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Weighted local Hardy spaces associated with operators

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Abstract. Let L be a self-adjoint positive operator on $L^2(\mathbb{R}^n)$. Assume that the semigroup e^{-tL} generated by $-L$ satisfies the Gaussian kernel bounds on $L^2(\mathbb{R}^n)$. In this article, we study weighted local Hardy space $h^1_{L,w}(\mathbb{R}^n)$ associated with L in terms of the area function characterization, and prove their atomic characters. Then, we introduce the weighted local BMO space $\mathrm{bmo}_{L,w}(\mathbb{R}^n)$ and prove that the dual of $h^1_{L,w}(\mathbb{R}^n)$ is $\mathrm{bmo}_{L,w}(\mathbb{R}^n)$. Finally a broad class of applications of these results is described.

Keywords. Weighted local Hardy space; non-negative self-adjoint operator; semigroups; local (1, 2, w)-atoms; weighted local BMO space.

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1. Introduction

The classical localized Hardy space $h^1(\mathbb{R}^n)$ introduced by Goldberg [10] plays an important role in various fields of analysis and partial differential equations. The theory of local Hardy spaces was extended to the weighted case by Bui [4], see also [14].

Recently, the Hardy spaces associated with operators were studied by many authors, see, for example, [1,2,7–9,13,15,16,18,22,23] and references therein. More specifically, we refer the reader to [2] for an extensive study of localized Hardy space $h^1_L(\Omega)$ adapted to an operator L which is either the Dirichlet or Neumann Laplacian on a Lipschitz domain Ω of $\mathbb{R}^n$, or to [18] and [13] for molecular and atomic decompositions of $h^1_L(\mathbb{R}^n)$ associated with the operators with heat kernel bounds. In this paper, we study weighted local Hardy space $h^1_{L,w}(\mathbb{R}^n)$ associated with the operators with heat kernel bounds in terms of the area function characterization, and prove their atomic characters. Moreover, we introduce the weighted local BMO space $\mathrm{bmo}_{L,w}(\mathbb{R}^n)$ and prove that the dual of $h^1_{L,w}(\mathbb{R}^n)$ is $\mathrm{bmo}_{L,w}(\mathbb{R}^n)$.

Unless otherwise specified in the sequel, we always assume that L is a non-negative self-adjoint operator on $L^2(\mathbb{R}^n)$ and that the semigroup e^{-tL}, generated by $-L$ on $L^2(\mathbb{R}^n)$, has the kernel $p_t(x, y)$ which satisfies the following Gaussian upper bound

$$|p_t(x, y)| \leq C t^{-n/2} \exp\left(-\frac{|x-y|^2}{c\,t}\right) \tag{GE}$$

for all $t > 0$, and $x, y \in \mathbb{R}^n$, where C and c are positive constants.

For every $f \in L^2(\mathbb{R}^n)$, consider a local version of the area function associated with the operator L by

$$s_{\mathrm{loc},L} f(x) = \left(\int_0^1 \int_{|x-y|<t} |t^2 L e^{-t^2 L} f(y)|^2 \frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\right)^{1/2}, \quad x \in \mathbb{R}^n. \tag{1.1}$$

DEFINITION 1.1

Suppose $w \in A_\infty$. The space $\mathbb{h}^1_{L,w}(\mathbb{R}^n)$ is defined as

$$\left\{f \in L^2(\mathbb{R}^n) : \|f\|_{\mathbb{h}^1_{L,w}(\mathbb{R}^n)} < \infty\right\}$$

with norm

$$\|f\|_{\mathbb{h}^1_{L,w}(\mathbb{R}^n)} = \|s_{\mathrm{loc},L} f\|_{L^1_w(\mathbb{R}^n)} + \|f\|_{L^1_w(\mathbb{R}^n)}.$$

The space $h^1_{L,w}(\mathbb{R}^n)$ is then defined as the completion of $\mathbb{h}^1_{L,w}(\mathbb{R}^n)$ with respect to this norm.

Let us introduce the notion of a local $(1, 2, w)$-*atom* associated with an operator L on $L^2(\mathbb{R}^n)$ with domain $\mathcal{D}(L)$.

DEFINITION 1.2

Suppose $w \in A_\infty$. A function a supported in a ball B of $\mathbb{R}^n$ is called a local $(1, 2, w)$-atom associated with L if $\|a\|_{L^2(\mathbb{R}^n)} \leq |B|^{1/2} w(B)^{-1}$ and either

(i) $r_B > 1$; or
(ii) $r_B \leq 1$ and a is a $(1, 2, w)$-atom, i.e., there exists a function $b \in \mathcal{D}(L)$ such that $a = Lb$, $\mathrm{supp}\,(L^k b) \subset B$, $k = 0, 1$, and

$$\left\|(r_B^2 L)^k b\right\|_{L^2(\mathbb{R}^n)} \leq r_B^2 |B|^{1/2} w(B)^{-1}$$

for all $k = 0, 1$.

The main result of this article is the following.

Theorem 1.3. *Let L be a non-negative self-adjoint operator such that the corresponding heat kernels satisfy Gaussian bound* (GE).

(i) Let $w \in A_1$. *If* $f \in \mathbb{h}^1_{L,w}(\mathbb{R}^n)$, *then there exists a family of local* $(1, 2, w)$-*atoms* $\{a_i\}_{i=0}^\infty$ *and a sequence of numbers of* $\{\lambda_i\}_{i=0}^\infty$ *such that* f *can be represented in the form*

$f = \sum_{i=0}^{\infty} \lambda_i a_i$, *and the sum converges in the sense of* $L^2(\mathbb{R}^n)$*-norm. Moreover,*

$$\sum_{i=0}^{\infty} |\lambda_i| \le C \|f\|_{\mathbb{h}^1_{L,w}(\mathbb{R}^n)}.$$

(*ii*) *Conversely, given* $w \in A_1 \cap RH_2$, *let* $f = \sum_{i=0}^{\infty} \lambda_i a_i$, *where* $\{\lambda_i\}_{i=0}^{\infty} \in \ell^1$, *the* a_i*'s are all local* $(1, 2, w)$*-atoms, and the sum converges in* $L^2(\mathbb{R}^n)$. *Then* $f \in \mathbb{h}^1_{L,w}(\mathbb{R}^n)$ *and*

$$\|f\|_{\mathbb{h}^1_{L,w}(\mathbb{R}^n)} \le C \sum_{i=0}^{\infty} |\lambda_i|.$$

The paper is organized as follows. In section 2, we introduce some notation and preliminary lemmas, which are useful in the sequel. In section 3, we prove Theorem 1.3 to obtain an atomic decomposition of functions in $h^1_{L,w}(\mathbb{R}^n)$. In section 4, we introduce the weighted local BMO space $\mathrm{bmo}_{L,w}(\mathbb{R}^n)$ and prove the dual of $h^1_{L,w}(\mathbb{R}^n)$ is $\mathrm{bmo}_{L,w}(\mathbb{R}^n)$. Finally, in the last section, we conclude this article with some applications to large classes of operators.

Throughout, the letter 'C' or 'c' will denote (possibly different) constants that are independent of the essential variables.

2. Preliminaries

Throughout this article, we shall denote $w(E) := \int_E w(x)dx$ for any set $E \subset \mathbb{R}^n$. For $1 < p < \infty$, denote p' the adjoint number of p, i.e., $1/p + 1/p' = 1$.

2.1 *Muckenhoupt weights*

We review some needed background on Muckenhoupt weights. We use the notation

$$⨍_E h(x)\mathrm{d}x = \frac{1}{|E|} \int_E h(x)\mathrm{d}x.$$

A weight w is a non-negative locally integrable function on $\mathbb{R}^n$. We say that $w \in A_p$, $1 < p < \infty$, if there exists a constant C such that for every ball $B \subset \mathbb{R}^n$,

$$\left(⨍_B w\mathrm{d}x\right)\left(⨍_B w^{-1/(p-1)}\mathrm{d}x\right)^{p-1} \le C.$$

For $p = 1$, we say that $w \in A_1$ if there is a constant C such that for every ball $B \subset \mathbb{R}^n$,

$$⨍_B w(y)\mathrm{d}y \le Cw(x) \quad \text{for a.e. } x \in B.$$

The reverse Hölder classes are defined in the following way: $w \in RH_q$, $1 < q < \infty$, if there is a constant C such that for any ball $B \subset \mathbb{R}^n$,

$$\left(⨍_B w^q\mathrm{d}x\right)^{\frac{1}{q}} \le C⨍_B w\mathrm{d}x.$$

The endpoint $q = \infty$ is given by the condition: $w \in RH_\infty$ whenever, there is a constant C such that for any ball $B \subset \mathbb{R}^n$,

$$w(y) \le C \fint_B w(x)\mathrm{d}x, \quad \text{for } a.e.\ y \in B.$$

Let $w \in A_\infty$ for $1 \le p < \infty$. The weighted Lebesgue spaces L^p_w can be defined by $\left\{f : \int_{\mathbb{R}^n} |f(x)|^p w(x)\mathrm{d}x < \infty\right\}$ with norm $\|f\|_{L^p_w} := \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\mathrm{d}x\right)^{1/p}$.

We sum up some of the properties of classes in the following results.

Lemma 2.1. *We have the following properties*:

- *(i)* $A_1 \subset A_p \subset A_q$ *for* $1 \le p \le q < \infty$.
- *(ii)* $RH_\infty \subset RH_q \subset RH_p$ *for* $1 < p \le q < \infty$.
- *(iii)* *If* $w \in A_p$, $1 < p < \infty$, *then there exists* $1 < q < p$ *such that* $w \in A_q$.
- *(iv)* *If* $w \in RH_q$, $1 < q < \infty$, *then there exists* $q < p < \infty$ *such that* $w \in RH_p$.
- *(v)* $A_\infty = \bigcup_{1 \le p < \infty} A_p = \bigcup_{1 < q \le \infty} RH_q$.
- *(vi)* *If* $1 < p < \infty$, $w \in A_p$ *if and only if* $w^{1-p'} \in A_{p'}$.
- *(vii)* *Let* $w \in A_p$, $p \ge 1$. *Then for any ball* B *and* $\lambda > 1$, *we have that* $w(\lambda B) \le C\lambda^{np} w(B)$ *for some constant* C *independent of* B *and* λ.

Proof. Properties (i)–(vi) are standard, see for instance, [21], [11] and [6]. For (vii), see [17]. □

Lemma 2.2. *Denote any ball* B *and any measurable subset* E *of* B. *Let* $w \in A_p$, $p \ge 1$. *Then, there exist constants* $C_1 > 0$ *such that*

$$C_1 \left(\frac{|E|}{|B|}\right)^p \le \frac{w(E)}{w(B)}.$$

If $w \in RH_r$, $r > 1$. *Then, there exist constants* $C_2 > 0$ *such that*

$$\frac{w(E)}{w(B)} \le C_2 \left(\frac{|E|}{|B|}\right)^{(r-1)/r}.$$

Proof. For the proof, we refer to [11] and [6]. □

2.2 *Finite speed propagation for the wave equation.*

Recall that, if L is a non-negative, self-adjoint operator on $L^2(\mathbb{R}^n)$, and $E_L(\lambda)$ denotes its spectral decomposition, then for every bounded Borel function $F : [0, \infty) \to \mathbb{C}$, one defines the bounded operator $F(L) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by the formula

$$F(L) := \int_0^\infty F(\lambda)\,\mathrm{d}E_L(\lambda). \tag{2.1}$$

In particular, the operator $\cos(t\sqrt{L})$ is then well-defined and bounded on $L^2(\mathbb{R}^n)$. Moreover, it follows from Theorem 3 of [5] and condition (GE) that there exists a finite, positive constant c_0 with the property that the Schwartz kernel $K_{\cos(t\sqrt{L})}$ of $\cos(t\sqrt{L})$ satisfies

$$\mathrm{supp} K_{\cos(t\sqrt{L})} \subseteq \left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \le c_0 t\right\}. \tag{2.2}$$

The precise value of c_0 is inessential and throughout the article we will choose $c_0 = 1$. By the Fourier inversion formula, whenever F is an even, bounded, Borel function with $\hat{F} \in L^1(\mathbb{R})$, we can write $F(\sqrt{L})$ in terms of $\cos(t\sqrt{L})$. Concretely, by recalling (2.1) we have

$$F(\sqrt{L}) = (2\pi)^{-1} \int_{-\infty}^{\infty} \hat{F}(t) \cos(t\sqrt{L})\, \mathrm{d}t, \tag{2.3}$$

which, when combined with (2.2), gives

$$K_{F(\sqrt{L})}(x, y) = (2\pi)^{-1} \int_{|t| \geq |x-y|} \hat{F}(t) K_{\cos(t\sqrt{L})}(x, y)\, \mathrm{d}t, \qquad \forall\, x, y \in \mathbb{R}^n, \tag{2.4}$$

where $\hat{F}$ denotes the Fourier transform of F.

Lemma 2.3. *Let* $\varphi \in C_0^\infty(\mathbb{R})$ *be an even function with* $\int \varphi = 2\pi$, $\operatorname{supp} \varphi \subset (-1, 1)$. *For every* $m = 0, 1, 2, \ldots,$ *set* $\Phi^{(m)}(\xi) := \frac{\mathrm{d}^m}{\mathrm{d}\xi^m} \hat{\varphi}(\xi)$. *Then for every* $\kappa, m \in \mathbb{N}$ *and* $\kappa + m \in 2\mathbb{N}$, *the kernel* $K_{(t\sqrt{L})^\kappa \Phi^{(m)}(t\sqrt{L})}$ *of* $(t\sqrt{L})^\kappa \Phi^{(m)}(t\sqrt{L})$ *satisfies*

$$\operatorname{supp} K_{(t\sqrt{L})^\kappa \Phi^{(m)}(t\sqrt{L})} \subseteq \left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq t\right\} \tag{2.5}$$

and

$$\left|K_{(t\sqrt{L})^\kappa \Phi^{(m)}(t\sqrt{L})}(x, y)\right| \leq C t^{-n} \tag{2.6}$$

for all $t > 0$ *and* $x, y \in \mathbb{R}^n$.

Proof. The proof can be obtained by making minor modifications with the proof of Lemma 2.3 of [12]. See also [15,20]. □

Finally, for $s > 0$, we define

$$\mathbb{F}(s) := \left\{ \psi : \mathbb{C} \to \mathbb{C} \text{ measurable} : \ |\psi(z)| \leq C \frac{|z|^s}{(1 + |z|^{2s})} \right\}.$$

Then for any non-zero function $\psi \in \mathbb{F}(s)$, we have that $\{\int_0^\infty |\psi(t)|^2 \mathrm{d}t/t\}^{1/2} < \infty$. Denote by $\psi_t(z) = \psi(tz)$. It follows from the spectral theory in [26] that for any $f \in L^2(\mathbb{R}^n)$,

$$\begin{aligned} \left\{ \int_0^\infty \|\psi(t\sqrt{L}) f\|^2_{L^2(\mathbb{R}^n)} \frac{\mathrm{d}t}{t} \right\}^{1/2} &= \left\{ \int_0^\infty \left\langle \overline{\psi}(t\sqrt{L})\, \psi(t\sqrt{L}) f, f \right\rangle \frac{\mathrm{d}t}{t} \right\}^{1/2} \\ &= \left\{ \left\langle \int_0^\infty |\psi|^2 (t\sqrt{L}) \frac{\mathrm{d}t}{t} f, f \right\rangle \right\}^{1/2} \\ &\leq \kappa \|f\|_{L^2(\mathbb{R}^n)}, \end{aligned} \tag{2.7}$$

where $\kappa = \left\{ \int_0^\infty |\psi(t)|^2 \mathrm{d}t/t \right\}^{1/2}$, an estimate which will be used often in the sequel.

3. Atomic characterization of $h^1_{L,w}(\mathbb{R}^n)$

In order to prove Theorem 1.3, we first establish an inhomogeneous Calderón type reproducing formula. For every $m \in \mathbb{N}$ and every $\ell = 0, 1, \ldots, m$, $j = 0, 1, \ldots, \left[(m - \ell)/2\right]$, define

$$c_1(m,\ell,j) := \frac{(-1)^{m-\ell-j}}{120} 2^{m-\ell-2j} \frac{m!}{(m-\ell)!\,\ell!} \frac{(m-\ell)!}{(m-\ell-2j)!\,j!} \tag{3.1}$$

and

$$c_2(m,\ell,j) = (-1)^m \frac{1}{(5-m)!} c_1(5-m,\ell,j). \tag{3.2}$$

Then we have the following result.

PROPOSITION 3.1 (Calderón reproducing formula I)

Let φ and Φ be as in Lemma 2.3. For every $m \in \mathbb{N}$ and every $\ell = 0, 1, \dots, m$, $j = 0, 1, \dots, \left[\frac{m-\ell}{2}\right]$, we set $\Psi^{(\ell)}_{m,j}(s) := s^{2(m-j)-\ell} \frac{\mathrm{d}^\ell}{\mathrm{d}s^\ell}\Phi(s)$. Then for every $f \in L^2(\mathbb{R}^n)$, we have

$$f = f_1 - f_2, \tag{3.3}$$

where

$$\begin{aligned} f_1 :&= \int_0^1 \left(s^6 \frac{\mathrm{d}^6}{\mathrm{d}s^6}\big(\Phi(s)e^{-s^2}\big)\right)\bigg|_{s=t\sqrt{L}} f \frac{\mathrm{d}t}{t} \\ &= \sum_{\ell=0}^{6} \sum_{j=0}^{\left[\frac{6-\ell}{2}\right]} c_1(6,\ell,j) \int_0^1 (t^2L)^2 \Psi^{(\ell)}_{6,\,j+3}\big(t\sqrt{L}\big) t^2 L\mathrm{e}^{-t^2L} f \frac{\mathrm{d}t}{t} \end{aligned} \tag{3.4}$$

and

$$f_2 := \sum_{m=0}^{5} \sum_{\ell=0}^{5-m} \sum_{j=0}^{\left[\frac{5-m-\ell}{2}\right]} c_2(m,\ell,j) \Psi^{(\ell)}_{5-m,j}(\sqrt{L})\mathrm{e}^{-L} f. \tag{3.5}$$

Proof. For the proof of this proposition, see [7, Lemma 3.9]. □

Now, we turn to prove Theorem 1.3.

Proof of Theorem 1.3(*i*). Let $f \in \mathbb{h}^1_{L,w}(\mathbb{R}^n)$. Firstly, we need a result which was proved in Theorem 1 and Theorem 2 of Chapter IV in [21]. Denote $s_{\mathrm{loc},\alpha}(f)(x) = \left(\int_0^1 \int_{|y-x|<\alpha t} |t^2L\mathrm{e}^{-t^2L} f(y)|^2 \frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\right)^{1/2}$. The result is as follows: Suppose $w \in A_\infty$, $\alpha \geq 1$ and $0 < q < \infty$, then

$$\left\| s_{\mathrm{loc},L}(f)\right\|_{L^q_w} \approx \left\| s_{\mathrm{loc},\alpha}(f)\right\|_{L^q_w}. \tag{3.6}$$

From Proposition 3.1, we can write $f = f_1 - f_2$, where f_i, $i = 1, 2$ are functions in (3.4) and (3.5), respectively. Define

$$F(y,t) = \begin{cases} t^2L\mathrm{e}^{-t^2L} f(y), & 0 < t \leq 1, \\ 0, & t > 1. \end{cases}$$

Then supp $F \subseteq \mathbb{R}^n \times (0, 1]$. Now define the family of sets $\mathcal{D}$ be dyadic cubes, $\Omega_k := \{x \in \mathbb{R}^n : s_{\mathrm{loc},10\sqrt{n}}(f)(x) > 2^k\}$ and $\mathcal{D}_k := \big\{Q \in \mathcal{D} :\ w(Q\cap\Omega_k) > w(Q)/2,\ w(Q\cap\Omega_{k+1}) \leq w(Q)/2\big\}$. We also denote by $\{Q^i_k\}$ the maximal cubes of $\mathcal{D}_k$ and denote

$$Q^+ := \big\{(x,t) :\ x \in Q,\ \ell(Q)/2 < t \leq \ell(Q)\big\},$$

where $\ell(Q)$ denote the sidelength of Q. Using the formula (3.4), we can write

$$\begin{aligned} f_1 &= \sum_{\ell=0}^{6}\sum_{j=0}^{[\frac{6-\ell}{2}]}\sum_{i,k} c_1(6,\ell,j) \sum_{\substack{Q\subset Q_k^i\\ Q\in\mathcal{D}_k}} \iint_{Q^+} K_{(t^2L)^2\Psi^{(\ell)}_{6,\,j+3}(t\sqrt{L})}(x,y)F(y,t)\frac{\mathrm{d}y\mathrm{d}t}{t} \\ &=: \sum_{\ell=0}^{6}\sum_{j=0}^{[\frac{6-\ell}{2}]}\sum_{i,k} \lambda_{\ell,j,i,k}a_{\ell,j,i,k}, \end{aligned} \tag{3.7}$$

where $a_{\ell,j,i,k} = Lb_{\ell,j,i,k}$,

$$b_{\ell,j,i,k} = \frac{c_1(6,\ell,j)}{\lambda_{\ell,j,i,k}} \sum_{\substack{Q\subset Q_k^i\\ Q\in\mathcal{D}_k}} \iint_{Q^+} K_{t^4L\Psi^{(\ell)}_{6,\,j+3}(t\sqrt{L})}(x,y)F(y,t)\frac{\mathrm{d}y\mathrm{d}t}{t}$$

and

$$\lambda_{\ell,j,i,k} = \frac{w(Q_k^i)}{|Q_k^i|^{1/2}}\left(\sum_{\substack{Q\subset Q_k^i\\ Q\in\mathcal{D}_k}} \iint_{Q^+} |F(y,t)|^2\frac{\mathrm{d}y\mathrm{d}t}{t}\right)^{1/2}.$$

We claim that, up to normalization by a fixed multiplicative constant, the $a_{\ell,j,i,k}$'s are $(1,2,w)$-atoms for L. To get started with the proof of the claim, we note that for every $m = 0,1$, there holds

$$L^m b_{\ell,j,i,k} = \frac{c_1(6,\ell,j)}{\lambda_{\ell,j,i,k}} \sum_{\substack{Q\subset Q_k^i\\ Q\in\mathcal{D}_k}} \iint_{Q^+} K_{t^4L^{m+1}\Psi^{(\ell)}_{6,\,j+3}(t\sqrt{L})}(x,y)F(y,t)\frac{\mathrm{d}y\mathrm{d}t}{t}.$$

Lemma 2.3 shows that for every $m = 0,1$,

$$\mathrm{supp}(L^m b_{\ell,j,i,k}) \subseteq 3Q_k^i. \tag{3.8}$$

To continue, for each cube Q_k^i consider some $h \in L^2(3Q_k^i)$ such that $\|h\|_{L^2(3Q_k^i)} = 1$. Then for every $m = 0,1$, there holds

$$\begin{aligned} &\left|\int_{\mathbb{R}^n} (\ell(Q_k^i)^2L)^m b_{\ell,j,i,k}(x)h(x)\mathrm{d}x\right| \\ &= \frac{c_1(6,\ell,j)\ell(Q_k^i)^{2m}}{\lambda_{\ell,j,i,k}}\left|\sum_{\substack{Q\subset Q_k^i\\ Q\in\mathcal{D}_k}} \int_{\mathbb{R}^n}\iint_{Q^+} K_{t^4L^{m+1}\Psi^{(\ell)}_{6,\,j+3}(t\sqrt{L})}(x,y)\right. \\ &\qquad\left. F(y,t)\frac{\mathrm{d}y\mathrm{d}t}{t}h(x)\mathrm{d}x\right| \\ &\le C\lambda^{-1}_{\ell,j,i,k}\ell(Q_k^i)^2\left(\sum_{\substack{Q\subset Q_k^i\\ Q\in\mathcal{D}_k}} \iint_{Q^+} \left|F(y,t)\right|^2\frac{\mathrm{d}y\mathrm{d}t}{t}\right)^{1/2} \end{aligned}$$

$$\left(\int_{\mathbb{R}^{n+1}_+} \left|(t^2L)^{m+1}\Psi^{(\ell)}_{6,\,j+3}(t\sqrt{L})h(y)\right|^2 \frac{\mathrm{d}y\mathrm{d}t}{t}\right)^{1/2}$$
$$\leq C\lambda^{-1}_{\ell,j,i,k}\ell(Q^i_k)^2 \left(\sum_{\substack{Q\subset Q^i_k\\ Q\in\mathcal{D}_k}} \iint_{Q^+} \left|F(y,t)\right|^2 \frac{dydt}{t}\right)^{1/2}$$
$$\leq C\ell(Q^i_k)^2|Q^i_k|^{1/2}w(Q^i_k)^{-1}.$$

Since h was arbitrary, this estimate entails $\|(\ell(Q^i_k)^2L)^m b_{\ell,j,i,k}\|_{L^2(\mathbb{R}^n)} \leq C\ell(Q^i_k)^2|Q^i_k|^{1/2}$ $w(Q^i_k)^{-1}$. Together with (3.8), this shows that each $a_{\ell,j,i,k}$ is, up to a fixed multiplicative constant, a $(1, 2, w)$-atom.

It remains to prove that $\sum_{k,i}|\lambda_{\ell,j,i,k}| \leq C$. We claim that

$$\sum_i \frac{w(Q^i_k)}{|Q^i_k|} \sum_{\substack{Q\subset Q^i_k\\ Q\in\mathcal{D}_k}} \iint_{Q^+} |F(y,t)|^2 \frac{\mathrm{d}y\mathrm{d}t}{t} \leq C2^{2k}w(\Omega_k). \tag{3.9}$$

Once (3.9) is proved, we apply Hölder's inequality to obtain

$$\sum_{k,i}|\lambda_{\ell,j,i,k}| = \sum_{k,i}\frac{w(Q^i_k)}{|Q^i_k|^{1/2}}\left(\sum_{\substack{Q\subset Q^i_k\\ Q\in\mathcal{D}_k}} \iint_{Q^+} |F(y,t)|^2 \frac{\mathrm{d}y\mathrm{d}t}{t}\right)^{1/2}$$
$$\leq C\sum_k\left(\sum_i w(Q^i_k)\right)^{1/2}$$
$$\left(\sum_i \frac{w(Q^i_k)}{|Q^i_k|} \sum_{\substack{Q\subset Q^i_k\\ Q\in\mathcal{D}_k}} \iint_{Q^+} |F(y,t)|^2 \frac{\mathrm{d}y\mathrm{d}t}{t}\right)^{1/2}$$
$$\leq C\sum_k w(\Omega_k)\cdot 2^k \leq C\left\|s_{\mathrm{loc},10\sqrt{n}}(f)\right\|_{L^1_w} \leq C\left\|s_{\mathrm{loc},L}(f)\right\|_{L^1_w}.$$

Let us prove (3.9). Suppose P is an arbitrary cube in $\mathbb{R}^n$. Denote

$$\mathcal{M}_w(g)(x) := \sup_{x\in P}\frac{1}{w(P)}\int_P |g(y)|w(y)\mathrm{d}y$$

and

$$\Omega^*_k := \left\{x\in\mathbb{R}^n : \mathcal{M}_w(\chi_{\Omega_k})(x) > \frac{1}{2}\right\}.$$

It follows from the operator $\mathcal{M}_w$ is weak type (1,1) with $w(z)\mathrm{d}z$ measure and that $w(\Omega^*_k) \leq Cw(\Omega_k)$. We observe that if $x\in Q$, $(y,t)\in Q^+$, then $|x-y| < 10\sqrt{n}t$. Further, if $Q\in\mathcal{D}_k$ and $Q\subset Q^i_k$, then $Q\subset\Omega^*_k$. This gives

$$\int \chi_{\Omega^*_k\setminus\Omega_{k+1}}(x)\chi\left(\frac{x-y}{10\sqrt{n}t}\right)w(x)\mathrm{d}x \geq w\big(Q\cap(\Omega^*_k\setminus\Omega_{k+1})\big)$$

$$= w(Q \cap \Omega_k^*) - w(Q \cap \Omega_{k+1})$$
$$\geq w(Q)/2.$$

Since $w \in A_1$ and $Q \subset Q_k^i$, by Lemma 2.2, we have that $\frac{|Q|}{|Q_k^i|} \leq C\big(\frac{w(Q)}{w(Q_k^i)}\big)$. Then estimate (3.9) is obtained because

$$\sum_i \frac{w(Q_k^i)}{|Q_k^i|} \sum_{\substack{Q\subset Q_k^i\\ Q\in\mathcal{D}_k}} \int_{Q^+} |F(y,t)|^2 \frac{\mathrm{d}y\mathrm{d}t}{t}$$
$$\leq \sum_i \sum_{\substack{Q\subset Q_k^i\\ Q\in\mathcal{D}_k}} \frac{w(Q)}{|Q|} \iint_{Q^+} |F(y,t)|^2 \frac{\mathrm{d}y\mathrm{d}t}{t}$$
$$\leq C \sum_i \sum_{\substack{Q\subset Q_k^i\\ Q\in\mathcal{D}_k}} \iint_{Q^+} \int \chi_{\Omega_k^*\setminus\Omega_{k+1}}(x)\chi\left(\frac{x-y}{10\sqrt{n}t}\right) w(x)\mathrm{d}x |F(y,t)|^2 \frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}$$
$$\leq C \int_{\Omega_k^*\setminus\Omega_{k+1}} s^2_{\mathrm{loc},10\sqrt{n}} f(x) w(x)\mathrm{d}x$$
$$\leq C2^{2k} w(\Omega_k^*) \leq C2^{2k} w(\Omega_k).$$

Further, we can see that the sum $f_1 = \sum_{\ell=0}^{6}\sum_{j=0}^{[\frac{6-\ell}{2}]}\sum_{i,k}\lambda_{\ell,j,i,k}a_{\ell,j,i,k}$ converges in $L^2(\mathbb{R}^n)$. In fact, we use the equality (3.7) to write

$$\Big\|\sum_{k=N}^{\infty}\sum_{i=N'}^{\infty}\lambda_{\ell,j,i,k}a_{\ell,j,i,k}(x)\Big\|_{L^2(\mathbb{R}^n)} \leq \sum_{k=N}^{\infty}\sum_{i=N'}^{\infty}\big\|\lambda_{\ell,j,i,k}a_{\ell,j,i,k}(x)\big\|_{L^2(\mathbb{R}^n)}$$
$$\leq C\sum_{k=N}^{\infty}\sum_{i=N'}^{\infty}\Big\|\sum_{\substack{Q\subset Q_k^i\\ Q\in\mathcal{D}_k}}\iint_{Q^+} K_{(t^2L)^2\Psi^{(\ell)}_{6,j+3}(t\sqrt{L})}(x,y)F(y,t)\frac{\mathrm{d}y\mathrm{d}t}{t}\Big\|_{L^2(\mathbb{R}^n)}$$
$$\leq C\sum_{k=N}^{\infty}\sum_{i=N'}^{\infty}\sup_{\|h\|_2\leq 1}\sum_{\substack{Q\subset Q_k^i\\ Q\in\mathcal{D}_k}}\iint_{Q^+}|F(y,t)||(t^2L)^2\Psi^{(\ell)}_{6,j+3}(t\sqrt{L})h(y)|\frac{\mathrm{d}y\mathrm{d}t}{t}$$
$$\leq C\sum_{k=N}^{\infty}\sum_{i=N'}^{\infty}\sum_{\substack{Q\subset Q_k^i\\ Q\in\mathcal{D}_k}}\left(\iint_{Q^+}\big|F(y,t)\big|^2\frac{\mathrm{d}y\mathrm{d}t}{t}\right)^{1/2}$$

Note that

$$\left(\iint_{\mathbb{R}^{n+1}_+}|t^2Le^{-t^2L}f(y)|^2\frac{\mathrm{d}y\mathrm{d}t}{t}\right)^{1/2} \leq C\|f\|_{L^2(\mathbb{R}^n)},$$

and $\mathbb{R}^{n+1}_+$ is the union set of all the disjoint sets $\sum_{\substack{Q\subset Q_k^i\\ Q\in\mathcal{D}_k}} Q^+$ for $k, i = 1, 2, \ldots$. Then we have

$$\Big\|\sum_{k=N}^{\infty}\sum_{i=N'}^{\infty}\lambda_{\ell,j,i,k}a_{\ell,j,i,k}(x)\Big\|_{L^2(\mathbb{R}^n)} \to 0$$

as $N \to \infty,\ N' \to \infty$.

Consider the term f_2. Denote $\mathcal{D}_k = \{Q \in \mathcal{D} :\ \ell(Q) = 2^k\}$. Then there exists a $k_0 \in \mathbb{Z}$ such that for any $Q \in \mathcal{D}_{k_0}$, $\mathrm{diam}(Q) \le 1/2$. Next, we let χ_Q denote the characterization function of Q. Then f_2 can be further decomposed as

$$f_2 = \sum_{m=0}^{5} \sum_{\ell=0}^{5-m} \sum_{j=0}^{\left[\frac{5-m-\ell}{2}\right]} \sum_{Q \in \mathcal{D}_{k_0}} c_2(m, \ell, j)\Psi^{(\ell)}_{5-m,j}(\sqrt{L})\big(\chi_Q\, e^{-L} f\big)$$

$$= \sum_{m=0}^{5} \sum_{\ell=0}^{5-m} \sum_{j=0}^{\left[\frac{5-m-\ell}{2}\right]} \sum_{Q \in \mathcal{D}_{k_0}} \gamma_Q\, a_{m,\ell,j,Q}, \tag{3.10}$$

where

$$\gamma_Q := \frac{w(Q)}{|Q|} \|\chi_Q \mathrm{e}^{-L} f\|_{L^1(\mathbb{R}^n)} \tag{3.11}$$

and

$$a_{m,\ell,j,Q} := \gamma_Q^{-1} c_2(m, \ell, j)\Psi^{(\ell)}_{5-m,j}(\sqrt{L})\big(\chi_Q\, \mathrm{e}^{-L} f\big). \tag{3.12}$$

Since $\mathrm{diam}(Q) \le 1/2$, it follows from Lemma 2.3 that $\mathrm{supp}\, a_{m,\ell,j,Q} \subseteq B(x_Q, 2)$. This, in combination with (2.6) and doubling condition, yields

$$\begin{aligned}\big\|a_{m,\ell,j,Q}\big\|^2_{L^2(\mathbb{R}^n)} &\le C\gamma_Q^{-2} \int_{\mathbb{R}^n} \big|\Psi^{(\ell)}_{5-m,j}(\sqrt{L})\big(\chi_Q\, \mathrm{e}^{-L} f\big)(x)\big|^2 \mathrm{d}x \\ &\le C\gamma_Q^{-2} \sup_{y \in Q} \int_{\mathbb{R}^n} \big|K_{\Psi^{(\ell)}_{5-m,j}(\sqrt{L})}(x, y)\big|^2 \mathrm{d}x\ \|\chi_Q\, \mathrm{e}^{-L} f\|^2_{L^1(\mathbb{R}^n)} \\ &\le C\frac{|Q|^2}{w(Q)^2} \le \frac{C|B(x_Q, 2)|}{w(B(x_Q, 2))^2}.\end{aligned}$$

This shows that up to a multiplication by a harmless constant, each $a_{m,\ell,j,Q}$ is a local $(1, 2, w)$ atom with $\mathrm{supp}\, a_{m,\ell,j,Q} \subseteq B(x_Q, 2)$. Thus, it remains to check that $\sum_{Q \in \mathcal{D}_{k_0}} |\gamma_Q| < \infty$. Note that for $w \in A_1$,

$$\begin{aligned}\sum_{Q \in \mathcal{D}_{k_0}} |\gamma_Q| &\le C \sum_{Q \in \mathcal{D}_{k_0}} \int_Q \big|\mathrm{e}^{-L} f(x)\big| w(x)\, \mathrm{d}x \\ &\le C \int_{\mathbb{R}^n} |K_{\mathrm{e}^{-L}}(x, y)| w(x) \mathrm{d}x |f(y)| \mathrm{d}y \\ &\le C \int_{\mathbb{R}^n} |f(y)| Mw(y) \mathrm{d}y \\ &\le C\|f\|_{L^1_w(\mathbb{R}^n)}.\end{aligned}$$

Altogether, this shows that the proof of (i) of Theorem 1.3 is complete.

Proof of Theorem 1.3(*ii*). Let us now move to part (ii) of Theorem 1.3. We now state an important lemma.

Lemma 3.2. *Fix $w \in A_\infty$. Assume that T is a sublinear operator, satisfying the weak-type* $(2, 2)$ *bound*

$$|\{x \in \mathbb{R}^n : |Tf(x)| > \eta\}| \le C_T\ \eta^{-2}\|f\|^2_{L^2(\mathbb{R}^n)}, \quad \forall s > 0, \tag{3.13}$$

and that for every local $(1, 2, w)$*-atom a, we have*

$$\|Ta\|_{L^1_w(\mathbb{R}^n)} \leq C \tag{3.14}$$

with constant C independent of a. Then T is bounded from $\mathbb{h}^1_{L,w}(\mathbb{R}^n)$ *to* $L^1_w(\mathbb{R}^n)$*, and*

$$\|Tf\|_{L^1_w(\mathbb{R}^n)} \leq C\|f\|_{\mathbb{h}^1_{L,w}(\mathbb{R}^n)}.$$

Proof. The proof is similar to that of Lemma 4.3 in [15] and so we skip it here. See also [16]. □

From Lemma 3.2, it remains to establish a uniform L^1_w bound on atoms, i.e., that there exists a constant $C > 0$ such that for every local $(1, 2, w)$-atom a associated to a ball $B = B(x_B, r_B)$ (x_B is the center of B, r_B is the radius of B),

$$\|s_{\mathrm{loc},L}(a)\|_{L^1_w(\mathbb{R}^n)} \leq C \quad \text{and} \quad \|a\|_{L^1_w(\mathbb{R}^n)} \leq C. \tag{3.15}$$

We apply Hölder's inequality to obtain

$$\|a\|_{L^1_w(\mathbb{R}^n)} \leq C\|a\|_{L^2(\mathbb{R}^n)}\left(\int_B w^2(x)\mathrm{d}x\right)^{1/2}.$$

Using $w \in A_1 \cap RH_2$, we have $\left(\int_B w^2(x)\mathrm{d}x\right)^{1/2} \leq Cw(B)|B|^{-1/2}$. Noting that $\|a\|_{L^2(\mathbb{R}^n)} \leq |B|^{1/2}w(B)^{-1}$, we have $\|a\|_{L^1_w(\mathbb{R}^n)} \leq C$.

Now we turn to prove $\|s_{\mathrm{loc},L}(a)\|_{L^1_w(\mathbb{R}^n)} \leq C$. If $r_B < 1$, then a standard argument as in Theorem 3.1 of [22] shows that $\|s_{\mathrm{loc},L}(a)\|_{L^1_w(\mathbb{R}^n)} \leq C$. If $r_B \geq 1$, we write

$$\begin{aligned}\int s_{\mathrm{loc},L}(a)(x)w(x)\mathrm{d}x &= \int_{2B} S_L(a)(x)w(x)\mathrm{d}x \\ &\quad + \sum_{k=1}^{+\infty}\int_{2^{k+1}B\setminus 2^kB} S_L(a)(x)w(x)\mathrm{d}x \\ &=: \mathrm{I}_1 + \sum_{k=1}^{+\infty}\mathrm{I}_k.\end{aligned}$$

We apply Hölder's inequality to obtain

$$\begin{aligned}\mathrm{I}_1 &\leq \left(\int_{2B} s^2_{\mathrm{loc},L}(a)(x)\mathrm{d}x\right)^{1/2}\left(\int_{2B} w^2(x)\mathrm{d}x\right)^{1/2} \\ &\leq C\|a\|_{L^2(\mathbb{R}^n)}\left(\int_{2B} w^2(x)\mathrm{d}x\right)^{1/2} \\ &\leq C.\end{aligned}$$

Turning to $\mathrm{I}_k,\ \ k = 1, 2, \ldots$. Using Hölder's inequality, we obtain

$$\mathrm{I}_k \leq C\left(\int_{2^kB\setminus 2^{k-1}B} |s_{\mathrm{loc},L}(a)(x)|^2\mathrm{d}x\right)^{1/2} \cdot \left(\int_{2^kB} w(x)^2\mathrm{d}x\right)^{1/2}.$$

Applying $w \in RH_2$ again, we get

$$\mathrm{I}_k \leq C\left(\int_{2^kB\setminus 2^{k-1}B} |s_{\mathrm{loc},L}(a)(x)|^2\mathrm{d}x\right)^{1/2} \frac{w(2^kB)}{|2^kB|^{1/2}}. \tag{3.16}$$

We observe that $x \in 2^k B \setminus 2^{k-1} B$, $z \in B$ and $|x-y| < r_B$ imply $|y-z| \geq C 2^k r_B$. Then, for any $x \in 2^k B \setminus 2^{k-1} B$, we have

$$\begin{aligned}
s^2_{\mathrm{loc},L}(a)(x) &\leq \int_0^{r_B} \int_{|x-y|<t} \left|t^2 L \mathrm{e}^{-t^2 L} a(y)\right|^2 \frac{\mathrm{d}y \mathrm{d}t}{t^{n+1}} \\
&\leq C \int_0^{r_B} \int_{|x-y|<t} \left(\int \frac{t}{(t+|y-z|)^{n+1}} |a(z)| \mathrm{d}z \right)^2 \mathrm{d}y \frac{\mathrm{d}t}{t^{n+1}} \\
&\leq C \frac{\|a\|^2_{L^1(\mathbb{R}^n)}}{(2^k r_B)^{2n+2}} \int_0^{r_B} \int_{|x-y|<t} t^2 \mathrm{d}y \frac{\mathrm{d}t}{t^{n+1}} \\
&\leq C \frac{1}{2^{k(2n+2)}|B|} \|a\|^2_{L^2(\mathbb{R}^n)} \leq C \frac{1}{2^{k(2n+2)} w(B)^2},
\end{aligned}$$

where in the last inequality we used the condition $\|a\|_{L^2(\mathbb{R}^n)} \leq |B|^{1/2} w(B)^{-1}$. This, combining with (vii) of Lemma 2.1 gives that

$$\mathrm{I}_k \leq C 2^{-k(n+1)} w(2^k B) w(B)^{-1} \leq C 2^{-k},$$

which completes the proof of (3.15) and the proof of Theorem 1.3. □

Theorem 3.3. *Let L be a non-negative self-adjoint operator such that the corresponding heat kernels satisfy Gaussian bound (GE). Let $w \in A_1 \cap RH_2$. If $f \in h^1_{L,w}(\mathbb{R}^n)$, then there exists a family of local $(1, 2, w)$-atoms $\{a_i\}_{i=0}^\infty$ and a sequence of numbers of $\{\lambda_i\}_{i=0}^\infty$ such that f can be represented in the form*

$$f = \sum_{i=0}^{\infty} \lambda_i a_i \quad \text{in } h^1_{L,w}(\mathbb{R}^n)$$

Moreover,

$$\sum_{i=0}^{\infty} |\lambda_i| \leq C \|f\|_{h^1_{L,w}(\mathbb{R}^n)}.$$

Proof. Firstly, consider $f \in h^1_{L,w}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. From Theorem 1.3, there exists a family of local $(1, 2, w)$-atoms $\{a_i\}_{i=0}^\infty$ and a sequence of numbers of $\{\lambda_i\}_{i=0}^\infty$ such that f can be represented in the form

$$f = \sum_{i=0}^{\infty} \lambda_i a_i \quad \text{in } L^2(\mathbb{R}^n).$$

Take $f_N = \sum_{i=0}^N \lambda_i a_i$. we have that $f_N \to f$ in $L^2(\mathbb{R}^n)$ as $N \to \infty$, and thus

$$\left\| s_{\mathrm{loc},L}(f_N - f) \right\|_{L^2(\mathbb{R}^n)} \to 0.$$

It follows by F. Riesz's theorem that we can take subsequence to obtain $s_{\mathrm{loc},L}(f_{N_k} - f) \to 0$ a.e., and

$$\lim_{k' \to \infty} s_{\mathrm{loc},L}(f_{N_k} - f_{N_{k'}}) = s_{\mathrm{loc},L}(f_{N_k} - f), \quad \text{a.e.}$$

By Theorem 1.3, we have that $\lim_{k,k'\to\infty} \|s_{\mathrm{loc},L}(f_{N_k} - f_{N_{k'}})\|_{L^1_w(\mathbb{R}^n)} \le C \lim_{k,k'\to\infty} \sum_{i=k}^{k'} |\lambda_i| = 0$. Consequently, for any given $\eta > 0$,

$$\begin{aligned}\|s_{\mathrm{loc},L}(f_{N_k} - f)\|_{L^1_w(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \lim_{k'\to\infty}\inf s_{\mathrm{loc},L}(f_{N_k} - f_{N_{k'}}) w(x)\mathrm{d}x \\ &\le \lim_{k'\to\infty}\inf \int_{\mathbb{R}^n} s_{\mathrm{loc},L}(f_{N_k} - f_{N_{k'}}) w(x)\mathrm{d}x \\ &\le \eta\end{aligned}$$

for k chosen large enough. This gives the proof in the special case.

Consider general $f \in h^1_{L,w}(\mathbb{R}^n)$. For any $f \in h^1_{L,w}(\mathbb{R}^n)$, there exists a sequence of functions $f_m \in h^1_{L,w}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that

$$f = \sum_m f_m \quad \text{in} \quad h^1_{L,w}(\mathbb{R}^n)$$

and $\sum_m \|f_m\|_{h^1_{L,w}(\mathbb{R}^n)} \le C\|f\|_{h^1_{L,w}(\mathbb{R}^n)}$. We have proved that

$$f_m = \sum_j \mu_j^m b_j^m \quad \text{in} \quad h^1_{L,w}(\mathbb{R}^n),$$

where the b_j^m are all local $(1, 2, w)$-atoms and $\sum_j |\mu_j^m| \le C\|f_m\|_{h^1_{L,w}(\mathbb{R}^n)}$. Therefore,

$$f = \sum_m \sum_j \mu_j^m b_j^m \quad \text{in} \quad h^1_{L,w}(\mathbb{R}^n)$$

and

$$\begin{aligned}\sum_{m,j} |\mu_j^m| &\le \sum_m \sum_j |\mu_j^m| \le C \sum_m \|f_m\|_{h^1_{L,w}(\mathbb{R}^n)} \\ &\le C\|f\|_{h^1_{L,w}(\mathbb{R}^n)}.\end{aligned}$$

The proof is complete. □

4. Weighted local BMO spaces $\mathbf{bmo}_{L,w}$

In this section, we introduce the weighted local BMO spaces $\mathrm{bmo}_{L,w}(\mathbb{R}^n)$ and establish the duality between $h^1_{L,w}(\mathbb{R}^n)$ and $\mathrm{bmo}_{L,w}(\mathbb{R}^n)$. Let us begin with some definitions.

DEFINITION 4.1

Suppose $w \in A_\infty$. A locally integrable function f is said to belong to $\mathrm{BMO}_{L,w}(\mathbb{R}^n)$ if for all balls $B \subset \mathbb{R}^n$,

$$\|f\|_{\mathrm{BMO}_{L,w}(\mathbb{R}^n)} = \sup_{B\subset\mathbb{R}^n} \frac{1}{w(B)} \int_B |f(y) - \mathrm{e}^{-r_B^2 L} f(y)|\mathrm{d}y < \infty,$$

where r_B is the radius of the ball B and the supremum is taken over all balls in $\mathbb{R}^n$.

Following Goldberg [10], the weighted local BMO spaces $\mathrm{bmo}_{L,w}(\mathbb{R}^n)$ is defined as follows.

DEFINITION 4.2

Suppose $w \in A_\infty$. A locally integrable function f is said to belong to $\mathrm{bmo}_{L,w}(\mathbb{R}^n)$ if

$$\begin{aligned}\|f\|_{\mathrm{bmo}_{L,w}(\mathbb{R}^n)} = &\sup_{B\subset\mathbb{R}^n, r_B<1} \frac{1}{w(B)} \int_B |f(y) - \mathrm{e}^{-r_B^2 L} f(y)|\mathrm{d}y \\ &+ \sup_{B\subset\mathbb{R}^n, r_B\geq 1} \frac{1}{w(B)} \int_B |f(y)|\mathrm{d}y < \infty,\end{aligned}$$

where r_B is the radius of the ball B and the supremum is taken over all balls in $\mathbb{R}^n$.

Lemma 4.3. *Suppose* $w \in A_\infty$. *Then* $\mathrm{bmo}_{L,w}(\mathbb{R}^n) \subset \mathrm{BMO}_{L,w}(\mathbb{R}^n)$, *and there exists a positive constant C such that for all* $f \in \mathrm{bmo}_{L,w}(\mathbb{R}^n)$, $\|f\|_{\mathrm{BMO}_{L,w}(\mathbb{R}^n)} \leq C\|f\|_{\mathrm{bmo}_{L,w}(\mathbb{R}^n)}$.

Proof. By definitions of $\mathrm{BMO}_{L,w}(\mathbb{R}^n)$ and $\mathrm{bmo}_{L,w}(\mathbb{R}^n)$, it suffices to show that there exists a positive constant C such that for all balls $B \subset \mathbb{R}^n$ with $r_B \geq 1$, $\frac{1}{w(B)} \int_B |\mathrm{e}^{-r_B^2 L} f(y)|dy < C\|f\|_{\mathrm{bmo}_{L,w}(\mathbb{R}^n)}$. By (GE), we have

$$\begin{aligned}\frac{1}{w(B)} \int_B |\mathrm{e}^{-r_B^2 L} f(y)|\mathrm{d}y &\leq \frac{C}{w(B)} \int_B \int_{\mathbb{R}^n} r_B^{-n} \mathrm{e}^{-\frac{|y-z|^2}{c r_B^2}} |f(z)|\mathrm{d}z\mathrm{d}y \\ &\leq \frac{C}{w(B)} \int_B \int_{2B} r_B^{-n} \mathrm{e}^{-\frac{|y-z|^2}{c r_B^2}} |f(z)|\mathrm{d}z\mathrm{d}y \\ &\quad + \sum_{k=1}^{\infty} \frac{C}{w(B)} \int_B \int_{2^{k+1}B\setminus 2^k B} r_B^{-n} \mathrm{e}^{-\frac{|y-z|^2}{c r_B^2}} |f(z)|\mathrm{d}z\mathrm{d}y \\ &\leq \frac{C}{w(B)} \int_{2B} |f(z)|\mathrm{d}z + \sum_{k=1}^{\infty} \frac{C\mathrm{e}^{-c_1 2^{2k}}}{w(B)} \int_{2^{k+1}B} |f(z)|\mathrm{d}z.\end{aligned}$$

Note that for $w \in A_\infty$, there exists a $1 \leq q_0 < \infty$ such that $w \in A_{q_0}$. Thus, from (vii) of Lemma 2.1, we have

$$\begin{aligned}\frac{1}{w(B)} \int_B |\mathrm{e}^{-r_B^2 L} f(y)|\mathrm{d}y &\leq \frac{C}{w(2B)} \int_{2B} |f(z)|\mathrm{d}z \\ &\quad + \sum_{k=1}^{\infty} \frac{C 2^{(k+1)q_0 n} \mathrm{e}^{-c_1 2^{2k}}}{w(2^{k+1}B)} \int_{2^{k+1}B} |f(z)|\mathrm{d}z \\ &\leq C\|f\|_{\mathrm{bmo}_{L,w}(\mathbb{R}^n)},\end{aligned}$$

which completes the proof of this lemma. □

It was proved in [25] that there exists a variant of the classical John–Nirenberg inequality for $\mathrm{BMO}_{L,w}(\mathbb{R}^n)$ space, namely, if $0 < c < w \in A_\infty$, then there exist positive constants

c_1 and c_2 such that for all $f \in \mathrm{BMO}_{L,w}(\mathbb{R}^n)$, balls B and $\lambda > 0$, we have

$$\left|\left\{x \in B : |f(x) - \mathrm{e}^{-r_B^2 L} f(x)| > \lambda \frac{w(B)}{|B|}\right\}\right| \le c_1 |B| \exp\left\{-\frac{c_2 \lambda}{\|f\|_{\mathrm{BMO}_{L,w}}}\right\}. \tag{4.1}$$

We have the following result for $\mathrm{bmo}_{L,w}(\mathbb{R}^n)$ space.

PROPOSITION 4.4

Suppose $0 < c < w \in A_\infty$. *Then there exist positive constants* C_1 *and* C_2 *such that for all* $f \in \mathrm{bmo}_{L,w}(\mathbb{R}^n)$, *balls* B *and* $\lambda > 0$, *we have*

$$\left|\left\{x \in B : |f(x) - \mathrm{e}^{-r_B^2 L} f(x)| > \lambda \frac{w(B)}{|B|}\right\}\right| \le C_1 |B| \exp\left\{-\frac{C_2 \lambda}{\|f\|_{\mathrm{bmo}_{L,w}}}\right\} \tag{4.2}$$

for $r_B < 1$ *and for* $r_B \ge 1$,

$$\left|\left\{x \in B : |f(x)| > \lambda \frac{w(B)}{|B|}\right\}\right| \le C_1 |B| \exp\left\{-\frac{C_2 \lambda}{\|f\|_{\mathrm{bmo}_{L,w}}}\right\}. \tag{4.3}$$

Proof. By Lemma 4.3 and (4.1), we have (4.2). To see (4.3), for the ball B with $r_B \ge 1$ and for all $x \in B$, we have

$$\begin{aligned}
|\mathrm{e}^{-r_B^2 L} f(x)| &\le C \int_{\mathbb{R}^n} r_B^{-n} \mathrm{e}^{-\frac{|x-y|^2}{c r_B^2}} |f(y)| \mathrm{d}y \\
&\le C \int_{2B} r_B^{-n} \mathrm{e}^{-\frac{|x-y|^2}{c r_B^2}} |f(y)| \mathrm{d}y \\
&\quad + \sum_{k=1}^{\infty} C \int_{2^{k+1}B \setminus 2^k B} r_B^{-n} \mathrm{e}^{-\frac{|x-y|^2}{c r_B^2}} |f(y)| \mathrm{d}y \\
&\le \frac{C}{|B|} \int_{2B} |f(y)| \mathrm{d}y + \sum_{k=1}^{\infty} \frac{C \mathrm{e}^{-c_1 2^{2k}}}{|B|} \int_{2^{k+1}B} |f(y)| \mathrm{d}y \\
&\le \frac{C w(B)}{|B|} \|f\|_{\mathrm{bmo}_{L,w}}.
\end{aligned}$$

If $\lambda > 2C\|f\|_{\mathrm{bmo}_{L,w}}$, then $\lambda > 2|\mathrm{e}^{-r_B^2 L} f(x)||B|/w(B)$ for all $x \in B$. Thus,

$$\left|\left\{x \in B : |f(x)| > \lambda \frac{w(B)}{|B|}\right\}\right| \le \left|\left\{x \in B : |f(x) - \mathrm{e}^{-r_B^2 L} f(x)| > \lambda/2 \frac{w(B)}{|B|}\right\}\right|,$$

which together with (4.1) implies that (4.3) holds. If $\lambda \le 2C\|f\|_{\mathrm{bmo}_{L,w}}$, we then have

$$\left|\left\{x \in B : |f(x)| > \lambda \frac{w(B)}{|B|}\right\}\right| \le |B| \le C_1 |B| \exp\left\{-\frac{C_2 \lambda}{\|f\|_{\mathrm{bmo}_{L,w}}}\right\}.$$

This completes the proof of this proposition. □

Applying Proposition 4.4, we obtain the following conclusions.

COROLLARY 4.5

Suppose $0 < c < w \in A_1$. *Let* $q \in (1, \infty)$. *Then* $\mathrm{bmo}^q_{L,w}(\mathbb{R}^n) = \mathrm{bmo}_{L,w}(\mathbb{R}^n)$ *with equivalent norms. A locally integrable function* f *is said to belong to* $\mathrm{bmo}^q_{L,w}(\mathbb{R}^n)$ *if*

$$\|f\|_{\mathrm{bmo}^q_{L,w}(\mathbb{R}^n)} = \sup_{B\subset\mathbb{R}^n, r_B<1}\left\{\frac{|B|^{q-1}}{w(B)^q}\int_B |f(y)-\mathrm{e}^{-r_B^2 L}f(y)|^q \mathrm{d}y\right\}^{1/q} + \sup_{B\subset\mathbb{R}^n, r_B\geq 1}\left\{\frac{|B|^{q-1}}{w(B)^q}\int_B |f(y)|^q \mathrm{d}y\right\}^{1/q} < \infty.$$

Proof. Suppose $f \in \mathrm{bmo}^q_{L,w}(\mathbb{R}^n)$. For the ball B with $r_B < 1$, by Hölder's inequality, we have

$$\int_B |f-\mathrm{e}^{-r_B^2 L}f|\mathrm{d}y \leq \left(\int_B |f-\mathrm{e}^{-r_B^2 L}f|^q w^{1-q}\mathrm{d}y\right)^{1/q}\left(\int_B w\mathrm{d}x\right)^{1/q'},$$

where $1/q + 1/q' = 1$. Note that for $w \in A_1$, $w(B)/|B| \leq Cw(x)$, a.e. $x \in B$. Thus,

$$\frac{1}{w(B)}\int_B |f-\mathrm{e}^{-r_B^2 L}f|\mathrm{d}y \leq \left(\frac{|B|^{q-1}}{w(B)^q}\int_B |f-\mathrm{e}^{-r_B^2 L}f|^q\mathrm{d}y\right)^{1/q}.$$

Similarly, for the ball B with $r_B \geq 1$, we have

$$\frac{1}{w(B)}\int_B |f|\mathrm{d}y \leq \left(\frac{|B|^{q-1}}{w(B)^q}\int_B |f|^q\mathrm{d}y\right)^{1/q}.$$

Therefore, $f \in \mathrm{bmo}_{L,w}(\mathbb{R}^n)$ and $\|f\|_{\mathrm{bmo}_{L,w}(\mathbb{R}^n)} \leq C\|f\|_{\mathrm{bmo}^q_{L,w}(\mathbb{R}^n)}$.

Conversely, suppose $f \in \mathrm{bmo}_{L,w}(\mathbb{R}^n)$. For the ball B with $r_B < 1$, from Proposition 4.4, we have

$$\begin{aligned}\int_B |f-\mathrm{e}^{-r_B^2 L}f|^q\mathrm{d}y &= q\int_0^\infty \lambda^{q-1}|\{x\in B:\ |f(x)-\mathrm{e}^{-r_B^2 L}f(x)|>\lambda\}|\mathrm{d}\lambda\\ &= q\frac{w(B)^q}{|B|^q}\int_0^\infty \lambda^{q-1}\\ &\qquad |\{x\in B:\ |f(x)-\mathrm{e}^{-r_B^2 L}f(x)|>\lambda w(B)/|B|\}|\mathrm{d}\lambda\\ &\leq C\frac{w(B)^q}{|B|^{q-1}}\int_0^\infty \lambda^{q-1}\exp\left\{-\frac{C_2\lambda}{\|f\|_{\mathrm{bmo}_{L,w}}}\right\}\mathrm{d}\lambda\\ &\leq C\frac{w(B)^q}{|B|^{q-1}}\|f\|^q_{\mathrm{bmo}_{L,w}}.\end{aligned}$$

Similarly, for the ball B with $r_B \geq 1$, we have $\int_B |f|^q \mathrm{d}y \leq C \frac{w(B)^q}{|B|^{q-1}} \|f\|^q_{\mathrm{bmo}_{L,w}}$. Thus, $f \in \mathrm{bmo}^q_{L,w}(\mathbb{R}^n)$ and $\|f\|_{\mathrm{bmo}^q_{L,w}(\mathbb{R}^n)} \leq C\|f\|_{\mathrm{bmo}_{L,w}(\mathbb{R}^n)}$. This completes the proof of this corollary. □

Now, we have the following theorem.

Theorem 4.6. *Suppose* $0 < c < w \in A_1 \cap RH_2$. *Then the dual space of* $h^1_{L,w}(\mathbb{R}^n)$ *is* $\mathrm{bmo}_{L,w}(\mathbb{R}^n)$.

Proof. Let $f \in \mathrm{bmo}_{L,w}(\mathbb{R}^n)$. For any $g \in h^1_{L,w}(\mathbb{R}^n)$, there exist local $(1, 2, w)$-atoms $\{a_i\}_{i=0}^{\infty}$ associated balls B^i with $r_{B^i} < 1$, local $(1, 2, w)$-atoms $\{b_j\}_{j=0}^{\infty}$ associated balls B_j with $r_{B_j} \geq 1$ and a sequence of numbers of $\{\kappa_i, \lambda_i\}_{i=0}^{\infty}$ such that

$$g = \sum_{i=0}^{\infty} \kappa_i a_i + \sum_{j=0}^{\infty} \lambda_j b_j =: g_1 + g_2$$

with $\sum_{i=0}^{\infty} |\kappa_i| + \sum_{j=0}^{\infty} |\lambda_j| \leq C\|g\|_{h^1_{L,w}(\mathbb{R}^n)}$. By the atomic characterization of $H^1_{L,w}(\mathbb{R}^n)$ in [22], we know that $g_1 \in H^1_{L,w}(\mathbb{R}^n)$. Note that $f \in \mathrm{BMO}_{L,w}$ (see Lemma 4.3) and $(H^1_{L,w})^* = \mathrm{BMO}_{L,w}$ (see [25, Theorem 1]). Thus, we can define $\ell(g_1) = \sum_{i=0}^{\infty} \kappa_i \int_{\mathbb{R}^n} f(x)a_i(x)\mathrm{d}x$. Moreover,

$$|\ell(g_1)| \leq \|f\|_{\mathrm{BMO}_{L,w}} \sum_{i=0}^{\infty} |\kappa_i| \|a_i\|_{H^1_{L,w}} \leq C\|f\|_{\mathrm{bmo}_{L,w}} \|g\|_{h^1_{L,w}}.$$

For g_2, by the Hölder inequality, we have

$$\begin{aligned}\left| \int_{\mathbb{R}^n} f g_2 \mathrm{d}x \right| &\leq \sum_{j=0}^{\infty} |\lambda_j| \int_{\mathbb{R}^n} |f(x)b_j(x)|\mathrm{d}x \leq C \sum_{j=0}^{\infty} |\lambda_j| \|f\|_{L^2(B_j)} \|b_j\|_{L^2(B_j)} \\ &\leq C \sum_{j=0}^{\infty} |\lambda_j| \|f\|_{L^2(B_j)} |B_j|^{1/2}/w(B_j) \leq C\|f\|_{\mathrm{bmo}_{L,w}} \|g\|_{h^1_{L,w}}.\end{aligned}$$

Thus, we can define $\ell(g_2) = \sum_{j=0}^{\infty} \lambda_j \int_{\mathbb{R}^n} f(x)b_j(x)\mathrm{d}x$. Thus, each $f \in \mathrm{bmo}_{L,w}(\mathbb{R}^n)$ induces a bounded linear functional on $h^1_{L,w}$.

Conversely, let $\ell \in (h^1_{L,w})^*$. For the ball B with $r_B \geq 1$ and for any $g \in L^2(B)$, using the ideas and skills as the proof of (ii) of Theorem 1.3, we have $\|g\|_{h^1_{L,w}} \leq Cw(B)/|B|^{1/2}\|g\|_{L^2(B)}$. Thus,

$$\ell(g) \leq \|\ell\| \|g\|_{h^1_{L,w}} \leq Cw(B)/|B|^{1/2} \|\ell\| \|g\|_{L^2(B)},$$

which implies that there exists $f^B \in L^2(B)$ such that $\ell(g) = \int_{\mathbb{R}^n} f^B g \mathrm{d}x$ with $\|f^B\|_{L^2(B)} \leq Cw(B)/|B|^{1/2}\|\ell\|$. If $B_1 \subset B_2$, since $L^2(B_1) \subset L^2(B_2)$), we then have that

$f^{B_2}(x) = f^{B_1}(x)$ for $x \in B_1$. Therefore we define $f(x) = \lim_{j\to\infty} f^{jB}(x)$ for $x \in \mathbb{R}^n$. Then we have that for any ball B with $r_B < 1$,

$$\begin{aligned}\|f - \mathrm{e}^{-r_B^2 L} f\|_{L^2_{w^{-1}}} &= \sup_{\|g\|_{L^2_w(\mathbb{R}^n)}\le 1} \left|\int_{\mathbb{R}^n} (I - \mathrm{e}^{-r_B^2 L}) f(x) g(x)\mathrm{d}x\right| \\ &= \sup_{\|g\|_{L^2_w(\mathbb{R}^n)}\le 1} \left|\int_{\mathbb{R}^n} f(x)(I - \mathrm{e}^{-r_B^2 L}) g(x)\mathrm{d}x\right| \\ &\le \sup_{\|g\|_{L^2_w(\mathbb{R}^n)}\le 1} \|\ell\| \|(I - \mathrm{e}^{-r_B^2 L}) g\|_{h^1_{L,w}}.\end{aligned}$$

Note that $\|(I - \mathrm{e}^{-r_B^2 L}) g\|_{H^1_{L,w}} \le C w(B)^{1/2} \|g\|_{L^2_w}$ (see the proof of Theorem 1 in [25]). Thus,

$$\begin{aligned}\|f - \mathrm{e}^{-r_B^2 L} f\|_{L^2_{w^{-1}}} &\le \sup_{\|g\|_{L^2_w(\mathbb{R}^n)}\le 1} \|\ell\| \|(I - \mathrm{e}^{-r_B^2 L}) g\|_{h^1_{L,w}} \\ &\le \sup_{\|g\|_{L^2_w(\mathbb{R}^n)}\le 1} \|\ell\| \|(I - \mathrm{e}^{-r_B^2 L}) g\|_{H^1_{L,w}} \le C\|\ell\| w(B)^{1/2}.\end{aligned}$$

By the proof of Corollary 4.5, we have

$$\frac{1}{w(B)} \int_B |f - \mathrm{e}^{-r_B^2 L} f|\mathrm{d}x \le \left(\frac{1}{w(B)} \int_B |f - \mathrm{e}^{-r_B^2 L} f|^2 w^{-1}\mathrm{d}x\right)^{1/2} \le C\|\ell\|.$$

Note that for ball B with $r_B \ge 1$, $\|f^B\|_{L^2(B)} \le C w(B)/|B|^{1/2} \|\ell\|$. Thus, we obtain that $\|f\|_{\mathrm{bmo}_{L,w}} \le C\|\ell\|$. This completes the proof of this theorem. □

5. Applications

In this section, we describe some situations where the operator L satisfies the Gaussian bound (GE).

(1) *Divergence form operator*. Let $A = \big((a_{ij}(x))\big)_{1\le i,j\le n}$ be an $n \times n$ matrix of complex with entries $a_{ij} \in L^\infty(\mathbb{R}^n)$ satisfying $\mathrm{Re} \sum a_{ij}\xi_i\xi_j \ge \lambda|\xi|^2$ for all $x \in \mathbb{R}^n, \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{C}^n$ and some $\lambda > 0$. We define divergence form operator

$$Lf \equiv -\mathrm{div}(A\nabla f),$$

which we interpret in the usual weak sense via a sesquilinear form.

It is known that the Gaussian upper bound (GE) is satisfied when A has real entries, or when $n = 1, 2$ in the case of complex entries, see [3, Chapter 1].

(2) *Schrödinger operators*. Let $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be a nonnegative function. The Schrödinger operator with potential V is defined by $L = -\Delta + V$ on $\mathbb{R}^n$, where $n \ge 3$. From the well-known Trotter–Kato product formula, it follows that the semigroup e^{-tL} has a kernel $p_t(x, y)$ satisfying

$$0 \le p_t(x, y) \le (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{|x-y|^2}{4t}\right) \quad \text{for all } t > 0,\ \ x, y \in \mathbb{R}^n. \tag{5.1}$$

See page 195 of [19].

There are many other interesting situations where the condition (GE) holds, see for example, [24].

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