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Transcendence of some power series for Liouville number arguments

FATMA ÇALIŞKAN

Department of Mathematics, Faculty of Science, Istanbul University, Vezneciler, Fatih, 34134 Istanbul, Turkey
E-mail: fatmac@istanbul.edu.tr

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Abstract. In this paper, we prove that some power series with rational coefficients take either values of rational numbers or transcendental numbers for the arguments from the set of Liouville numbers under certain conditions in the field of complex numbers. We then apply this result to an algebraic number field. In addition, we establish the p-adic analogues of these results and show that these results have analogues in the field of p-adic numbers.

Keywords. Mahler and Koksma classification; Liouville number; algebraic number field; p-adic number.

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1. Introduction

In 1932, Mahler introduced a classification of complex numbers and divided them into four disjointed classes called the A-numbers, S-numbers, T-numbers and U-numbers [12]. Two years later, Mahler also came up with an analogous classification of p-adic numbers which is the p-adic completion of rational numbers with respect to the p-adic valuation for a given prime number p, and the numbers in this field were again divided into four disjointed classes called the A-numbers, S-numbers, T-numbers and U-numbers [13].

In 1939, Koksma proposed another classification based on approximation by algebraic numbers which was similar to the one developed by Mahler in 1932 [9]. This classification was also appropriate for use with p-adic numbers.

Several mathematicians have studied the transcendence of values of infinite products and series. For example, in 1971, Baron and Braune [2] investigated the transcendence of Fourier series with rational coefficients at algebraic points. In 1973, this result was extended to the trigonometric series by Bundschuh [4]. Later, in 2002, Zhu [24] generalized this result using a condition dependent on the coefficients. On the other hand, in 1986, Oryan [14] considered the values of certain power series with rational coefficients for the arguments from the set of Liouville numbers and showed that they can be either rational numbers or Liouville numbers and then generalized this result taking coefficients from an algebraic number field. Other examples for the transcendence of values of infinite products and series can also be given [1,3,6,22].

Many questions in the approximation theory which have been studied in the field of complex numbers can also be transposed to the field of p-adic numbers [5,15,20,21].

In the present paper, we deal with some power series under certain conditions for obtaining new transcendental numbers. We show that the power series with rational coefficients for the arguments from the set of Liouville numbers take values of either rational numbers or transcendental numbers under certain conditions in the field of complex numbers. Later, we give a generalized form of this result for some power series with algebraic coefficients and then prove that some power series with algebraic coefficients from a certain algebraic number field K take values of either algebraic numbers in K or transcendental numbers for the arguments from the set of Liouville numbers under certain conditions in the field of complex numbers. Later, in section 4, we establish p-adic analogues of the results from section 3. In particular, the results in section 3 contains the theorems of [16] and [8] as a special case. Besides, the results in section 3 and section 4 are partially generalizations of the theorems of [14] and [15].

2. Preliminaries

In 1932, Mahler [12] developed a classification for complex numbers as follows: Let $P(x) = a_n x^n + \cdots + a_0$ represent a polynomial with integer coefficients. The number $H(P) = \max\{|a_n|, \ldots, |a_0|\}$ is called the height of P. Let ξ be a complex number. We can then define the quantity $w_n(H, \xi) = \min\{|P(\xi)| : P(x) \in Z[x],\ H(P) \leq H,\ \deg(P) \leq n,\ P(\xi) \neq 0\}$, where n and H are natural numbers. It is obvious that $0 < w_n(H, \xi) \leq 1$ for $n \geq 1$ and $H \geq 1$. Then we can establish that

$$w_n(\xi) = \limsup_{H \to +\infty} \frac{\log \frac{1}{w_n(H,\xi)}}{\log H} \quad \text{and} \quad w(\xi) = \limsup_{n \to +\infty} \frac{w_n(\xi)}{n}.$$

It is clear that $0 \leq w_n(\xi) \leq +\infty$ and $0 \leq w(\xi) \leq +\infty$ for $n \geq 1$. If $w_n(\xi) = +\infty$ for some integers n, then $\mu(\xi)$ is defined as the smallest such integer. In addition, if $w_n(\xi) < +\infty$ for every n, then $\mu(\xi) = +\infty$. Both $\mu(\xi)$ and $w(\xi)$ cannot be simultaneously finite. Therefore, the following four possibilities exist for the complex number ξ:

(1) an A-number if $w(\xi) = 0$ and $\mu(\xi) = +\infty$,
(2) an S-number if $0 < w(\xi) < +\infty$ and $\mu(\xi) = +\infty$,
(3) a T-number if $w(\xi) = +\infty$ and $\mu(\xi) = +\infty$,
(4) a U-number if $w(\xi) = +\infty$ and $\mu(\xi) < +\infty$.

The A-numbers are composed of all algebraic numbers. If $\mu(\xi) = m$, then ξ is called a U-number of degree m ($m \geq 1$), with U_m denoting the set of U-numbers of degree m. $U_m \cap U_n = \emptyset$ if $m \neq n$, and $U = \cup_{m=1}^{\infty} U_m$. Another classification of complex numbers was identified by Koksma [9]. If α is an algebraic number, then $P(z)$ is the minimal polynomial of α such that its coefficients are rational integers and relative prime numbers, with the highest coefficient being positive. Then the height $H(\alpha)$ of the algebraic number α is defined by $H(\alpha) = H(P)$, and the degree $\deg(\alpha)$ of α is defined as the degree of $P(z)$. For the complex number ξ and the natural number n, we define the quantity as $w_n^*(H, \xi) = \min\{|\xi - \alpha| : H(\alpha) \leq H,\ \deg(\alpha) \leq n,\ \xi \neq \alpha\}$, and hence we get

$$w_n^*(\xi) = \limsup_{H\to+\infty} \frac{\log \frac{1}{Hw_n^*(H,\xi)}}{\log H} \quad \text{and} \quad w^*(\xi) = \limsup_{n\to+\infty} \frac{w_n^*(\xi)}{n}.$$

The inequalities $0 \leq w_n^*(\xi) \leq +\infty$ and $0 \leq w^*(\xi) \leq +\infty$ hold true for $n \geq 1$. If $w_n^*(\xi) = +\infty$ for some integers n, then $\mu^*(\xi)$ is defined as the smallest such integer, and if $w_n^*(\xi) < +\infty$ for every n, then $\mu^*(\xi) = +\infty$. Both $\mu^*(\xi)$ and $w^*(\xi)$ cannot be simultaneously finite. Therefore, four possibilities exist for a complex number ξ:

(1) an A^*-number if $w^*(\xi) = 0$ and $\mu^*(\xi) = +\infty$,
(2) an S^*-number if $0 < w^*(\xi) < +\infty$ and $\mu^*(\xi) = +\infty$,
(3) a T^*-number if $w^*(\xi) = +\infty$ and $\mu^*(\xi) = +\infty$,
(4) a U^*-number if $w^*(\xi) = +\infty$ and $\mu^*(\xi) < +\infty$.

If $\mu^*(\xi) = m$, then ξ is called a U^*-number of degree m ($m \geq 1$). It follows from the results of Wirsing [23] that the classifications of Mahler and Koksma are equivalent; that is, the A, S, T, U-numbers are the same as the A^*, S^*, T^*, U^*-numbers. Moreover, every U-number of degree m is also a U^*-number of degree m, with the converse also being true. A complex number ξ is called a Liouville number if for each natural number n, there exists the rational integer numbers p_n and q_n ($q_n > 1$) such that the inequality

$$0 < \left| \xi - \frac{p_n}{q_n} \right| < q_n^{-n}$$

holds true. Every Liouville number is a U-number of degree 1, and every U-number of degree 1 is a Liouville number.

Let p be a fixed prime number. The notations $|.|_p$ and $\mathbb{Q}_p$ denote the p-adic valuation of the rational numbers and the field of p-adic numbers, respectively. In 1935, Mahler [13] developed a way to classify p-adic numbers as follows: Let ξ be a p-adic number and we can then define the quantity $w_n(H, \xi) = \min\{|P(\xi)|_p : P(x) \in Z[x],\ H(P) \leq H,\ \deg(P) \leq n,\ P(\xi) \neq 0\}$, where n and H are natural numbers. From this, it is clear that $0 < w_n(H, \xi) \leq 1$ for $n \geq 1$ and $H \geq 1$. Then we can establish that

$$w_n(\xi) = \limsup_{H\to+\infty} \frac{\log \frac{1}{w_n(H,\xi)}}{\log H} \quad \text{and} \quad w(\xi) = \limsup_{n\to+\infty} \frac{w_n(\xi)}{n}.$$

The inequalities $0 \leq w_n(\xi) \leq +\infty$ and $0 \leq w(\xi) \leq +\infty$ hold true for $n \geq 1$. If $w_n(\xi) = +\infty$ for some integers n, then $\mu(\xi)$ is defined as the smallest such integer, and if $w_n(\xi) < +\infty$ for every n, then $\mu(\xi) = +\infty$. Both $\mu(\xi)$ and $w(\xi)$ cannot be simultaneously finite. Therefore, there are four possibilities for the p-adic number ξ:

(1) an A-number if $w(\xi) = 0$ and $\mu(\xi) = +\infty$,
(2) an S-number if $0 < w(\xi) < +\infty$ and $\mu(\xi) = +\infty$,
(3) a T-number if $w(\xi) = +\infty$ and $\mu(\xi) = +\infty$,
(4) a U-number if $w(\xi) = +\infty$ and $\mu(\xi) < +\infty$.

The A-numbers are composed of all p-adic algebraic numbers. If $\mu(\xi) = m$, then ξ is called a U-number of degree m ($m \geq 1$), with U_m denoting the set of U-numbers of degree m. $U_m \cap U_n = \emptyset$ if $m \neq n$, and $U = \cup_{m=1}^{\infty} U_m$. The classification of complex numbers given by Koksma [9] can also be used for $\mathbb{Q}_p$. If α is an algebraic number in $\mathbb{Q}_p$, then $P(z)$ is the minimal polynomial of α such that its coefficients are rational integers and relative prime numbers, with the highest coefficient begin positive. Then the height $H(\alpha)$ of an algebraic number α in $\mathbb{Q}_p$ is defined by $H(\alpha) = H(P)$, and the degree deg (α) of

α is defined as the degree of $P(z)$. For the p-adic number ξ and natural numbers n, we define the quantity $w_n^*(H,\xi) = \min\{|\xi-\alpha|_p : H(\alpha) \le H,\ \deg(\alpha) \le n,\ \xi \ne \alpha\}$, and then we can establish that

$$w_n^*(\xi) = \limsup_{H\to+\infty} \frac{\log \frac{1}{H w_n^*(H,\xi)}}{\log H} \quad \text{and} \quad w^*(\xi) = \limsup_{n\to+\infty} \frac{w_n^*(\xi)}{n}.$$

The inequalities $0 \le w_n^*(\xi) \le +\infty$ and $0 \le w^*(\xi) \le +\infty$ hold true for $n \ge 1$. If $w_n^*(\xi) = +\infty$ for some integers n, then $\mu^*(\xi)$ is defined as the smallest such integer, and if $w_n^*(\xi) < +\infty$ for every n, then $\mu^*(\xi) = +\infty$. Both $\mu^*(\xi)$ and $w^*(\xi)$ cannot be simultaneously finite. Therefore, only four possibilities for the p-adic number ξ are possible:

(1) an A^*-number if $w^*(\xi) = 0$ and $\mu^*(\xi) = +\infty$,
(2) an S^*-number if $0 < w^*(\xi) < +\infty$ and $\mu^*(\xi) = +\infty$,
(3) a T^*-number if $w^*(\xi) = +\infty$ and $\mu^*(\xi) = +\infty$,
(4) a U^*-number if $w^*(\xi) = +\infty$ and $\mu^*(\xi) < +\infty$.

If $\mu^*(\xi) = m$, then ξ is called a U^*-number of degree m ($m \ge 1$). It follows from the results of Schlikewei [18] and Long [11] that the classifications of Mahler and Koksma are equivalent. In other words, the p-adic $A,\ S,\ T$ and U-numbers are the same as the p-adic $A^*,\ S^*,\ T^*$ and U^*-numbers. Moreover, every p-adic U-number of degree m is also a p-adic U^*-number of degree m, with the converse also being true. A p-adic number ξ is called a p-adic Liouville number if for each natural number n, there exists the rational integer numbers p_n and q_n ($q_n > 1$) such that the inequality

$$0 < \left|\xi - \frac{p_n}{q_n}\right|_p < H_n^{-n} \ \ (H_n = \max\{|p_n|, |q_n|\})$$

holds true. Every p-adic Liouville number is a p-adic U-number of degree 1 and every p-adic U-number of degree 1 is a p-adic Liouville number.

Roth [17] obtained the best possible exponent for approximation of algebraic irrationals by rational numbers. Roth's theorem can be given as follows: If α is an irrational algebraic number and $\epsilon > 0$ is arbitrarily small, then there are only finitely many integer solutions p and $q > 0$ of the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}.$$

The p-adic analogue of the Roth's theorem has been established by Lang [10].

3. Some power series in the field of complex numbers

We consider the power series

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \tag{1}$$

with non-zero rational coefficients $c_n = \frac{b_n}{a_n}$ (a_n and b_n are integers; $a_n > 1$ for sufficiently large n) satisfying the following two conditions:

$$\sigma := \liminf_{n\to+\infty} \frac{\log a_{n+1}}{\log a_n} > 1 \tag{2}$$

and

$$\theta := \limsup_{n\to+\infty} \frac{\log|b_n|}{\log a_n} < 1. \tag{3}$$

From (2), we know that there is a sufficiently small $\varepsilon_1(>0)$ such that $\sigma_1 = \sigma - \varepsilon_1 > 1$. Since $a_n > 1$ for sufficiently large n, then from (2), we get

$$\log a_{n+1} > \sigma_1 \log a_n \quad (n \geq N_1) \tag{4}$$

for a sufficiently large $N_1 = N_1(\varepsilon_1)$. Hence, we have

$$\log a_n > \sigma_1^{n-N_1} \log a_{N_1} \tag{5}$$

for $n \geq N_1$. Therefore, we can infer that

$$\lim_{n\to+\infty} a_n = +\infty, \lim_{n\to+\infty} \frac{\log a_n}{n} = +\infty \quad \text{and} \quad \lim_{n\to+\infty} \frac{\log a_n}{n^2} = +\infty. \tag{6}$$

From (3), there exists a sufficiently small $\varepsilon_2(>0)$ such that

$$|b_n| < a_n^{\theta+\varepsilon_2} \tag{7}$$

for sufficiently large n, where $\theta + \varepsilon_2 < 1$. Therefore, from (6) and (7) it follows that the radius of convergence of the power series $f(x)$ is infinity. From (4), we obtain

$$a_m < a_n^{\left(\frac{1}{\sigma_1}\right)^{(n-m)}} \quad (n > m \geq N_1), \tag{8}$$

and since $0 < \frac{1}{\sigma_1} < 1$, we have

$$a_n \leq A_n \leq C_0\, a_n^{\left(\frac{\sigma_1}{\sigma_1-1}\right)} \quad (n > N_1), \tag{9}$$

where C_0 is a suitable positive number and $A_n = [a_0, a_1, \ldots, a_n]$. From (9), we then get $1 \leq \limsup_{n\to+\infty} \frac{\log A_n}{\log a_n} \leq \frac{\sigma-\varepsilon_1}{\sigma-\varepsilon_1-1}$. If we choose a sufficiently small $\varepsilon_1(>0)$, then we have

$$1 \leq u \leq \frac{\sigma}{\sigma-1}, \tag{10}$$

where $u := \limsup \frac{\log A_n}{\log a_n}$. Furthermore, since $1-\theta-\varepsilon_2 > 0$, from (5) it follows that

$$0 < \left(\frac{a_{n+1}}{a_{n+\nu+1}}\right)^{1-\theta-\varepsilon_2} |\alpha|^\nu < \left(\frac{1}{2}\right)^\nu \quad (\nu = 1, 2, \ldots) \tag{11}$$

for sufficiently large n.

In the next two theorems, we consider the Liouville number α for which the following properties hold: There exist rational numbers p_n/q_n $(q_n > 1)$ and a sequence $w(n)$ such that equations

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^{nw(n)}} \quad \left(\lim_{n\to+\infty} w(n) = +\infty\right) \tag{12}$$

and

$$a_n^{\delta_1} \le q_n^n \le a_n^{\delta_2} \tag{13}$$

hold for sufficiently large n, where δ_1 and δ_2 are real numbers such that $e < \delta_1 \le \delta_2$ for a real number e.

Theorem 1. *If $e = u$ and $\sigma(1-\theta) > 4\delta_2$ for the power series (1), then $f(\alpha)$ is either a rational number or a transcendental number.*

Proof. We can consider the polynomials $f_n(x) = \sum_{\nu=0}^{n} c_\nu x^\nu$ $(n = 1, 2, \ldots)$. From (7) and (11), we then get

$$\begin{aligned} |f(\alpha) - f_n(\alpha)| &\le \sum_{\nu=n+1}^{\infty} \frac{1}{a_\nu^{1-\theta-\varepsilon_2}} |\alpha|^\nu \\ &\le \frac{|\alpha|^{n+1}}{a_{n+1}^{1-\theta-\varepsilon_2}} \left\{1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right\} = \frac{2|\alpha|^{n+1}}{a_{n+1}^{1-\theta-\varepsilon_2}} \end{aligned} \tag{14}$$

for sufficiently large n. In addition, from (6), there is a sufficiently small $\varepsilon_3 (> 0)$ such that $4|\alpha|^{n+1} < a_{n+1}^{\varepsilon_3}$ for sufficiently large n. Therefore, from (4) and (14), we have

$$|f(\alpha) - f_n(\alpha)| \le \frac{1}{2a_n^{(\sigma-\varepsilon_1)(1-\theta-\varepsilon_2-\varepsilon_3)}} \tag{15}$$

for sufficiently large n. Since $u = \limsup \frac{\log A_n}{\log a_n}$ and $u < \delta_1 \le \delta_2$, using (15), we obtain

$$|f(\alpha) - f_n(\alpha)| \le \frac{1}{2(A_n q_n^n)^{\frac{(\sigma-\varepsilon_1)(1-\theta-\varepsilon_2-\varepsilon_3)}{2\delta_2}}} \tag{16}$$

for sufficiently large n. From (12), we get $\left|\frac{p_n}{q_n}\right| < |\alpha| + 1$ for sufficiently large n. Hence, from this information, (6), (7) and the equation $u = \limsup \frac{\log A_n}{\log a_n}$, it follows that

$$\begin{aligned} \left|f_n(\alpha) - f_n\left(\frac{p_n}{q_n}\right)\right| &\le \frac{1}{2} q_n^{-nw(n)} \left(\frac{q_n^n}{a_n^{u+\varepsilon_4}}\right)^{\frac{w(n)}{2}} \\ &\le \frac{1}{2} q_n^{-nw(n)} \left(\frac{q_n^n}{A_n}\right)^{\frac{w(n)}{2}} = \frac{1}{2\left(A_n q_n^n\right)^{\frac{w(n)}{2}}} \end{aligned} \tag{17}$$

for sufficiently large n, where $\varepsilon_4(>0)$ is chosen such that $0 < u + \varepsilon_4 < \delta_1$. Hence it follows from $\lim_{n\to+\infty} w(n) = +\infty$, (16) and (17),

$$\left|f(\alpha) - f_n\left(\frac{p_n}{q_n}\right)\right| \leq \frac{1}{\left(A_n q_n^n\right)^{\frac{(\sigma-\varepsilon_1)(1-\theta-\varepsilon_2-\varepsilon_3)}{2\delta_2}}} \tag{18}$$

for sufficiently large n. If we choose sufficiently small ε_1, ε_2 and ε_3, then a suitable positive number ε exists such that $\frac{(\sigma-\varepsilon_1)(1-\theta-\varepsilon_2-\varepsilon_3)}{2\delta_2} > \frac{\sigma(1-\theta)}{2\delta_2} - \varepsilon$. Since $\sigma(1-\theta) > 4\delta_2$, it is then possible to choose a positive number ε such that

$$\frac{\sigma(1-\theta)}{2\delta_2} - \varepsilon \geq 2 + \varepsilon. \tag{19}$$

We can take $f_n\left(\frac{p_n}{q_n}\right) = \frac{h_n}{A_n q_n^n}$, where $h_n, A_n \in \mathbb{Z}$. Therefore, from (18) and (19), we get

$$\left|f(\alpha) - \frac{h_n}{A_n q_n^n}\right| < \frac{1}{\left(A_n q_n^n\right)^{2+\varepsilon}} \tag{20}$$

for sufficiently large n, where ε is a suitable positive number dependent on ε_1, ε_2 and ε_3. If the sequence $\left\{\frac{h_n}{A_n q_n^n}\right\}$ is constant, then $f(\alpha)$ is a rational number. Otherwise, $f(\alpha)$ is a transcendental number according to Roth's theorem [17]. □

Let K be an algebraic number field of degree m. We now consider the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{\eta_n}{a_n} x^n \tag{21}$$

with non-zero algebraic coefficients, where η_n is an algebraic integer in K and a_n is a rational integer ($a_n > 1$ for sufficiently large n) satisfying the following two conditions:

$$\sigma := \liminf_{n\to+\infty} \frac{\log a_{n+1}}{\log a_n} > 1 \tag{22}$$

and

$$\theta := \limsup_{n\to+\infty} \frac{\log H(\eta_n)}{\log a_n} < 1, \tag{23}$$

where $H(\eta_n)$ represents the height of the algebraic number η_n.

From (22), the equations (4), (5), (6), (8), (9), (11) for the sequence $\{a_n\}$ hold. In addition, there is a sufficiently small $\varepsilon_2(>0)$ such that $\theta + \varepsilon_2 < 1$. Furthermore, from (23) we know that

$$H(\eta_n) < a_n^{\theta+\varepsilon_2} \tag{24}$$

for sufficiently large n. Therefore, from Lemma 2, we have for sufficiently large n,

$$|\eta_n| < 2a_n^{\theta+\varepsilon_2}. \tag{25}$$

Hence, the radius of convergence of the power series $f(x)$ is infinity. Two lemmas are needed to prove the next theorem.

Lemma 2 [19]. *Let ξ be an algebraic number of height h. Then*

$$|\xi| \leq h+1.$$

Lemma 3 [7]. *Let $\alpha_1, \ldots, \alpha_k$ $(k \geq 1)$ be algebraic numbers in an algebraic number field K of degree g, and let $F(y, x_1, \ldots, x_k)$ be a polynomial with integral coefficients so that its degree in y is at least* 1. *If ξ is an algebraic number such that $F(\xi, \alpha_1, \ldots, \alpha_k) = 0$, then the degree of ξ is $\leq dg$, and*

$$H(\xi) \leq 3^{2dg+(l_1+\cdots+l_k)g} H^g H(\alpha_1)^{l_1 g} \ldots H(\alpha_k)^{l_k g},$$

where $H(\xi)$ is the height of ξ, $H(\alpha_i)$ is the height of α_i $(i = 1, \ldots, k)$, H is the maximum of the absolute values of the coefficients of the polynomial F, l_i is the degree of F in $x_i (i = 1, \ldots, k)$ and d is the degree of F in y.

Theorem 4. *If $e = 1$ and $2(\sigma(1+\theta) + (\sigma-1)\delta_2)m < \sigma(1-\theta)(\sigma-1)$ for the power series* (21), *then $f(\alpha)$ is either an algebraic number in K or a transcendental number.*

Proof. We need to consider the polynomials $f_n(x) = \sum_{\nu=0}^{n} \frac{\eta_\nu}{a_\nu} x^\nu$ $(n = 1, 2, \ldots)$. Let $\gamma_n := f_n(\frac{p_n}{q_n})$, where γ_n $(n = 1, 2, \ldots)$ is an algebraic number in K; therefore $\deg(\gamma_n) \leq m$. Now we must obtain an upper bound for the height of γ_n, and to do this, we can use Lemma 3 with the polynomial

$$F(y, x_0, x_1, \ldots, x_n) = A_n q_n^n y - A_n q_n^n \sum_{\nu=0}^{n} \frac{1}{a_\nu} \left(\frac{p_n}{q_n}\right)^\nu x_\nu,$$

where $F(\gamma_n, \eta_0, \eta_1, \ldots, \eta_n) = 0$. Here,

$$H = \max_{\nu=0}^{n} \left\{ A_n q_n^n, A_n q_n^n \frac{1}{a_\nu} \left|\frac{p_n}{q_n}\right|^\nu \right\} \leq A_n q_n^n d_0^n, \tag{26}$$

where $|\frac{p_n}{q_n}| \leq d_0$ for a sufficiently large integer d_0. Now from Lemma 3, we know that $H(\gamma_n) \leq 3^{2m+(n+1)m} H^m H(\eta_0)^m \ldots H(\eta_n)^m$. Using (24) and (26), we can obtain $H(\gamma_n) \leq 3^{(n+3)m} (A_n q_n^n d_0^n)^m C_1^m (a_0 a_1 \ldots a_n)^{(\theta+\varepsilon_2)m}$, for sufficiently large n, in which C_1 is a sufficiently large positive integer. From (6) and (9), we get $H(\gamma_n) \leq C_2^{nm} q_n^{nm} a_n^{(\frac{\sigma_1}{\sigma_1-1}+\varepsilon_3)(1+\theta+\varepsilon_2)m}$ for sufficiently large n, in which C_2 is a sufficiently large positive integer and ε_3 is a sufficiently small positive constant. Finally, using this information along with (6) and (13), we have

$$H(\gamma_n) \leq a_n^{\left(\left(\frac{\sigma-\varepsilon_1}{\sigma-\varepsilon_1-1}+\varepsilon_3\right)(1+\theta+\varepsilon_2)+\delta_2+\varepsilon_4\right)m} \tag{27}$$

for sufficiently large n, in which ε_4 is a sufficiently small positive constant. Next, from (12), we know that

$$\left|f_n(\alpha) - f_n\left(\frac{p_n}{q_n}\right)\right| \le n^2 q_n^{-nw(n)}(|\alpha|+1)^{n-1} \max_{1\le\nu\le n}\{|\eta_\nu|\}$$

for sufficiently large n. Thus, using (6), (13), (25) and (27) we have

$$\begin{aligned}\left|f_n(\alpha) - f_n\left(\frac{p_n}{q_n}\right)\right| &\le \frac{2n^2(|\alpha|+1)^{n-1}}{a_n^{\delta_1 w(n)-\theta-\varepsilon_2}} \le \frac{\frac{1}{2}}{a_n^{\delta_1 w(n)-(1+\theta+\varepsilon_2)}} \\ &< \frac{\frac{1}{2}}{H(\gamma_n)^{\frac{\delta_1 w(n)-(1+\theta+\varepsilon_2)}{\left(\left(\frac{\sigma-\varepsilon_1}{\sigma-\varepsilon_1-1}+\varepsilon_3\right)(\theta+\varepsilon_2+1)+\delta_2+\varepsilon_4\right)m}}}\end{aligned} \tag{28}$$

for sufficiently large n. From (4), (6), (11) and (25),

$$\begin{aligned}|f(\alpha)-f_n(\alpha)| &\le \frac{2|\alpha|^{n+1}}{a_{n+1}^{1-\theta-\varepsilon_2}}\left\{1+\frac{1}{2}+\frac{1}{2^2}+\dots\right\} \\ &\le \frac{4|\alpha|^{n+1}}{a_{n+1}^{1-\theta-\varepsilon_2}} < \frac{\frac{1}{2}}{a_n^{(\sigma-\varepsilon_1)(1-\theta-\varepsilon_2-\varepsilon_5)}}\end{aligned}$$

for sufficiently large n. Finally we can use this and (27) to get

$$|f(\alpha)-f_n(\alpha)| \le \frac{\frac{1}{2}}{H(\gamma_n)^{\frac{(\sigma-\varepsilon_1)(1-\theta-\varepsilon_2-\varepsilon_5)}{\left(\left(\frac{\sigma-\varepsilon_1}{\sigma-\varepsilon_1-1}+\varepsilon_3\right)(1+\theta+\varepsilon_2)+\delta_2+\varepsilon_4\right)m}}} \tag{29}$$

for sufficiently large n. Hence, from (28), (29) and $\lim_{n\to+\infty} w(n) = +\infty$, it follows that

$$\left|f(\alpha) - f_n\left(\frac{p_n}{q_n}\right)\right| \le \frac{1}{H(\gamma_n)^{\frac{(\sigma-\varepsilon_1)(1-\theta-\varepsilon_2-\varepsilon_5)}{\left(\left(\frac{\sigma-\varepsilon_1}{\sigma-\varepsilon_1-1}+\varepsilon_3\right)(1+\theta+\varepsilon_2)+\delta_2+\varepsilon_4\right)m}}} \tag{30}$$

for sufficiently large n. If we choose sufficiently small $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ and ε_5, then there is a suitable positive number ε such that

$$\frac{(\sigma-\varepsilon_1)(1-\theta-\varepsilon_2-\varepsilon_5)}{\left(\left(\frac{\sigma-\varepsilon_1}{\sigma-\varepsilon_1-1}+\varepsilon_3\right)(1+\theta+\varepsilon_2)+\delta_2+\varepsilon_4\right)m} > \frac{\sigma(1-\theta)}{\left(\left(\frac{\sigma}{\sigma-1}\right)(1+\theta)+\delta_2\right)m} - \varepsilon.$$

Since $m < \frac{\sigma(1-\theta)(\sigma-1)}{2(\sigma(1+\theta)+(\sigma-1)\delta_2)}$, it is possible to choose a positive number ε so that

$$\frac{\sigma(1-\theta)(\sigma-1)}{(\sigma(1+\theta)+(\sigma-1)\delta_2)m} - \varepsilon \ge 2+\varepsilon. \tag{31}$$

Since $f_n(\frac{p_n}{q_n}) = \gamma_n \in K$ in which $\deg(\gamma_n) \le m \quad (n = 1, 2, \dots)$, from (30) and (31), we know that

$$|f(\alpha)-\gamma_n| < \frac{1}{H(\gamma_n)^{2+\varepsilon}} \tag{32}$$

for sufficiently large n, where ε is a suitable positive number dependent on $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ and ε_5. If the sequence $\{\gamma_n\}$ is constant, then $f(\alpha)$ is an algebraic number in K. Otherwise, $f(\alpha)$ is a transcendental number according to Roth's theorem [17]. □

4. Some power series in the field of *p*-adic numbers

In the p-adic field $\mathbb{Q}_p$, we consider the power series

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \tag{33}$$

with non-zero rational coefficients $c_n = \frac{b_n}{a_n}$ (a_n and b_n are integers; $b_n \neq 0$, $(a_n, b_n) = 1$; $a_n > 1$ for sufficiently large n) that satisfies the following conditions:

$$\sigma := \liminf_{n\to+\infty} \frac{u_{n+1}}{u_n} > 1 \tag{34}$$

and

$$\lambda := \limsup_{n\to+\infty} \frac{\log_p \ A_n B_n}{u_n} < +\infty, \tag{35}$$

where $|c_n|_p \leq p^{-u_n}$, $\lambda \geq 0$, $B_n = \max_{0\leq i\leq n}\{|b_i|\}$, $A_n = [a_0, a_1, \ldots, a_n]$ and $u_n > 0$ for sufficiently large n.

From (34), there exists a sufficiently small $\varepsilon_1 > 0$ such that $\sigma_1 = \sigma - \varepsilon_1 > 1$. Since $a_n > 1$ for sufficiently large n, from (34), we get

$$u_{n+1} > \sigma_1 u_n \quad (n \geq N_1) \tag{36}$$

for sufficiently large natural number $N_1 = N_1(\varepsilon_1)$. Then

$$u_n > \sigma_1^{n-N_1} u_{N_1} \tag{37}$$

for $n \geq N_1$. Therefore, we have

$$\lim_{n\to+\infty} u_n = +\infty \quad \text{and} \quad \lim_{n\to+\infty} \frac{u_n}{n} = +\infty. \tag{38}$$

Hence, the radius of convergence of the power series $f(x)$ is infinity.

In this section, we consider the p-adic Liouville number α for which the following holds: There exist rational numbers p_n/q_n ($q_n > 1$) and a sequence $w(n)$ such that equations

$$\left|\alpha - \frac{p_n}{q_n}\right|_p < \frac{1}{H_n^{nw(n)}} \quad \left(\lim_{n\to+\infty} w(n) = +\infty\right) \tag{39}$$

and

$$p^{u_n\delta_1} \le H_n^n \le p^{u_n\delta_2} \tag{40}$$

hold for sufficiently large n, where δ_1 and δ_2 are real numbers such that $0 < \delta_1 \le \delta_2$ and $H_n = \max\{|p_n|, |q_n|\}$.

Theorem 5. *If $\lambda + \delta_2 > 1$ and $\sigma > 4(\lambda + \delta_2)$ for the power series (33), then $f(\alpha)$ is either a rational number or a p-adic transcendental number.*

Proof. We consider the polynomials $f_n(x) = \sum_{\nu=0}^{n} c_\nu x^\nu$ $(n = 1, 2, \ldots)$. There is an integer number h such that $|\alpha|_p = p^h$. From (38), we can deduce that there is a sufficiently small positive number ε_2 such that

$$1 - \frac{nh}{u_n} > 1 - \varepsilon_2 \tag{41}$$

for sufficiently large n. Since $|c_n|_p \le p^{-u_n}$, we know from (41) that

$$|c_\nu \alpha^\nu|_p \le p^{-(1-\varepsilon_2)u_\nu} \qquad (\nu = n+1, n+2, \ldots) \tag{42}$$

for sufficiently large n. In addition, from (36) and (42), we know that

$$|f(\alpha) - f_n(\alpha)|_p \le p^{-(1-\varepsilon_2)(\sigma-\varepsilon_1)u_n} \tag{43}$$

for sufficiently large n. We can determine $f_n(\frac{p_n}{q_n}) := \frac{P_n}{Q_n}$ and we have $\eta_n = \max(|P_n|, |Q_n|) \le (n+1)A_n B_n H_n^n$. Using (35), we can then deduce that there is a sufficiently small positive number ε_3 such that $A_n B_n \le p^{u_n(\lambda+\varepsilon_3)}$ for sufficiently large n. From this information and (40), it follows that

$$\eta_n \le (n+1)p^{u_n(\lambda+\varepsilon_3+\delta_2)} \tag{44}$$

for sufficiently large n. Furthermore, from (39), we know that $\left|\frac{p_n}{q_n}\right|_p \le |\alpha|_p + 1$ for sufficiently large n. Since $\{u_n\}$ is a monotone increasing sequence, it follows that $p^{-u_\nu} < C$ $(\nu = 0, 1, \ldots)$, where C is a suitable positive number. Therefore, we have

$$\left|f_n(\alpha) - f_n\left(\frac{p_n}{q_n}\right)\right|_p \le \left|\alpha - \frac{p_n}{q_n}\right|_p (|\alpha|_p + 1)^{n-1}\ C \le C_1^n p^{-u_n\delta_1 w(n)}$$

for sufficiently large n, where C_1 represents a suitable positive constant. Therefore, we know from (38) that there is a sufficiently small positive number ε_4 such that

$$\left|f_n(\alpha) - f_n\left(\frac{p_n}{q_n}\right)\right|_p \le p^{-u_n(\delta_1 w(n)-\varepsilon_4)} \tag{45}$$

for sufficiently large n. Therefore, from (38), (43), (45) and $\lim_{n\to+\infty} w(n) = +\infty$, we know that

$$\left|f(\alpha) - f_n\left(\frac{p_n}{q_n}\right)\right|_p \le \max\{p^{-(1-\varepsilon_2)(\sigma-\varepsilon_1)u_n}, p^{-u_n(\delta_1 w(n)-\varepsilon_4)}\}$$
$$= p^{-(1-\varepsilon_2)(\sigma-\varepsilon_1)u_n} \le ((n+1)p^{u_n})^{\frac{-(1-\varepsilon_2)(\sigma-\varepsilon_1)}{2}}$$

for sufficiently large n. In addition, from (44) and $\lambda+\delta_2>1$, we get

$$\left|f(\alpha) - f_n\left(\frac{p_n}{q_n}\right)\right|_p \le \frac{1}{\eta_n^{\frac{(1-\varepsilon_2)(\sigma-\varepsilon_1)}{2(\lambda+\varepsilon_3+\delta_2)}}} \tag{46}$$

for sufficiently large n. If ε_1, ε_2 and ε_3 are sufficiently small, then we can obtain a suitable positive number ε such that $\frac{(1-\varepsilon_2)(\sigma-\varepsilon_1)}{2(\lambda+\varepsilon_3+\delta_2)} > \frac{\sigma}{2(\lambda+\delta_2)} - \varepsilon$. Since $\sigma > 4(\lambda+\delta_2)$, it is possible to choose a positive number ε so that

$$\frac{\sigma}{2(\lambda+\delta_2)} - \varepsilon \ge 2+\varepsilon. \tag{47}$$

Therefore, it follows from (46) and (47) that for sufficiently large n,

$$\left|f(\alpha) - \frac{P_n}{Q_n}\right|_p < \frac{1}{\eta_n^{2+\varepsilon}}, \tag{48}$$

where ε is a suitable positive number that is dependent on ε_1, ε_2 and ε_3. If the sequence $\left\{\frac{P_n}{Q_n}\right\}$ is constant, then $f(\alpha)$ is a rational number. Otherwise, $f(\alpha)$ is a p-adic transcendental number according to Lang's theorem [10]. □

Let K be a p-adic algebraic number field of degree m. In the p-adic field $\mathbb{Q}_p$, we consider the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{\eta_n}{a_n} x^n \tag{49}$$

with the p-adic algebraic coefficients, in which η_n is a p-adic algebraic integer in K and a_n is a rational integer ($a_n > 1$ for sufficiently large n) that satisfies the following conditions:

$$\sigma := \liminf_{n\to+\infty} \frac{u_{n+1}}{u_n} > 1 \tag{50}$$

and

$$\lambda := \limsup_{n\to+\infty} \frac{\log_p A_n H(\eta_n)}{u_n} < +\infty, \tag{51}$$

where $|c_n|_p \le p^{-u_n}$, $\lambda \ge 0$, $A_n = [a_0, a_1, \ldots, a_n]$ and $u_n > 0$ for sufficiently large n. In addition, $H(\eta_n)$ is the height of the algebraic numbers η_n.

From (50), for the sequence $\{u_n\}$, (36), (37) and (38) hold true. Hence, the radius of convergence of the power series $f(x)$ is infinity.

We will employ Theorem 7 after the following lemma.

Lemma 6 [7]. *Let $\alpha_1, \ldots, \alpha_k$ $(k \geq 1)$ be p-adic algebraic numbers in a p-adic algebraic number field K of degree g, and let $F(y, x_1, \ldots, x_k)$ be a polynomial with integral coefficients so that its degree in y is at least* 1. *If ξ is a p-adic algebraic number such that $F(\xi, \alpha_1, \ldots, \alpha_k) = 0$, then the degree of ξ is $\leq dg$, and*

$$H(\xi) \leq 3^{2dg+(l_1+\cdots+l_k)g} H^g H(\alpha_1)^{l_1 g} \ldots H(\alpha_k)^{l_k g},$$

where $H(\xi)$ is the height of ξ, $H(\alpha_i)$ is the height of α_i $(i = 1, \ldots, k)$, H is the maximum of the absolute values of the coefficients of F, l_i is the degree of F in x_i $(i = 1, \ldots, k)$ and d is the degree of F in y.

Theorem 7. *If $4m(\sigma\lambda + (\sigma - 1)\delta_2) < \sigma(\sigma - 1)$ for the power series* (49), *then $f(\alpha)$ is either a p-adic algebraic number in K or a p-adic transcendental number.*

Proof. Consider the polynomials $f_n(x) = \sum_{\nu=0}^{n} \frac{\eta_\nu}{a_\nu} x^\nu$, $(n = 1, 2, \ldots)$. Let $\gamma_n := f_n(\frac{p_n}{q_n})$ $(n = 1, 2, \ldots)$. Since γ_n $(n = 1, 2, \ldots)$ is a p-adic algebraic number in K, $\deg(\gamma_n) \leq m$. We can then determine an upper boundary for the height of γ_n using Lemma 6 with the polynomial

$$F(y, x_0, x_1, \ldots, x_n) = A_n q_n^n y - A_n q_n^n \sum_{\nu=0}^{n} \frac{1}{a_\nu} \left(\frac{p_n}{q_n}\right)^\nu x_\nu,$$

where $F(\gamma_n, \eta_0, \eta_1, \ldots, \eta_n) = 0$. By applying Lemma 6, we get

$$H(\gamma_n) \leq 3^{2m+(n+1)m} H^m H(\eta_0)^m \ldots H(\eta_n)^m,$$

where $H \leq A_n H_n^n$. Therefore, we can obtain

$$H(\gamma_n) \leq 3^{4nm} (A_n H_n^n)^m H(\eta_0)^m \ldots H(\eta_n)^m. \tag{52}$$

Next, from (51), there is a sufficiently small positive number ε_2 such that $A_n H(\eta_n) < p^{u_n(\lambda+\varepsilon_2)}$, for sufficiently large n. Then from (36), we get $u_\nu \leq \frac{1}{\sigma_1^{n-\nu}} u_n$ $(n \geq \nu \geq N_1)$, which leads to $u_{N_3} + \cdots + u_n \leq \frac{\sigma_1}{\sigma_1 - 1} u_n$ $(n \geq N)$, where N is a suitable natural number. Hence, from this information (38), (40) and (52), we obtain

$$\begin{aligned} H(\gamma_n) &\leq C_0 3^{4nm} H_n^{nm} p^{(u_{N_3}+\cdots+u_n)(\lambda+\varepsilon_2)m} \\ &\leq C_1^n H_n^{nm} p^{\frac{\sigma_1}{\sigma_1-1}(\lambda+\varepsilon_2)m u_n} \leq p^{\left(\frac{(\sigma-\varepsilon_1)(\lambda+\varepsilon_2)}{\sigma-\varepsilon_1-1}+\delta_2+\varepsilon_3\right)m u_n}, \end{aligned} \tag{53}$$

for sufficiently large n, in which $C_0 = (H(\eta_0) \ldots H(\eta_{N-1}))^m$ and C_1 is a sufficiently large positive number. From (38), we can deduce that $\frac{-u_n}{2} \geq -u_n + n \log_p |\alpha|_p$ for sufficiently large n, and from (54), we can obtain $|\frac{\eta_n}{a_n} \alpha^n|_p \leq p^{\frac{-u_n}{2}}$ for sufficiently large n. Since $\{u_n\}$ is a monotone increasing sequence, we then have

$$|f(\alpha) - f_n(\alpha)|_p \leq p^{\frac{-u_{n+1}}{2}} \tag{54}$$

for sufficiently large n. Therefore, from (36), (53) and (54), we can obtain

$$|f(\alpha) - f_n(\alpha)|_p \leq \frac{1}{H(\gamma_n)^{\frac{\sigma-\varepsilon_1}{2m\left(\frac{(\sigma-\varepsilon_1)(\lambda+\varepsilon_2)}{\sigma-\varepsilon_1-1}+\delta_2+\varepsilon_3\right)}}} \tag{55}$$

for sufficiently large n. From (38), it follows that $p^{-u_\nu} < C_2$ $(\nu = 0, 1, \ldots)$, where C_2 is a suitable positive number. Therefore, from (39) and (40), we have

$$\begin{aligned}\left|f_n(\alpha) - f_n\left(\frac{p_n}{q_n}\right)\right|_p &\leq \max_{1\leq\nu\leq n}\left\{\left|\frac{\eta_\nu}{a_\nu}\right|_p \left|\alpha^\nu - \left(\frac{p_n}{q_n}\right)^\nu\right|_p\right\}\\ &\leq C_2\left|\alpha - \frac{p_n}{q_n}\right|_p (|\alpha|_p + 1)^{n-1} \leq C_3^n p^{-u_n\delta_1 w(n)}\end{aligned}$$

for sufficiently large n, in which C_3 is a suitable positive constant. In addition, from (38), we know that there exists a sufficiently small positive number ε_4 such that

$$\left|f_n(\alpha) - f_n\left(\frac{p_n}{q_n}\right)\right|_p \leq p^{-u_n(\delta_1 w(n)-\varepsilon_4)} \tag{56}$$

for sufficiently large n. Furthermore, from (53) and (56), we can then obtain

$$\left|f_n(\alpha) - f_n\left(\frac{p_n}{q_n}\right)\right|_p \leq \frac{1}{H(\gamma_n)^{\frac{\delta_1 w(n)-\varepsilon_4}{m\left(\frac{(\sigma-\varepsilon_1)(\lambda+\varepsilon_2)}{\sigma-\varepsilon_1-1}+\delta_2+\varepsilon_3\right)}}} \tag{57}$$

for sufficiently large n. Thus, since $\lim_{n\to+\infty} w(n) = +\infty$ and $\delta_1 > 0$, from (55) and (57), we have

$$\left|f(\alpha) - f_n\left(\frac{p_n}{q_n}\right)\right|_p \leq \frac{1}{H(\gamma_n)^{\frac{\sigma-\varepsilon_1}{2m\left(\frac{(\sigma-\varepsilon_1)(\lambda+\varepsilon_2)}{\sigma-\varepsilon_1-1}+\delta_2+\varepsilon_3\right)}}} \tag{58}$$

for sufficiently large n. If we choose sufficiently small ε_1, ε_2 and ε_3, then there is a suitable positive number ε such that

$$\frac{\sigma - \varepsilon_1}{2m\left(\frac{(\sigma-\varepsilon_1)(\lambda+\varepsilon_2)}{\sigma-\varepsilon_1-1} + \delta_2 + \varepsilon_3\right)} > \frac{\sigma}{2m\left(\frac{\sigma\lambda}{\sigma-1} + \delta_2\right)} - \varepsilon.$$

Since $4m(\sigma\lambda + (\sigma - 1)\delta_2) < \sigma(\sigma - 1)$, it is possible to choose a positive number ε such that

$$\frac{\sigma(\sigma - 1)}{2m(\sigma\lambda + (\sigma - 1)\delta_2)} - \varepsilon \geq 2 + \varepsilon. \tag{59}$$

Therefore, it follows from (58) and (59) that

$$\left| f(\alpha) - f_n\left(\frac{p_n}{q_n}\right)\right|_p < \frac{1}{H(\gamma_n)^{2+\varepsilon}} \tag{60}$$

for sufficiently large n, in which ε is a suitable positive number that is dependent on ε_1, ε_2 and ε_3. If the sequence $\{\gamma_n\}$ is constant, then $f(\alpha)$ is a p-adic algebraic number in K. Otherwise, $f(\alpha)$ is a p-adic transcendental number according to Lang's theorem [10]. □

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