{3.13}$$

Thus, by (3.9) and (3.13), we conclude that

$$\prod_{q\in S}\left(\frac{\left|L_{1,q}\left(\mathbf{x}^{(n)}\right)\right|_q}{\|\mathbf{x}^{(n)}\|_q}\frac{\left|L_{2,q}\left(\mathbf{x}^{(n)}\right)\right|_q}{\|\mathbf{x}^{(n)}\|_q}\frac{\left|L_{3,q}\left(\mathbf{x}^{(n)}\right)\right|_q}{\|\mathbf{x}^{(n)}\|_q}\right) \\ \le \frac{1}{H^3(\mathbf{x}^{(n)})}\prod_{i=1}^{3}\prod_{p\in S}|L_{i,p}(\mathbf{x}^{(n)})|_p \le \frac{1}{H^3(\mathbf{x}^{(n)})b^{\ell_n}}. \tag{3.14}$$

By (1.4), we see that

$$\frac{(2+\epsilon)(\log(j_n+k_n)+\log\log b)}{\log b} \leq \ell_n \iff \log^{(2+\epsilon)}(b^{j_n+k_n}) \leq b^{\ell_n} \iff \log^{2+\epsilon} H(\mathbf{x}^{(n)}) \leq b^{\ell_n}.$$

By putting this information in (3.14), we get

$$\prod_{q\in S}\left(\frac{\left|L_{1,q}\left(\mathbf{x}^{(n)}\right)\right|_q}{\|\mathbf{x}^{(n)}\|_q}\frac{\left|L_{2,q}\left(\mathbf{x}^{(n)}\right)\right|_q}{\|\mathbf{x}^{(n)}\|_q}\frac{\left|L_{3,q}\left(\mathbf{x}^{(n)}\right)\right|_q}{\|\mathbf{x}^{(n)}\|_q}\right) \leq \frac{1}{H^3(\mathbf{x}^{(n)})\log^{2+\epsilon} H(\mathbf{x}^{(n)})}.$$

Thus for all n, the nonzero lattice points $\mathbf{x}^{(n)} \in \mathbb{Z}^3$ satisfies the hypothesis of Subspace Lang Conjecture. Thus, by Conjecture 1, the integer lattice points $\mathbf{x}^{(n)} = (b^{j_n+k_n}, b^{j_n}, p_n)$ lie in finitely many proper subspaces of $\mathbb{Q}^3$ for all integer $n \geq 1$. Therefore, there exists a proper subspace of $\mathbb{Q}^3$ containing the integer lattice point $\mathbf{x}^{(n)} = (b^{j_n+k_n}, b^{j_n}, p_n)$ for infinitely many values of n. That is, there exist integers a_1, a_2 and a_3 with $(a_1, a_2, a_3) \neq (0, 0, 0)$ such that

$$a_1 b^{j_n+k_n} + a_2 b^{j_n} + a_3 p_n = 0 \tag{3.15}$$

holds true for infinitely many values of n's.

First note that $a_3 \neq 0$. If possible, we assume that $a_3 = 0$. Then, clearly, we have $a_1 \neq 0$ and $a_2 \neq 0$. Thus, for infinitely many values of n's, we get

$$a_1 b^{j_n+k_n} + a_2 b^{j_n} = 0 \implies b^{k_n} = -a_2/a_1,$$

which implies that k_n is a constant for infinitely many values of n, which is a contradiction to (3.4). Therefore, we conclude that $a_3 \neq 0$. Now, by (3.15), we consider

$$\lim_{n\to\infty}\left(a_1\frac{b^{j_n+k_n}}{b^{j_n}(b^{k_n}-1)} + a_2\frac{b^{j_n}}{b^{j_n}(b^{k_n}-1)} + a_3\frac{p_n}{b^{j_n}(b^{k_n}-1)}\right) = 0$$

$$\lim_{n\to\infty}\left(a_1\frac{b^{j_n+k_n}}{b^{j_n}(b^{k_n}-1)} + a_2\frac{b^{j_n}}{b^{j_n}(b^{k_n}-1)} + a_3\alpha_n\right) = 0$$

$$\implies \lim_{n\to\infty}\left(\frac{a_1 b^{k_n}}{b^{k_n}-1} + \frac{a_2}{b^{k_n}-1} + a_3\alpha_n\right) = 0$$

$$\implies a_1 + a_3\alpha = 0,$$

which implies that α is a rational number. This proves the theorem. □

Proof of Theorem 1. Let $S = \{n \in \mathbb{N} : k_n = \ell_n \text{ and } j_n \leq K\}$ (for some integer $K > 1$) be a subset of $\mathbb{N}$. By assumption, we have $|S| = \infty$. Therefore, by the Pigeon Hole Principle, there exists a subset T of S with $|T| = \infty$ such that $j_n = j_m = j_0$ for all $m, n \in T$.

Let $n \in T$ be any element. Then, by (3.2), we have

$$|\alpha - \alpha_n| = \left|\alpha - \frac{p_n}{b^{j_n}(b^{k_n}-1)}\right| < \frac{1}{b^{j_n+k_n+\ell_n}} = \frac{1}{b^{j_n}b^{2k_n}}.$$

Therefore, we get

$$\left| b^{j_n}\alpha - \frac{p_n}{b^{k_n}-1} \right| < \frac{1}{(b^{k_n}-1)^2}.$$

Since $|T| = \infty$, we see that the number $b^{j_n}\alpha = b^{j_0}\alpha$ is an irrational number and hence α is irrational. Since (1.2) and (1.3) are satisfied, by Theorem A, the result follows. □

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References

[1] Adamczewski B and Bugeaud Y, On the complexity of algebraic numbers - I, Expansions in integer bases, *Ann. Math.*, **165** (2007) 547–565

[2] Adamczewski B, Bugeaud Y and Luca F, Sur la complexité des nombres algébriques, *C. R. Acad. Sci. Paris*, **339** (2004) 11–14

[3] Dixit A B, Rath P and Shankar A, Lang's conjecture and complexity of algebraic numbers, preprint, http://www.cmi.ac.in/~adixit/Langborel.pdf

[4] Ridout D, Rational approximations to algebraic numbers, *Mathematika*, **4** (1957) 125–131

[5] Roth K F, Rational approximations to algebraic numbers, *Mathematika*, **2** (1955) 1–20; Corrigendum, 168

[6] Schlickewei H P, The $\mathfrak{p}$-adic Thue–Siegel–Roth–Schmidt theorem, *Arch. Math. (Basel)* **29** (1977) 267–270

[7] Schmidt W M, Diophantine Approximation, Lecture Notes in Mathematics 785 (1980) (Springer)

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