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# Arithmetical Fourier and limit values of elliptic modular functions

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**Abstract.** Here, we shall use the first periodic Bernoulli polynomial $\bar{B}_1(x) = x-[x]-\frac{1}{2}$ to resurrect the instinctive direction of B Riemann in his posthumous fragment II on the limit values of elliptic modular functions à la C G J Jacobi, Fundamenta Nova §40 (1829). In the spirit of Riemann who considered the odd part, we use a general Dirichlet–Abel theorem to condense Arias–de-Reyna's theorems 8–15 into 'a bigger theorem' in Sect. 2 by choosing a suitable $R$-function in taking the radial limits. We supplement Wang (*Ramanujan J*. **24** (2011) 129–145). Furthermore, the same method is applied to obtain in Sect. 3 a correct representation for the 'trigonometric series', i.e., we prove that for every rational number $x$ the trigonometric series (3.5) is represented by $\sum_{n=1}^{\infty}(-1)^n \frac{\bar{B}_1(nx)}{n}$ as Dedekind suggested but not by $\sum_{n=1}^{\infty} \frac{\bar{B}_1(nx)}{n}$ as Riemann stated.

**Keywords.** Elliptic modular function; Dedekind eta function; trigonometric series; Dirichlet–Abel theorem; Riemann's posthumous fragment II.

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## 1. Introduction

Riemann's posthumous fragment [8] consists of two parts: Fragment I and Fragment II. Dedekind succeeded in elucidating the genesis of all the formulas in Fragment II by introducing the most celebrated Dedekind eta-function [1]. All the results in Fragment II deal with the asymptotic behavior of those modular functions from Jacobi's Fundamenta Nova, §40 [4] for which the variable tends to rational points on the unit circle. After Dedekind, several authors including Smith [11], Hardy [3], Rademacher [6,7] made some more incorporations of Fragment II. In 2004, Arias–de-Reyna [2] analysed all the formulas in Fragment II again. As Wang [12] pointed out, a serious defect is that Arias–de-Reyna overlooked the more informative paper of Wintner [13] published in 1941, which already gave a close analysis of Fragment I and some far-reaching comments on Fragment II. What Riemann did was to eliminate the singular part which is the Clausen function, by taking the odd part. Accordingly, Wang [12, Theorem 3] (see Lemma 2 below) chose a suitable $R$-function by taking the radial limits and applied the new Dirichlet–Abel theorem, whereby he almost immediately got the expression in terms of the (differences of) polylogarithm function of order 1, without singularity and claimed that this is what Riemann intended to do but could not do because of lack of time.

In order to remove singularities, Riemann used a well-known device of taking the odd part (3.2) or an alternate sum (3.3) to be stated in §3.

In §2, we shall reveal that the limit values of elliptic modular functions in Riemann's fragment II by evaluating the differences of polylogarithm function $l_1(x)$ of order 1 (cf. Lemma 2 below), which can be made more concise, by applying the identities

$$l_1(x) = A_1(x) - \pi i \bar{B}_1(x),\ 0 < x < 1,\ A_1(x) = -\log 2|\sin 2\pi x|, \tag{1.1}$$

the odd part (3.2) being

$$l_1(x) - l_1(-x) = -2\pi i \bar{B}_1(x),\ 0 < x < 1, \tag{1.2}$$

where $\bar{B}_1(x) = x - [x] - \frac{1}{2}$ is the first periodic Bernoulli polynomial having the Fourier expansion denoted by $\psi(x)$ in [13]. Then incorporating the Bernoulli formula

$$\bar{B}_1(2x) - \bar{B}_1(x) = \bar{B}_1\left(x + \frac{1}{2}\right) \tag{1.3}$$

whose right-hand side is the Fourier series

$$-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin 2\pi n\left(x + \frac{1}{2}\right),$$

which is $\bar{B}_1\left(x + \frac{1}{2}\right)$ for $x \notin \mathbb{Z}$ and is 0 for $x \in \mathbb{Z} + \frac{1}{2}$ and is denoted by $\varphi(x)$ in [2]. We shall rewrite the 'bigger theorem' of Wang ([12, Theorem 3]) in the form presented there. In §3, we shall consider the trigonometric series (3.5) (cf. [10]) at every rational point $x$, which, Riemann asserts, is the function $f(x) = \sum_{n=1}^{\infty}\frac{\bar{B}_1(nx)}{n}$, but Dedekind [9, pp. 270–271] adds a note where he criticizes this statement of Riemann. Arias–de-Reyna [2, pp. 115–120] gave a proof of this statement of Riemann, which should be replaced by our theorem in §3, Theorem 2.

## 2. Riemann's posthumous fragment II revisited

All the subsequent theorems that Riemann considers in the second fragment are rephrases of the results of Jacobi and we state them as follows:

DEFINITION 1

The elliptic modular functions $k = k(z),\ K = K(z),\ k' = k'(z)$ are defined respectively by

$$\log k - \log 4\sqrt{z} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{4z^n}{1+z^n}, \tag{2.1}$$

$$\log\frac{2K}{\pi} = \sum_{p=1}^{\infty}\frac{4z^p}{p(1+z^p)} \tag{2.2}$$

and

$$-\log k' = \sum_{p=1}^{\infty} \frac{8z^p}{p(1-z^{2p})}, \tag{2.3}$$

where in the last two sums, we follow Riemann and let $p$ run through odd integers, i.e. these sums are odd parts.

To condense the 'bigger theorem' of [12] (Lemma 2 below) or Arias–de-Reyna's theorem 8–15 of [2], we use the coincident notations as [12] and prove some elementary lemmas.

*Lemma* 1. *Let $\alpha = \mathrm{e}^{\pi i/Q}$ be the first $2Q$-th primitive root of unity, and let $M$, $Q$ be integers with $M$ even and $Q > 1$. We have*

$$\sum_{r=1}^{Q-1}(-1)^r\left(l_1\left(\frac{Mr}{Q}\right)-l_1\left(\frac{Mr}{2Q}\right)\right) = -\pi i(-1)^{Q-1}\left[\frac{Q}{2}\right]\frac{M}{2Q}+\pi i\sum_{r=1}^{Q-1}(-1)^r\left[\frac{Mr+Q}{2Q}\right], \quad (Q\ \text{odd}) \tag{2.4}$$

$$\sum_{r=1}^{Q-1}(-1)^r\frac{r}{Q}\left(2l_1\left(\frac{Mr}{2Q}\right)-l_1\left(\frac{Mr}{Q}\right)-2l_1\left(\frac{-Mr}{2Q}\right)+l_1\left(\frac{-Mr}{Q}\right)\right) = \pi i\sum_{r=1}^{Q-1}(-1)^r\frac{r}{Q}(-1)^{\left[\frac{Mr}{2Q}\right]}, \tag{2.5}$$

$$\sum_{r=0}^{\left[\frac{Q}{2}\right]-1}\frac{2r+1}{Q}\left(2l_1\left(\frac{M(2r+1)}{2Q}\right)-2l_1\left(\frac{-M(2r+1)}{2Q}\right) - l_1\left(\frac{M(2r+1)}{Q}\right)+l_1\left(\frac{-M(2r+1)}{Q}\right)\right) = 2\pi i\sum_{r=0}^{\left[\frac{Q}{2}\right]-1}\frac{2r+1}{2Q}(-1)^{\left[\frac{M(2r+1)}{2Q}\right]}, \tag{2.6}$$

$$\sum_{r=1}^{\frac{Q}{2}-1}(-1)^r\frac{2r}{Q}\left(l_1\left(\frac{Mr}{2Q}\right)-l_1\left(\frac{-Mr}{2Q}\right)\right) = -2\pi i\sum_{r=1}^{\frac{Q}{2}-1}(-1)^r\frac{2r}{Q}\left(\frac{Mr}{2Q}-\left[\frac{Mr}{2Q}\right]-\frac{1}{2}\right) \tag{2.7}$$

*Proof.* Applying (1.1) and (1.3), we have

$$l_1\left(\frac{Mr}{Q}\right)-l_1\left(\frac{Mr}{2Q}\right)=-\log 2\left|\cos\frac{Mr}{2Q}\pi\right|-\pi i\left(\bar{B}_1\left(\frac{Mr}{Q}\right)-\bar{B}_1\left(\frac{Mr}{2Q}\right)\right)$$
$$=-\log 2\left|\cos\frac{Mr}{2Q}\pi\right|-\pi i\bar{B}_1\left(\frac{Mr+Q}{2Q}\right),$$

and sum these over $r=1,\ldots,Q-1$ to deduce that

$$\sum_{r=1}^{Q-1}(-1)^r\left(-\log 2\left|\cos\frac{Mr}{2Q}\pi\right|\right)=0$$

by symmetry. Note that $\bar{B}_1(x)=x-[x]-\frac{1}{2}$. We have

$$\sum_{r=1}^{Q-1}(-1)^r\left(l_1\left(\frac{Mr}{Q}\right)-l_1\left(\frac{Mr}{2Q}\right)\right)$$
$$=-\pi i\sum_{r=1}^{Q-1}(-1)^r\left(\frac{Mr}{2Q}-\left(\left[\frac{Mr}{Q}\right]-\left[\frac{Mr}{2Q}\right]\right)\right).$$

Replacing $\left[\frac{Mr}{Q}\right]-\left[\frac{Mr}{2Q}\right]$ by $\left[\frac{Mr+Q}{2Q}\right]$ and $\sum_{r=1}^{Q-1}(-1)^r r$ by $(-1)^{Q-1}\left[\frac{Q}{2}\right]$, we conclude (2.4).

Similarly, applying (1.2), the LHS of (2.5), respectively (2.6) reads as

$$\text{LHS}=-2\pi i\sum_{r=1}^{Q-1}(-1)^r\frac{r}{Q}\left(2\bar{B}_1\left(\frac{Mr}{2Q}\right)-\bar{B}_1\left(\frac{Mr}{Q}\right)\right),$$

respectively.

$$\text{LHS}=-2\pi i\sum_{r=0}^{\left[\frac{Q}{2}\right]-1}\frac{2r+1}{Q}\left(2\bar{B}_1\left(\frac{M(2r+1)}{2Q}\right)-\bar{B}_1\left(\frac{M(2r+1)}{Q}\right)\right),$$

and by the identity

$$\bar{B}_1\left(\frac{Mr}{Q}\right)-2\bar{B}_1\left(\frac{Mr}{2Q}\right)=2\left[\frac{Mr}{2Q}\right]-\left[\frac{Mr}{Q}\right]+\frac{1}{2}=(-1)^{\left[\frac{Mr}{2Q}\right]}\frac{1}{2}$$

we have (2.5) respectively (2.6). Hence (2.7) is trivial, thus completing the proof. □

*Lemma* 2 [7, Theorem 3]. *Let* $\xi = \frac{M}{Q}$ *be a rational number with* $M$ *even and* $Q > 1$, *and let* $z = y\mathrm{e}^{\pi i \xi},\ y \in [0, 1)$. *Then we have*

$$\log k = \frac{1}{2}\log y + \frac{M\pi}{2Q}i + \omega(y) - 2\sum_{r=1}^{Q-1}(-1)^r\left(l_1\left(\frac{Mr}{Q}\right) - l_1\left(\frac{Mr}{2Q}\right)\right),$$

$$\log\frac{2K}{\pi} = -\log(1-y) + \omega(y) + \log\frac{\pi}{Q} + \sum_{r=1}^{Q-1}(-1)^r\left(l_1\left(\frac{Mr}{Q}\right) - 2l_1\left(\frac{Mr}{2Q}\right)\right),$$

$$-\log k' = \frac{\pi^2}{2Q^2(1-y)} - \frac{\pi^2}{4Q^2} - \log 4 + \omega(y) + 2\sum_{r=0}^{\frac{Q-1}{2}-1}\frac{2r+1}{Q} \times \left(2l_1\left(\frac{M(2r+1)}{2Q}\right) - l_1\left(\frac{M(2r+1)}{Q}\right) - 2l_1\left(\frac{-M(2r+1)}{2Q}\right) + l_1\left(\frac{-M(2r+1)}{Q}\right)\right),$$

*for* $Q$ *odd*;

$$\log k = \frac{1}{2}\log y + \frac{M\pi}{2Q}i + \frac{2\pi^2}{Q^2}\frac{1}{(1-y)} + \omega(y) - \frac{\pi^2}{Q^2} - \log 4 - 2\sum_{r=1}^{\frac{Q}{2}-1}(-1)^r\frac{2r}{Q}\left(l_1\left(\frac{Mr}{2Q}\right) - l_1\left(\frac{-Mr}{2Q}\right)\right),$$

$$\log\frac{2K}{\pi} = -\frac{2\pi^2}{Q^2(1-y)} - \log(1-y) + \frac{\pi^2}{Q^2} + \log\frac{8\pi}{Q} + \omega(y) - \sum_{r=1}^{\frac{Q}{2}-1}(-1)^r\frac{2r}{Q}\left(2l_1\left(\frac{Mr}{2Q}\right) - 2l_1\left(\frac{-Mr}{2Q}\right) - l_1\left(\frac{Mr}{Q}\right) + l_1\left(\frac{-Mr}{Q}\right)\right),$$

$$-\log k' = -\frac{2\pi^2}{Q^2(1-y)} + \frac{\pi^2}{Q^2} + \log 4 + \omega(y) - 2\sum_{r=0}^{\frac{Q}{2}-1}\frac{2r+1}{Q} \times \left(2l_1\left(\frac{M(2r+1)}{2Q}\right) - 2l_1\left(\frac{-M(2r+1)}{2Q}\right) - l_1\left(\frac{M(2r+1)}{Q}\right) + l_1\left(\frac{-M(2r+1)}{Q}\right)\right),$$

*for $Q$ even and $\frac{Q}{2}$ odd*;

$$\log k = \frac{1}{2}\log y - \frac{2\pi^2}{Q^2}\frac{1}{(1-y)} + \omega(y) + \frac{\pi^2}{Q^2} + \log 4 + \frac{M\pi}{2Q}i - 2\sum_{r=1}^{\frac{Q}{2}-1}(-1)^r\frac{2r}{Q}\left(l_1\left(\frac{Mr}{2Q}\right) - l_1\left(\frac{-Mr}{2Q}\right)\right)$$

$$\log\frac{2K}{\pi} = -\log(1-y) + \log\frac{2\pi}{Q} + \omega(y) + 2\sum_{r=1}^{\frac{Q}{2}-1}(-1)^r\frac{2r}{Q} \times \left(2l_1\left(\frac{Mr}{2Q}\right) - 2l_1\left(\frac{-Mr}{2Q}\right) - l_1\left(\frac{Mr}{Q}\right) + l_1\left(\frac{-Mr}{Q}\right)\right),$$

$$-\log k' = \omega(y) - 2\sum_{r=0}^{\frac{Q}{2}-1}\frac{2r+1}{Q} \times \left(2l_1\left(\frac{M(2r+1)}{2Q}\right) - 2l_1\left(\frac{-M(2r+1)}{2Q}\right) - l_1\left(\frac{M(2r+1)}{Q}\right) + l_1\left(\frac{-M(2r+1)}{Q}\right)\right),$$

*for $\frac{Q}{2}$ even; where $2\pi i\frac{M}{Q}$ is one of the values of $\log e^{2\pi i\frac{M}{Q}}$, and $\omega(y)$ is a continuation function on $I = [0, 1]$ with $\omega(1) = 0$ which maybe different in different place.*

We are in a position to state our theorem.

**Theorem 1.** *Let $\xi = \frac{M}{Q}$ be a rational number with $M$ even and $Q > 1$, and let $z = ye^{\pi i\xi}$, $y \in [0, 1)$. Then we have*

$$\log k = \frac{1}{2}\log y + \frac{M\pi}{2Q}i + \omega(y) + 2\pi i\left[\frac{Q}{2}\right]\frac{M}{2Q} - 2\pi i\sum_{r=1}^{Q-1}(-1)^r\left[\frac{Mr+Q}{2Q}\right]$$

*and*

$$\log\frac{2K}{\pi} = -\log(1-y) + \omega(y) + \log\frac{\pi}{Q} - \pi i\sum_{r=1}^{Q-1}\frac{r}{Q}(-1)^{r+\left[\frac{Mr}{2Q}\right]},$$

*and*

$$-\log k' = \frac{\pi^2}{2Q^2(1-y)} - \frac{\pi^2}{4Q^2} - \log 4 + \omega(y) + 2\pi i\sum_{r=0}^{\frac{Q-1}{2}-1}\frac{2r+1}{Q}(-1)^{\left[\frac{M(2r+1)}{2Q}\right]},$$

*for $Q$ odd*;

$$\log k = \frac{1}{2}\log y + \frac{M\pi}{2Q}i + \frac{2\pi^2}{Q^2}\frac{1}{(1-y)} + \omega(y) - \frac{\pi^2}{Q^2} - \log 4 + 4\pi i \sum_{r=1}^{\frac{Q}{2}-1}(-1)^r \frac{2r}{Q}\left(\frac{Mr}{2Q} - \left[\frac{Mr}{2Q}\right] - \frac{1}{2}\right)$$

*and*

$$\log\frac{2K}{\pi} = -\frac{2\pi^2}{Q^2(1-y)} - \log(1-y) + \frac{\pi^2}{Q^2} + \log\frac{8\pi}{Q} + \omega(y) - \pi i \sum_{r=1}^{\frac{Q}{2}-1}\frac{2r}{Q}(-1)^{r+\left[\frac{Mr}{2Q}\right]},$$

$$-\log k' = -\frac{2\pi^2}{Q^2(1-y)} + \frac{\pi^2}{Q^2} + \log 4 + \omega(y) - 2\pi i \sum_{r=0}^{\frac{Q}{2}-1}\frac{2r+1}{Q}(-1)^{\left[\frac{M(2r+1)}{2Q}\right]},$$

*for $Q$ even and $\frac{Q}{2}$ odd*;

$$\log k = \frac{1}{2}\log y - \frac{2\pi^2}{Q^2}\frac{1}{(1-y)} + \omega(y) + \frac{\pi^2}{Q^2} + \log 4 + \frac{M\pi}{2Q}i + 4\pi i \sum_{r=1}^{\frac{Q}{2}-1}(-1)^r \frac{2r}{Q}\left(\frac{Mr}{2Q} - \left[\frac{Mr}{2Q}\right] - \frac{1}{2}\right)$$

*and*

$$\log\frac{2K}{\pi} = -\log(1-y) + \log\frac{2\pi}{Q} + \omega(y) + 2\pi i \sum_{r=1}^{\frac{Q}{2}-1}\frac{2r}{Q}(-1)^{r+\left[\frac{Mr}{2Q}\right]},$$

*and*

$$-\log k' = \omega(y) - 4\pi i \sum_{r=0}^{\frac{Q}{2}-1}\frac{2r+1}{2Q}(-1)^{\left[\frac{M(2r+1)}{2Q}\right]},$$

*for $\frac{Q}{2}$ even; where $2\pi i\frac{M}{Q}$ is one of the values of $\log e^{2\pi i\frac{M}{Q}}$, and $\omega(y)$ is a continuation function on $I = [0, 1]$ with $\omega(1) = 0$ which maybe different in different place.*

*Proof.* The proofs are trivial, i.e., distinguishing the parities of the integers $M$ and $Q$, then replacing the last summation terms of each equations in Lemma 2 by the same expression summations of the differences of polylogarithm function $l_1(x)$ in Lemma 1, and, after some calculus, we obtain Theorem 1. □

## 3. A 'non integrable function' represented by a trigonometric series

In [10], Riemann asserts that the function $f(x) = \sum_{n=1}^{\infty} \frac{\bar{B}_1(nx)}{n}$ is represented by the trigonometric series

$$\sum_{n=1}^{\infty} \left( \sum_{d|n} -(-1)^d \right) \frac{\sin 2\pi nx}{n\pi},$$

at each rational point $x$. Dedekind [9, pp. 270–271] added the following: Man findet diese Entwicklung (wenn auch auf einem nicht ganz einwurfsfreien Wege), wenn man die Function $\varphi(x)$ durch die bekannte Formel

$$-\sum_{m=1}^{\infty} (-1)^m \frac{\sin 2m\pi x}{m\pi}$$

ausdrückt, dies in die Summe $\sum \frac{(nx)}{n}$ einsetzt und die Ordnung der Summationen vertauscht [1], where he criticizes the statement of Riemann. He assumed that Riemann follows here a false argument. But Riemann does not say anything about how he proves his assertion. At the end of the posthumous fragment I, Riemann proved the theorem of Abel about the radial limit of a power series and then ends this fragment with the words:

From this theorem, – that when the above had already been written (September 14th 1852), Prof. Dirichlet tells that it is due to Prof. Abel – easily follows. . .

Smith [11] asserts that it is not easy to see how he proposed to complete the demonstration. By Arias–de-Reyna [2, pp. 115–120] and Wintner [13], we prove the following theorem.

**Theorem 2.** *Let $M$, $Q$ be co-prime integers and $x = \frac{M}{Q}$ with $Q > 1$. Then*

$$f(x) = \sum_{n=1}^{\infty} (-1)^n \frac{\bar{B}_1(nx)}{n} = \sum_{n=1}^{\infty} \left( \sum_{d|n} -(-1)^d \right) \frac{\sin 2n\pi x}{n\pi}. \tag{3.1}$$

*Remark* 1. The odd part and the alternating sum referred to in §1 are described respectively by

$$\sum_{2\nmid n} a_n = \sum_{n} a_n - \sum_{2|n} a_n \tag{3.2}$$

and

$$\sum_{n} (-1)^n a_n = \sum_{2|n} a_n - \sum_{2\nmid n} a_n = 2\sum_{2|n} a_n - \sum_{n} a_n, \tag{3.3}$$

by (3.2), where $n$ runs over a finite range or the series are absolutely convergent. For some more details, readers may refer to [8–13].

By the identities

$$-\sum_{m=1}^{\infty}(-1)^m \frac{\sin 2m\pi x}{m\pi} = -\sum_{m=1}^{\infty} \frac{\sin 2m\pi \left(x+\frac{1}{2}\right)}{m\pi} = \bar{B}_1\left(x+\frac{1}{2}\right)$$

and

$$\sum_{n=1}^{\infty}(-1)^n \frac{\bar{B}_1(nx)}{n} = \sum_{n=1}^{\infty} \frac{\bar{B}_1(2nx)-\bar{B}_1(nx)}{n} = \sum_{n=1}^{\infty} \frac{\bar{B}_1\left(nx+\frac{1}{2}\right)}{n}, \tag{3.4}$$

we see that Theorem 2 is exactly as in Dedekind's notes where he criticizes the statement of Riemann. Note that (3.4) follows from (3.3) and (1.3).

*Proof.* We apply a generalization of Dirichlet's test to prove that the series

$$\sum_{n=1}^{\infty}\left(\sum_{d|n} -(-1)^d\right) \frac{\sin 2n\pi x}{n\pi} \tag{3.5}$$

converges: the series $\sum_{n=1}^{\infty} a_n(x) b_n(s)$ is uniformly convergent in $\sigma > 0$ if we check that the partial sums of $a_n(x)$ are bounded uniformly in $x$, $\lim_{n\to\infty} b_n(s) = 0$ uniformly in $\sigma > 0$ and $|b_n(s) - b_{n+1}| \le c_n$ and $\sum_{n=1}^{\infty} c_n < \infty$.

Since $b_n(1) = n^{-1}$, $\lim_{n\to\infty} b_n(1) = 0$ and $|b_n(1) - b_{n+1}(1)| = \left|\int_n^{n+1} t^{-2}\mathrm{d}t\right| = O(n^{-2})$, we have $\sum_{n=1}^{\infty} c_n = \zeta(2) < \infty$.

The boundedness of the partial sum of $a_n(x) = \left(\sum_{d|n} -(-1)^d\right) \frac{\sin 2n\pi x}{\pi}$ follows if there exists a constant $C$ such that for every $N$ (cf. [2, pp. 116–118]),

$$|S_N| = \left|\frac{1}{\pi}\sum_{t=1}^{N}\left(\sum_{d|t} -(-1)^d\right)\sin 2\pi t x\right| \le C\sqrt{N}$$

and

$$\sum_{t=N}^{M}\left(\sum_{d|t} -(-1)^d\right)\frac{\sin 2\pi t x}{t} \le \frac{C}{\sqrt{N}} + C\sum_{t=N}^{M-1}\frac{\sqrt{t}}{t(t+1)} + \frac{C}{\sqrt{M}}.$$

This can be made arbitrarily small by taking $M \ge N \ge N_0$ with $N \to \infty$.

Let $z = y\mathrm{e}^{2\pi i x}$. By the new Dirichlet–Abel theorem [12, Theorem 1], the value of $f(x)$ is the radial limit

$$f(x) = \sum_{n=1}^{\infty}\left(\sum_{d|n} -(-1)^d\right)\frac{\sin 2n\pi x}{n\pi} = \lim_{y\to 1^-}\sum_{n=1}^{\infty}\left(\sum_{d|n} -(-1)^d\right)\frac{\sin 2n\pi x}{n\pi} y^n$$

or

$$f(x) = \lim_{z\to \mathrm{e}^{2\pi i x}} \sum_{n=1}^{\infty} \mathrm{Im}\left(\sum_{d|n} -(-1)^d\right)\frac{z^n}{n\pi},$$

where $z \to e^{2\pi ix}$ is the radial limit, and the series is absolutely convergent for $|z| < 1$. Therefore by Euler identity $e^{i\theta} = \cos\theta + i\sin\theta$ and the Fourier series of $\bar{B}_1(x)$, we have

$$\operatorname{Im}\sum_{n=1}^{\infty}\left(\sum_{d|n} -(-1)^d\right)\frac{e^{2\pi inx}}{n\pi} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sum_{m=1}^{\infty} -\frac{\sin 2\pi mnx}{m\pi}$$
$$= \sum_{n=1}^{\infty}\frac{(-1)^n\bar{B}_1(nx)}{n},$$

thus completing the proof. □

The proof of the Theorem 2 is due to [2,13] by applying the new Drichlet–Abel theorem [12, Theorem 1] and the Lambert series [13, pp. 633–634]. In what follows, we shall prove a radial limit theorem (Theorem 3 below) due to Wintner [13] and show two examples.

Let $\{a_n\} \subset \mathbb{C}$ be such that

$$\limsup |a_n|^{1/n} \le 1,$$

i.e. such that the power series $\sum_{n=1}^{\infty} a_n z^n$ is absolutely convergent in $|z| < 1$. Then the Lambert series

$$g(z) = \sum_{n=1}^{\infty} a_n \frac{z^n}{1 - z^n} \tag{3.6}$$

is absolutely convergent in $|z| < 1$ and represents an analytic function and moreover the power series of this function can be obtained by formal rearrangement of (3.6), i.e.,

$$f(z) = \sum_{n=1}^{\infty} b_n z^n \quad (|z| < 1),$$

where

$$b_n = \sum_{d|n} a_d.$$

**Theorem 3** **[13]**. *Suppose*

$$a_n = O(n^{\lambda-\delta}) \tag{3.7}$$

*form some $\delta > 0$ and a fixed $0 < \lambda \le \frac{1}{2}$. Then the boundary function $F(e^{i\theta})$ exists and is measurable such that*

$$F(re^{i\theta}) \to F(e^{i\theta}) \quad \text{a.e. as } r \to 1 \tag{3.8}$$

*along the Stoltz path. If $\lambda < \frac{1}{2}$, then $F(e^{i\theta})$ is of class $L^{1/\lambda}$ and if in (3.7), the exponent can be taken arbitrarily small, then it is of class $L^{\infty}$.*

*Proof.* Since

$$d(n) = \sum_{d|n} 1 = O(n^{\varepsilon})$$

for every $\varepsilon > 0$ (e.g. [3]), it follows that

$$c_n = O(n^{\lambda-1-\delta})$$

for some $\delta > 0$. Hence if $\lambda < \frac{1}{2}$, then the $L^p$-condition (3.7) is satisfied and if $\lambda = \frac{1}{2}$, then the series for $F(re^{i\theta})$ is Cauchy in $L^2$ and so there exists a function $F(e^{i\theta})$ of class $L^2$ such that

$$F(e^{i\theta}) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\theta}.$$

This together with the condition (3.7) implies that

$$F(e^{i\theta}) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta} \quad \text{a.e..}$$

Hence (3.8) follows by Abel's continuity theorem. □

*Example* 1.

(i) In the case $a_{2n} = 0$, $a_{2n+1} = 4(-1)^{n+1}$, we obtain the Lambert series

$$f(z) = \frac{2K}{\pi} = 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^{2n+1}}{1 - z^{2n+1}} + 1,$$

which in the notation of (3) reads as

$$f(z) = \frac{2K}{\pi} = 4 \sum_{\ell,m,n=0}^{\infty} d_{4,1}(n) z^{2^{\ell}(4m-1)^2 n} + 1,$$

with $d_{4,1}(n)$ denoting the number of divisors of $n$ of the form $4k + 1$. Hence

$$F(z) = \sum_{n=0}^{\infty} \frac{d_{4,1}(n)}{n} z^n, \quad |z| < 1.$$

Hence by Theorem 3, the boundary function $F(e^{i\theta})$ exists and

$$F(e^{i\theta}) = \sum_{n=0}^{\infty} \frac{d_{4,1}(n)}{n} e^{in\theta} \quad \text{a.e.}$$

(ii) In the case $a_n = 1$, we obtain the Lambert series considered by Lambert [5]

$$f(z) = 4\sum_{n=1}^{\infty} \frac{z^n}{1-z^n} = \sum_{n=1}^{\infty} d(n) z^n, \quad |z| < 1.$$

Hence

$$F(z) = \sum_{n=0}^{\infty} \frac{d(n)}{n} z^n, \quad |z| < 1.$$

Hence by Theorem 3, the boundary function $F(\mathrm{e}^{i\theta})$ exists and

$$F(\mathrm{e}^{i\theta}) = \sum_{n=0}^{\infty} \frac{d(n)}{n} \mathrm{e}^{in\theta} \quad \text{a.e.}$$

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