Proc. Indian Acad. Sci. (Math. Sci.) (2018) 128:22
https://doi.org/10.1007/s12044-018-0397-0

Positive solutions with single and multi-peak for semilinear elliptic equations with nonlinear boundary condition in the half-space

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MS received 20 September 2015; revised 4 January 2016; accepted 19 January 2016; published online 24 April 2018

Abstract. We consider the existence of single and multi-peak solutions of the following nonlinear elliptic Neumann problem

$$\begin{cases} -\Delta u + \lambda^2 u = Q(x)|u|^{p-2}u & \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial n} = f(x,u) & \text{on } \partial\mathbb{R}^N_+, \end{cases}$$

where λ is a large number, $p \in (2, \frac{2N}{N-2})$ for $N \geq 3$, $f(x,u)$ is subcritical about u and Q is positive and has some non-degenerate critical points in $\mathbb{R}^N_+$. For λ large, we can get solutions which have peaks near the non-degenerate critical points of Q.

Keywords. Elliptic equation; multi-peak solutions; singular perturbation; nonlinear boundary condition.

2010 Mathematics Subject Classification. 26A33, 65M12, 65M06.

1. Introduction

In this paper, we consider the multiplicity of single and multi-peak positive solutions of the following problem:

$$\begin{cases} -\Delta u + \lambda^2 u = Q(x)|u|^{p-2}u & \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial n} = f(x,u) & \text{on } \partial\mathbb{R}^N_+ \cong \mathbb{R}^{N-1}, \end{cases} \tag{1.1}$$

where $\lambda \in \mathbb{R}$ and $\frac{\partial u}{\partial n}$ denotes the outward unit normal on $\partial\mathbb{R}^N_+$; $2 < p < 2^*$ ($2^* = \frac{2N}{N-2}$), $N \geq 3$; $Q(x)$ and $f(x,t)$ satisfy the following conditions:

(Q_k): $Q \geq 0$, $Q \in C^3(\mathbb{R}^N_+)$, $D^i Q$ are bounded on $\mathbb{R}^N_+$, $i = 0, 1, 2, 3$; there exist k points $a^j \in \mathbb{R}^N_+$, $j = 1, \ldots, k$ such that $Q(a^j) > 0$, $DQ(a^j) = 0$, $\det(D^2 Q(a^j)) \neq 0$;

(f1): $f(x,0)=0,\ f_t(x,0)=0,\ f_t(x,t)\in C(\mathbb{R}^{N-1}\times\mathbb{R})$;
(f2): $\exists\,\gamma>0$, such that $\lim_{t\to 0}\frac{|f_t(x,t)|}{|t|^{\gamma}}<\infty$, uniformly for $x\in\mathbb{R}^{N-1}$;
(f3) : $\lim_{t\to\infty}\frac{|f_t(x,t)|}{|t|^{q-1}}=0$, $q\in(1,2_*-1)$ $(2_*=\frac{2N-2}{N-2})$, uniformly for $x\in\mathbb{R}^{N-1}$.

The semilinear elliptic equations in $\mathbb{R}^N_+$ with nonlinear boundary condition have received a great deal of research in recent years, for example [8,15–17,19]. In [15], Wu Tsung-fang has investigated the following equation

$$\begin{cases}-\Delta u+u=g(x)|u|^{p-2}u & \text{in } \mathbb{R}^N_+,\\ \frac{\partial u}{\partial n}+f(x)|u|^{q-2}u=0 & \text{on } \partial\mathbb{R}^N_+\cong\mathbb{R}^{N-1},\end{cases}\tag{P}$$

where $1<q<\min\{p,2_*\}$, $2<p<2^*$, $g(x)=1+\lambda a(x)$. He got that under suitable decay assumptions on f and a,

$$0\le f\le C_1\exp(-r_f|x|);\quad a(x)\ge C_2\exp(-r_a|x|);\quad a(x)\to 0,\ |x|\to\infty$$

for some constants $C_1, C_2, r_f, r_a>0$, there exists a positive number λ_* such that the equation (P) has at least three positive solutions for $\lambda\in(0,\lambda_*)$ and one positive solution for $\lambda\in\{0\}\cup[\lambda_*,\infty)$. In [16], Wu considered the multiplicity of positive solutions of (P) with sign-changing coefficient in which $g(x)=a(x)+\mu b(x)$, $f(x)=-\lambda c_+(x)-c_-(x)$ and $a(x), b(x), c(x)$ also satisfy some asymptotic properties.

For problem (1.1) in $\mathbb{R}^N$, the existence of positive solutions has been established by several authors. For example, in [1], the authors have got a positive solution in which $Q(x)$ is required to satisfy

$$Q(x)\to\bar{Q},\quad Q(x)-\bar{Q}\ge -C\exp(-2-\delta|x|),\quad \text{as } |x|\to\infty,$$

for some constants $\bar{Q}, C, \delta>0$. In [2], the authors considered that (1.1) in unbounded domain has at least one positive solution under condition

$$Q(x)-\bar{Q}\ge -C\exp(-\delta|x|),\quad \text{as } |x|\to\infty.$$

The above results are derived by variational method, and because of the loss of compactness on unbounded domain, the authors recovered the compactness partially by adding asymptotic decay condition to coefficients. We have to mention that in [3,5,9,11,18], this kind of asymptotic property is removed, for example, Cao in [5] has considered the existence of positive multi-peak solutions for equation (1.1) in $\mathbb{R}^N$ under condition (Q_k). If x_0 is a strict local minimum point of $Q(x)$, it is proved in [11] that the problem (1.1) in $\mathbb{R}^N$ has a multi-peak solution with all its peaks near x_0. The results in [9] tell us that the above result is not true for the strict local maximum point of Q.

We also mention that, when $f=0$, and Ω is a bounded smooth domain in $\mathbb{R}^N$, the problem (1.1) has also been considered by several authors in [4,6,7,12,14]. In [7], Cao *et al*. considered (1.1) subject to homogeneous Dirichlet boundary condition, if the coefficient $Q(x)$ has k non-degenerate critical points, then the problem has only k single-peak solutions. In [4], the following problem

$$\begin{cases}-\varepsilon\Delta u+u=u^{p-1} & \text{in } \Omega,\\ \dfrac{\partial u}{\partial n}=0 & \text{on } \partial\Omega\end{cases}$$

has a solution whose peak near the local strict maximum or minimum point of the mean curvature of the boundary $\partial\Omega$ for ε small enough. In these papers, both the topology of Ω and the shape of Q play an important role in the existence of positive solutions of (1.1) and on the location of their peaks.

In the present paper, stimulated by [3,15,18], we consider the single and multi-peak solutions of (1.1). To state our results more precisely, we introduce some notation first. Let U be the ground state solution of the problem

$$\begin{aligned} &-\Delta u + u = |u|^{p-2}u && \text{in } \mathbb{R}^N,\\ &u \in H^1(\mathbb{R}^N), && u(0) = \max_{x\in\mathbb{R}^N} u(x). \end{aligned} \tag{1.2}$$

It is known in [10] that U is a positive decreasing radial function, and

$$\lim_{|x|\to\infty} U(|x|)e^{|x|}|x|^{(N-1)/2} = \bar{C} > 0.$$

For $a \in \mathbb{R}^N_+$, U_a denotes the translate of U by a. Our main result is the following.

Theorem 1.1. *Assume condition* (Q_k) *holds*, $2 < p < 2^*$, $N \geq 3$, (*f*1) (*f*2) *and* (*f*3) *hold. Then there exists* $\lambda_0 > 0$ *large enough such that for* $\lambda > \lambda_0$, (1.1) *has* $2^k - 1$ *positive solutions satisfying*: *for any* $l = 1, \ldots, k$, *there are* $C_k^l := \frac{k(k-1)\cdots(k-l+1)}{l!}$ *solutions of the form*

$$u_{J,\lambda} = \lambda^{2/p-2}\left[\sum_{i=1}^{l} \alpha_{j_i} U(\lambda(y - a^{j_i} - x^{j_i}(\lambda))) + w_{J,\lambda}(\lambda y)\right],$$

in which $J = (j_1, \ldots, j_l)$, $j_i \in \{1, \ldots, k\}$, $j_i \neq j_m$, *if* $j \neq m$; $x^{j_i}(\lambda) \in \mathbb{R}^N$, $w_{J,\lambda} \in H^1(\mathbb{R}^N_+)$, *and satisfying*

$$x^{j_i}(\lambda) \to 0, \ \ \alpha_{j_i} \to (Q(a^{j_i}))^{-1/(p-2)}, \ \ w_{J,\lambda} \to 0 \ \mathit{strongly\ in}\ H^1(\mathbb{R}^N_+),$$

as $\lambda \to \infty$.

Set $v(y) = \lambda^{-\frac{2}{p-2}}u(\frac{y}{\lambda})$, then (1.1) becomes

$$\begin{cases} -\Delta v + v = Q\left(\frac{y}{\lambda}\right)|v|^{p-2}v & \text{in } \mathbb{R}^N_+,\\ \frac{\partial v}{\partial n} = \lambda^{\frac{-p}{p-2}} f\left(\frac{y}{\lambda}, \lambda^{\frac{2}{p-2}}v\right) & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{1.3}$$

If u_λ is a solution of (1.3), and by a translation for u_λ we set $v_\lambda(y) = u_\lambda(y + \lambda a^j)$, $y \in \mathbb{R}^N_+ - \{\lambda a^j\}$, then v_λ satisfies the following equation:

$$\begin{cases} -\Delta v_\lambda + v_\lambda = Q\left(\frac{y}{\lambda} + a_j\right)|v_\lambda|^{p-2}v_\lambda & \text{in } \mathbb{R}^N_+ - \{\lambda a^j\},\\ \frac{\partial v_\lambda}{\partial n} = \lambda^{\frac{-p}{p-2}} f\left(\frac{y}{\lambda} + a^j, \lambda^{\frac{2}{p-2}}v_\lambda\right) & \text{on } \partial\mathbb{R}^N_+ - \{\lambda a^j\}. \end{cases} \tag{1.4}$$

When $\lambda \to \infty$, for $y \in \partial\mathbb{R}^N_+ - \{\lambda a^j\}$, we have $|y| \to \infty$, and (1.4) becomes

$$-\Delta u + u = Q(a^j)|u|^{p-2}u \qquad \text{in } \mathbb{R}^N.$$

Now we are trying to find the solutions of (1.3) in the form $\sum_{j=1}^{l} \alpha_j(\lambda)U(y - \lambda a^j - x^j_\lambda) + w_\lambda(y)$, in which a^j is the non-degenerate critical point of Q, x^j_λ and w_λ are small perturbations. If $v(y) = \sum_{j=1}^{l} \alpha_j(\lambda)U(y - \lambda a^j - x^j_\lambda) + w_\lambda(y)$ is a solution of (1.3), then $u_\lambda(y) = \lambda^{\frac{2}{p-2}} v(\lambda y) = \lambda^{\frac{2}{p-2}}\left[\sum_{j=1}^{l} \alpha_j(\lambda)U(\lambda y - \lambda a^j - x^j_\lambda) + w_\lambda(y)\right]$ is a solution of (1.1).

Throughout this paper, we denote $H^1(\mathbb{R}^N_+)$ by H^1. Define $\|u\|_{H^1(\Omega)} = (\int_\Omega |\nabla u|^2 + u^2 \mathrm{d}x)^{\frac{1}{2}}$. Let $\langle\cdot,\cdot\rangle_\Omega$ denote the usual scalar product in $H^1(\Omega)$. For simplicity, we write $\|\cdot\| = \|\cdot\|_{H^1(\mathbb{R}^N_+)}$ and $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{\mathbb{R}^N_+}$. Set

$$\begin{aligned}
&B_\rho(0) = \{x \in \mathbb{R}^N, |x| \le \rho\}, \quad \text{and fix} \\
&\qquad\qquad\qquad \rho > 0 \text{ small enough so that } \{B_{4\rho}(a^i)\} \text{ are disjoint;} \\
&\Sigma_\rho = \{(\alpha_1, \ldots, \alpha_l, x^1, \ldots, x^l) : \\
&\alpha_j \in (-\rho + Q(a^j)^{-\frac{1}{p-2}}, \rho + Q(a^j)^{-\frac{1}{p-2}}), x^j \in B_\rho(0), j = 1, \ldots, l\}; \\
&M^l_{\lambda,\eta} = \{(\alpha_1, \ldots, \alpha_l, x^1, \ldots, x^l, w) : \\
&(\alpha_1, \ldots, \alpha_l, x^1, \ldots, x^l) \in \Sigma_{4\rho}, w \in E_{\lambda,x^1,\ldots,x^l}, \|w\| < \eta\}; \\
&E_{\lambda,x^1,\ldots,x^l} = \{u \in H^1, \langle u, U_{\lambda a^j + x^j}\rangle = \left\langle u, \frac{\partial U_{\lambda a^j + x^j}}{\partial x^j_i}\right\rangle = 0, \\
&j = 1, \ldots, l;\ i = 1, \ldots, N\}.
\end{aligned}$$

Define the corresponding functional K_λ of problem (1.3) as follows:

$$K_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}\int_{\mathbb{R}^N_+} Q\left(\frac{y}{\lambda}\right)|u|^p \mathrm{d}y - \lambda^{\frac{-p}{p-2}} \int_{\partial\mathbb{R}^N_+} F\left(\frac{y}{\lambda}, \lambda^{\frac{2}{p-2}} u\right)\mathrm{d}\sigma,$$

in which $F(x,t) = \int_0^t f(x,s)\mathrm{d}s$. For $(\alpha_1, \ldots, \alpha_l, x^1, \ldots, x^l, w) \in M^l_{\lambda,\eta}$, we define

$$J_\lambda(\alpha, x, w) = K_\lambda(\Sigma_{j=1}^{l}\alpha_j U_{\lambda a^j + x^j} + w).$$

PROPOSITION 1.1

There exist $\eta_0 > 0$ small enough and $\lambda_0 > 0$ large enough such that for $\lambda \ge \lambda_0$, $\eta \in (0, \eta_0)$, $(\alpha_1, \ldots, \alpha_l, x^1, \ldots, x^l, w) \in M^l_{\lambda,\eta}$ is a critical point of J_λ in $M^l_{\lambda,\eta}$ if and only if $u = \sum_{j=1}^{l} \alpha_j U_{\lambda a^j + x^j} + w$ is a critical point of K_λ in $H^1(\mathbb{R}^N_+)$.

The proof of Proposition 1.1 is quite similar to the proof of Lemma A5 in [5], and so is omitted. Notice that by Lagrange multiplier theorem, $(\alpha, x, w) \in M^l_{\lambda,\eta}$ is a critical point of J_λ if and only if there exist constants $\beta_j,\ \gamma_{mj},\ j = 1, \ldots, l;\ m = 1, \ldots, N$ such that

$$\begin{cases} \frac{\partial J_\lambda}{\partial \alpha_j} = 0, & j = 1, \ldots, l; \\ \frac{\partial J_\lambda}{\partial x_i^j} = \sum_{m=1}^{N} \gamma_{mj} \langle w, \frac{\partial^2 U_{\lambda a^j + x^j}}{\partial x_m^j \partial x_i^j} \rangle, & i = 1, \ldots, N; \quad j = 1, \ldots, l; \\ \langle \frac{\partial J_\lambda}{\partial w}, \varphi \rangle = \sum_{m=1}^{l} \beta_j \langle \varphi, U_{\lambda a^j + x^j} \rangle + \sum_{m=1}^{N} \sum_{j=1}^{l} \gamma_{mj} \langle \varphi, \frac{\partial U_{\lambda a^j + x^j}}{\partial x_m^j} \rangle, \varphi \in H^1 \left(\mathbb{R}_+^N\right). \end{cases} \tag{1.5}$$

Our arguments are essentially similar to those in [5,13] and based on a reduction method and fixed point argument. In section 2, we establish the existence of one peak solutions of (1.3), and multi-peak solutions in section 3.

2. The existence of single peak solution

Theorem 2.1. *Assume condition (Q_1) holds with $a \in \mathbb{R}_+^N$. Then there exists a constant $\lambda_0 > 0$ such that for any $\lambda \geq \lambda_0$, (1.1) has a solution of the form*

$$u_\lambda(y) = \lambda^{\frac{2}{p-2}} \big(\alpha_\lambda U(\lambda y - \lambda a - x_\lambda) + w_\lambda(y)\big),$$

satisfying that as $\lambda \to \infty$, $\alpha_\lambda \to (Q(a))^{-\frac{1}{p-2}}$, $x_\lambda \to 0$, $w_\lambda \to 0$ in H^1, $w_\lambda \in E_{\lambda, x_\lambda}$.

Without loss of generality, we may assume that $Q(a) = 1$. If $u_\lambda(y)$ is a solution of (1.1) in the form $\lambda^{\frac{2}{p-2}} \big(\alpha_\lambda U(\lambda y - \lambda a - x_\lambda) + w_\lambda(y)\big)$, then by easy computation, it is known that $\alpha_\lambda U_{\lambda a + x_\lambda}(y) + w_\lambda(y)$ is a solution of (1.3). Hence, to find the solution of (1.1), it suffices to find a critical point of K_λ in the form $u(y) = \alpha U_{\lambda a + x}(y) + w_\lambda(y)$, $w_\lambda \in E_{\lambda, x}$. By Proposition 1.1, this is equivalent to finding a critical point $(\alpha, x, w) \in M^1_{\lambda, \eta}$ of J_λ. Thus we only need to prove that for small η and large λ, there exist scalars γ, $\beta_j (j = 1, \ldots, N)$, $x_\lambda \in B_\rho(0)$ and $w \in E_{\lambda,\, x_\lambda}$, such that

$$\begin{cases} \dfrac{\partial J_\lambda}{\partial \alpha} = 0, & (3_\alpha) \\ \dfrac{\partial J_\lambda}{\partial x_i} = \displaystyle\sum_{j=1}^{N} \beta_j \left\langle w, \frac{\partial^2 U_{\lambda a + x}}{\partial x_j \partial x_i} \right\rangle, \quad i = 1, \ldots, N, & (3_\mathrm{i}) \\ \left\langle \dfrac{\partial J_\lambda}{\partial w}, \varphi \right\rangle = \gamma \, \langle U_{\lambda a + x}, \varphi \rangle + \displaystyle\sum_{j=1}^{N} \beta_j \left\langle \frac{\partial U_{\lambda a + x}}{\partial x_j}, \varphi \right\rangle, \quad \varphi \in H^1. & (3_\mathrm{w}) \end{cases}$$

Now we will solve the equations in (3_α), (3_i), (3_w). The proof of Theorem 2.1 will be divided into several propositions in the following.

PROPOSITION 2.1

Assume that condition (Q_1) holds with $x_0 = a \in \mathbb{R}_+^N$. Then there exist $\lambda_0 > 0$, $\rho > 0$ small and a continuous map $\mathfrak{A}$ such that for any (λ, x), $\lambda \geq \lambda_0$, $|x| < \rho$, $\mathfrak{A}(\lambda, x) = w_{\lambda,x} \in E_{\lambda,x}$ in which $w_{\lambda,x}$ satisfies (3_w) for some $(\gamma, \beta_1, \ldots, \beta_N) \in \mathbb{R}^{N+1}$. Moreover, there is a constant $C > 0$, such that

$$\|w_{\lambda,x}\| \leq \frac{C}{\lambda^2}. \tag{2.1}$$

Proof. We follow the argument due to Cao [3]. Expand J_λ in a neighbourhood of $w = 0$ in $E_{\lambda,x}$, and by the definition of J_λ, we have

$$\begin{aligned}
&J_\lambda(\alpha, x, w) = J_\lambda(\alpha, x, 0) + L_\lambda(w) + F_\lambda(w) + R(w)\\
&\quad L_\lambda(w) = -\int_{\mathbb{R}^N_+} Q\left(\frac{y}{\lambda}\right)(\alpha U_{\lambda a+x})^{p-1} w\mathrm{d}y - \lambda^{\frac{-p}{p-2}}\\
&\quad \int_{\partial\mathbb{R}^N_+} f\left(\frac{y}{\lambda}, \lambda^{\frac{2}{p-2}}\alpha U_{\lambda a+x}\right) w\mathrm{d}\sigma,\\
&F_\lambda(w) = \frac{1}{2}\|w\|^2 - \frac{p-1}{2}\int_{\mathbb{R}^N_+} Q\left(\frac{y}{\lambda}\right)(\alpha U_{\lambda a+x})^{p-2} w^2\mathrm{d}y - \lambda^{\frac{-p}{p-2}}\\
&\quad \int_{\partial\mathbb{R}^N_+} f'\left(\frac{y}{\lambda}, \lambda^{\frac{2}{p-2}}\alpha U_{\lambda a+x}\right) w^2\mathrm{d}\sigma,\\
&\quad R(w) = O\big(\|w\|^{\bar{p}}\big), \quad \bar{p} = \min\{p, 3\}.
\end{aligned}$$

Claim 1.

$$\|L_\lambda\| = O\left(\frac{1}{\lambda^2}\right). \tag{2.2}$$

In fact, by the property of U, we have

$$\begin{aligned}
\int_{\mathbb{R}^N_+} (U_{\lambda a+x})^{p-1} w\mathrm{d}y &= \langle w, U_{\lambda a+x}\rangle - \int_{\partial\mathbb{R}^N_+} \frac{\partial U_{\lambda a+x}}{\partial n} w\mathrm{d}\sigma\\
&= \int_{\partial\mathbb{R}^N_+} \frac{\partial U_{\lambda a+x}}{\partial y_N} w\mathrm{d}\sigma = O(\mathrm{e}^{-\lambda\tau})\|w\|.
\end{aligned} \tag{2.3}$$

for some suitable $\tau > 0$ small enough and it may change from line to line. If λ is large enough, then it follows from assumptions on Q, (2.3) and with mean value theorem that,

$$\begin{aligned}
&\int_{\mathbb{R}^N_+} Q\left(\frac{y}{\lambda}\right)(\alpha U_{\lambda a+x})^{p-1} w\mathrm{d}y\\
&\quad = \int_{z_N > -\lambda a_N - x_N} Q\left(\frac{z+x}{\lambda} + a\right)(\alpha U(z))^{p-1} w(z + \lambda a + x)\mathrm{d}z\\
&\quad \leq \int_{z_N > -\lambda a_N - x_N} \left(Q\left(\frac{z+x}{\lambda} + a\right) - Q(a)\right)(\alpha U(z))^{p-1} w(z + \lambda a + x)\mathrm{d}z\\
&\qquad + O(\mathrm{e}^{-\lambda\tau})\|w\|\\
&\quad \leq \frac{C}{\lambda^2}\int_{z_N > -\lambda a_N - x_N} |z+x|^2 (\alpha U(z))^{p-1} w(z + \lambda a + x)\mathrm{d}z + O(\mathrm{e}^{-\lambda\tau})\|w\|\\
&\quad \leq \frac{C}{\lambda^2}\|w\|.
\end{aligned} \tag{2.4}$$

By the assumption (f2) and (f3), we have

$$\forall \varepsilon > 0,\ \exists A_\varepsilon > 0,\ \text{s.t. } |f(x,t)| \leq \varepsilon |t| + A_\varepsilon |t|^q.$$

Thus

$$\begin{aligned}
&\lambda^{\frac{-p}{p-2}} \int_{\partial\mathbb{R}^N_+} f\left(\frac{y}{\lambda}, \lambda^{\frac{2}{p-2}} \alpha U_{\lambda a+x}\right) w \mathrm{d}\sigma \\
&\leq \lambda^{\frac{-p}{p-2}} \int_{\partial\mathbb{R}^N_+} \varepsilon \lambda^{\frac{2}{p-2}} \alpha U_{\lambda a+x} |w| \mathrm{d}\sigma + \lambda^{\frac{-p}{p-2}} \int_{\partial\mathbb{R}^N_+} A_\varepsilon \left(\lambda^{\frac{2}{p-2}} \alpha U_{\lambda a+x}\right)^q |w| \mathrm{d}\sigma \\
&\leq \frac{C}{\lambda} \mathrm{e}^{-\tau\lambda} \|w\| + C\lambda^{\frac{2q-p}{p-2}} \mathrm{e}^{-\tau\lambda} \|w\|. \qquad (2.5)
\end{aligned}$$

Hence we have, for λ large enough,

$$\lambda^{\frac{-p}{p-2}} \int_{\partial\mathbb{R}^N_+} f\left(\frac{y}{\lambda}, \lambda^{\frac{2}{p-2}} \alpha U_{\lambda a+x}\right) w \mathrm{d}\sigma \leq \frac{C}{\lambda^2} \|w\|$$

and (2.2) follows.

Claim 2. For some $\delta > 0$, we have

$$\|w\|^2 - (p-1) \int_{\mathbb{R}^N_+} (U_{\lambda a+x})^{p-2}\, w^2 \mathrm{d}y \geq \delta \|w\|^2,\ \forall w \in E_{\lambda,x},$$

(see Appendix). As in (2.4), we have

$$\int_{\mathbb{R}^N_+} Q\left(\frac{y}{\lambda}\right) (\alpha U_{\lambda a+x})^{p-2}\, w^2 \mathrm{d}y = \int_{\mathbb{R}^N_+} (\alpha U_{\lambda a+x})^{p-2}\, w^2 \mathrm{d}y + O\left(\frac{1}{\lambda^2}\right) \|w\|^2.$$

For λ large enough, and due to (f2),

$$\begin{aligned}
\lambda^{\frac{-p}{p-2}} \int_{\partial\mathbb{R}^N_+} f'\left(\frac{y}{\lambda}, \lambda^{\frac{2}{p-2}} \alpha U_{\lambda a+x}\right) w^2 \mathrm{d}\sigma &\leq C\lambda^{\frac{-p}{p-2}} \int_{\partial\mathbb{R}^N_+} \left(\lambda^{\frac{2}{p-2}} \alpha U_{\lambda a+x}\right)^\gamma w^2 \mathrm{d}\sigma \\
&\leq C\lambda^{\frac{2\gamma-p}{p-2}} \mathrm{e}^{-\lambda\tau\gamma} \|w\|^2.
\end{aligned}$$

By $(\alpha, x, w) \in M^1_{\lambda,\eta}$, we have $|\alpha - 1| \leq \rho$. Hence for $\rho > 0$ small, there hold

$$\begin{aligned}
F_\lambda(w) &\geq \frac{\delta'}{2} \|w\|^2 - \lambda^{\frac{-p}{p-2}} \int_{\partial\mathbb{R}^N_+} f'\left(\frac{y}{\lambda}, \lambda^{\frac{2}{p-2}} \alpha U_{\lambda a+x}\right) \\
w^2 \mathrm{d}\sigma &\geq \frac{\delta'}{2} \|w\|^2 - C\lambda^{\frac{2\gamma-p}{p-2}} \mathrm{e}^{-\lambda\tau\gamma} \|w\|^2.
\end{aligned}$$

For λ large enough, we get $F_\lambda(w) \geq \rho' \|w\|^2$. Then by Riesz representation theorem, we can denote L_λ and F_λ by $L_\lambda(w) = \langle L, w\rangle$, $L \in E_{\lambda,x}$, $\forall w \in E_{\lambda,x}$ and $F_\lambda(w) = \langle Aw, w\rangle$, $\forall w \in E_{\lambda,x}$, respectively. Here A is a continuous and coercive symmetric bilinear form on $E_{\lambda,x}$. Therefore,

$$\frac{\partial J_\lambda}{\partial w}\Big|_{E_{\lambda,x}} = L + 2Aw + R'(w),$$

and (3_w) is equivalent to $L + 2Aw + R'(w) = 0$.

Set $F_{\lambda,x}(L, w) = L + 2Aw + R'(w) : E_{\lambda,x} \times E_{\lambda,x} \to E_{\lambda,x}$, then we have

$$F(0,0) = 0, \quad \frac{\partial F}{\partial w}(L, w) = 2A + R''(w), \quad R''(w) = O(\|w\|^{p-2}).$$

If $\|w\|$ is small. Then $\frac{\partial F}{\partial w}$ is invertible, and by implicit theorem, there exists $(L, w_{\lambda,x}(L))$ such that $F(L, w_{\lambda,x}(L)) = 0$. That is, $\langle L, w\rangle + 2\langle Aw, w\rangle + \langle R'(w), w\rangle = 0$ and $\|w\| \leq C\|L\|$.

Now (2.1) follows from (2.2) and this completes the proof. □

PROPOSITION 2.2

Let ρ, λ and $w_{\lambda,x}$ be such that Proposition 2.1 holds, we have the equivalence that (3_α) holds, if and only if $\alpha = 1 + W(\lambda, x, \alpha)$, in which W is a continuous function of λ, x, α and $W(\lambda, x, \alpha) = O(\frac{1}{\lambda^2})$. Moreover, there exists an α such that (3_α) holds for $|\alpha - 1| \leq \rho$.

Proof.

$$\begin{aligned}\frac{\partial J_\lambda}{\partial \alpha} &= \alpha\|U_{\lambda a+x}\|^2 + \langle w, U_{\lambda a+x}\rangle \\ &\quad - \int_{\mathbb{R}^N_+} Q\left(\frac{y}{\lambda}\right)|\alpha U_{\lambda a+x} + w|^{p-2}(\alpha U_{\lambda a+x} + w)U_{\lambda a+x}\mathrm{d}y \\ &\quad - \lambda^{-1}\int_{\partial\mathbb{R}^N_+} f\left(\frac{y}{\lambda}, \lambda^{\frac{2}{p-2}}\alpha U_{\lambda a+x} + w\right)U_{\lambda a+x}\mathrm{d}\sigma,\end{aligned}$$

$$\begin{aligned}&\int_{\mathbb{R}^N_+} Q\left(\frac{y}{\lambda}\right)|\alpha U_{\lambda a+x} + w|^{p-2}(\alpha U_{\lambda a+x} + w)U_{\lambda a+x}\mathrm{d}y \\ &\quad = \int_{\mathbb{R}^N_+} Q(\frac{y}{\lambda})(\alpha^{p-1}U^p_{\lambda a+x} + (p-1)\alpha^{p-2}U^{p-1}_{\lambda a+x}w + U_{\lambda a+x}O(|w|^{\overline{p}-1}))\mathrm{d}y \\ &\quad = I_1 + I_2 + I_3.\end{aligned}$$

As in (2.3) (2.4) and (2.5), we have

$$\begin{aligned}&\int_{\mathbb{R}^N_+} Q\left(\frac{y}{\lambda}\right)U^p_{\lambda a+x}\mathrm{d}y = \int_{\mathbb{R}^N_+} U^p_{\lambda a+x}\mathrm{d}y + O\left(\frac{1}{\lambda^2}\right), \\ &\quad \int_{\mathbb{R}^N_+} Q\left(\frac{y}{\lambda}\right)U^{p-1}_{\lambda a+x}w\mathrm{d}y \leq \frac{C}{\lambda^2}\|w\|, \\ &\quad \int_{\mathbb{R}^N_+} Q\left(\frac{y}{\lambda}\right)U_{\lambda a+x}O(|w|^{\overline{p}-1})\mathrm{d}y = O(|w|^{\overline{p}-1}), \\ &\quad \lambda^{-1}\int_{\partial\mathbb{R}^N_+} f\left(\frac{y}{\lambda}, \lambda^{\frac{2}{p-2}}\alpha U_{\lambda a+x} + w\right)U_{\lambda a+x}\mathrm{d}\sigma \leq \lambda^{-1}\int_{\partial\mathbb{R}^N_+}\end{aligned}$$

$$\left[\varepsilon\lambda^{\frac{2}{p-2}}(\alpha U_{\lambda a+x}+w)+A_{\varepsilon}\left(\lambda^{\frac{2}{p-2}}(\alpha U_{\lambda a+x}+w)\right)^{q}\right]U_{\lambda a+x}\mathrm{d}\sigma$$
$$=O(\mathrm{e}^{-\lambda\tau}).$$

Let $A_\lambda = \|U_{\lambda a+x}\|^2$, $B_\lambda = \int_{\mathbb{R}^N_+} U^p_{\lambda a+x}\mathrm{d}y$, $A = \|U\|_{H^1(\mathbb{R}^N)}$. Then we have as $\lambda \to \infty$, $A_\lambda \to A^2$, $B_\lambda \to A^2$ and $A_\lambda = B_\lambda + O(\mathrm{e}^{-\tau\lambda})$, for some τ small.

Then we get $\frac{\partial J_\lambda}{\partial\alpha} = \alpha A_\lambda - \alpha^{p-1}B_\lambda + O(\frac{1}{\lambda^2})$. Therefore, $\frac{\partial J_\lambda}{\partial\alpha} = 0$ is equivalent to $\alpha = 1 + O(\frac{1}{\lambda^2})$. Hence it follows from the Brouwer fixed point theorem that there exists an α such that (3_α) holds. □

PROPOSITION 2.3

Let ρ, λ and $w_{\lambda,x}$ be such that Proposition 2.1 holds and α be such that Proposition 2.2 holds $\forall\ k = 1, \dots, N$, we have

$$\frac{\partial J_\lambda}{\partial x_k} = \frac{-\alpha^p B}{2\lambda^2}\sum_{m=1}^{N} D_{k,m}Q(a)x_m + o\left(\frac{1}{\lambda^2}\right) \text{ and } (3_k) \Leftrightarrow x = W(x,\lambda),$$

where B is some positive constant and $W : \overline{B_\rho} \to \mathbb{R}^N$ is a continuous mapping satisfying $W(x,\lambda) \to 0$, as $\lambda \to \infty$ uniformly in $\overline{B_\rho}$.

Proof. By the definition of J_λ and direct calculations, we have

$$\begin{aligned}
\frac{\partial J_\lambda}{\partial x_k} &= \alpha^2\left\langle \frac{\partial U_{\lambda a+x}}{\partial x_k}, U_{\lambda a+x}\right\rangle \\
&\quad - \alpha\int_{\mathbb{R}^N_+} Q\left(\frac{y}{\lambda}\right)|\alpha U_{\lambda a+x}+w|^{p-2}(\alpha U_{\lambda a+x}+w)\frac{\partial U_{\lambda a+x}}{\partial x_k}\mathrm{d}y \\
&\quad - \alpha\lambda^{-1}\int_{\partial\mathbb{R}^N_+} f\left(\frac{y}{\lambda}, \lambda^{\frac{2}{p-2}}\alpha U_{\lambda a+x}+w\right)\frac{\partial U_{\lambda a+x}}{\partial x_k}\mathrm{d}\sigma \\
&= \alpha^2\left\langle \frac{\partial U_{\lambda a+x}}{\partial x_k}, U_{\lambda a+x}\right\rangle - \alpha^p\int_{\mathbb{R}^N_+} Q\left(\frac{y}{\lambda}\right)U^{p-1}_{\lambda a+x}\frac{\partial U_{\lambda a+x}}{\partial x_k}\mathrm{d}y \\
&\quad - \alpha^{p-1}\int_{\mathbb{R}^N_+} Q\left(\frac{y}{\lambda}\right)U^{p-2}_{\lambda a+x}\frac{\partial U_{\lambda a+x}}{\partial x_k}w\mathrm{d}y - \alpha\lambda^{-1} \\
&\quad \int_{\partial\mathbb{R}^N_+} f\left(\frac{y}{\lambda}, \lambda^{\frac{2}{p-2}}\alpha U_{\lambda a+x}+w\right)\frac{\partial U_{\lambda a+x}}{\partial x_k}\mathrm{d}\sigma + O(\|w\|^{\overline{p}-1}).
\end{aligned}$$

Claim 1.

$$\left\langle \frac{\partial U_{\lambda a+x}}{\partial x_k}, U_{\lambda a+x}\right\rangle = O(\mathrm{e}^{-\tau\lambda}).$$

It is known that $\left\langle \frac{\partial U_{\lambda a+x}}{\partial x_k}, U_{\lambda a+x} \right\rangle_{\mathbb{R}^N} = 0$, so we may derive,

$$\begin{aligned}
\left\langle \frac{\partial U_{\lambda a+x}}{\partial x_k}, U_{\lambda a+x} \right\rangle &= -\int_{z_N<-\lambda a_N - x_N} \nabla \frac{\partial U}{\partial x_k} \nabla U + \frac{\partial U}{\partial x_k} U \\
&= O(\mathrm{e}^{-\tau\lambda}) \int_{\mathbb{R}^N_+} Q\left(\frac{y}{\lambda}\right) U^{p-2}_{\lambda a+x} \frac{\partial U_{\lambda a+x}}{\partial x_k} w \mathrm{d}y \\
&= \int_{\mathbb{R}^N_+} \left(Q\left(\frac{y}{\lambda}\right) - 1\right) U^{p-2}_{\lambda a+x} \frac{\partial U_{\lambda a+x}}{\partial x_k} w \mathrm{d}y \\
&\quad + \int_{\mathbb{R}^N_+} U^{p-2}_{\lambda a+x} \frac{\partial U_{\lambda a+x}}{\partial x_k} w \mathrm{d}y \\
&= O\left(\frac{1}{\lambda^2}\right) \|w\| + \int_{\mathbb{R}^N_+} U^{p-2}_{\lambda a+x} \frac{\partial U_{\lambda a+x}}{\partial x_k} w \mathrm{d}y.
\end{aligned}$$

For $y \in \partial\mathbb{R}^N_+$, $|y-\lambda a - x| \geq \lambda a_N + x_N$, so $|\frac{\partial^2 U_{\lambda a+x}}{\partial x_k^2}| \leq C\,\mathrm{e}^{-\lambda a_N - x_N}$, by Hölder inequality and Sobolev embedding theorem,

$$\begin{aligned}
(p-1) \int_{\mathbb{R}^N_+} U^{p-2}_{\lambda a+x} \frac{\partial U_{\lambda a+x}}{\partial x_k} w \mathrm{d}y &= \left\langle \frac{\partial U_{\lambda a+x}}{\partial x_k}, w \right\rangle - \int_{\partial\mathbb{R}^N_+} \frac{\partial^2 U_{\lambda a+x}}{\partial x_k^2} w \mathrm{d}\sigma \\
&= -\int_{\partial\mathbb{R}^N_+} \frac{\partial^2 U_{\lambda a+x}}{\partial x_k^2} w \mathrm{d}\sigma = O(\mathrm{e}^{-\tau\lambda})\|w\|.
\end{aligned}$$

As before, we have

$$\begin{aligned}
&\int_{\partial\mathbb{R}^N_+} f\left(\frac{y}{\lambda}, \lambda^{\frac{2}{p-2}} \alpha U_{\lambda a+x} + w\right) \frac{\partial U_{\lambda a+x}}{\partial x_k} \mathrm{d}\sigma \\
&\quad \leq \int_{\partial\mathbb{R}^N_+} [\varepsilon \lambda^{\frac{2}{p-2}} (\alpha U_{\lambda a+x} + w) + A_\varepsilon (\lambda^{\frac{2}{p-2}} (\alpha U_{\lambda a+x} + w))^q] \frac{\partial U_{\lambda a+x}}{\partial x_k} \mathrm{d}\sigma \\
&\quad = O(\mathrm{e}^{-\tau\lambda}), \\
&\frac{\partial J_\lambda}{\partial x_k} = -\alpha^p \int_{\mathbb{R}^N_+} Q\left(\frac{y}{\lambda}\right) U^{p-1}_{\lambda a+x} \frac{\partial U_{\lambda a+x}}{\partial x_k} \mathrm{d}x + O\left(\frac{1}{\lambda^2}\right) \|w\| \\
&\qquad + O(\|w\|^{\overline{p}-1}) + O(\mathrm{e}^{-\tau\lambda}) \\
&\quad = -\alpha^p \int_{\mathbb{R}^N_+} \left(Q\left(\frac{y}{\lambda}\right) - 1\right) U^{p-1}_{\lambda a+x} \frac{\partial U_{\lambda a+x}}{\partial x_k} \mathrm{d}x \\
&\qquad - \alpha^p \int_{\mathbb{R}^N_+} U^{p-1}_{\lambda a+x} \frac{\partial U_{\lambda a+x}}{\partial x_k} \mathrm{d}x + o\left(\frac{1}{\lambda^2}\right) \\
&\quad = \frac{-\alpha^p}{\lambda^2} \sum_{j,m=1}^{N} D_{j,m} Q(a) \int_{y_N > -\lambda a_N - x_N} (y_j + \lambda a_j + x_j)(y_m + \lambda a_m + x_m) \\
&\qquad U^{p-1} \frac{\partial U}{\partial y_k} \mathrm{d}y + o(\frac{1}{\lambda^2})
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-\alpha^p}{\lambda^2} \sum_{j,m=1}^{N} D_{j,m} Q(a) \int_{\mathbb{R}^{\mathbb{N}}} (y_j + \lambda a_j + x_j)(y_m + \lambda a_m + x_m) U^{p-1} \frac{\partial U}{\partial y_k} \mathrm{d}y \\
&\quad + \frac{\alpha^p}{\lambda^2} \sum_{j,m=1}^{N} D_{j,m} Q(a) \int_{y_N \le -\lambda a_N - x_N} (y_j + \lambda a_j + x_j)(y_m + \lambda a_m + x_m) \\
&\qquad U^{p-1} \frac{\partial U}{\partial y_k} \mathrm{d}y + o\left(\frac{1}{\lambda^2}\right) \\
&:= I_1 + I_2 + o\left(\frac{1}{\lambda^2}\right).
\end{aligned}
$$

By the same calculation as in [5], we have $I_1 = \frac{-\alpha^p B}{2\lambda^2} \sum_{m=1}^{N} D_{k,m} Q(a) x_m$, in which $B = -\int_{\mathbb{R}^{\mathbb{N}}} y_i \frac{\partial U}{\partial y_i} \mathrm{d}y > 0$. As for I_2, by the exponential decay of U and $\frac{\partial U}{\partial y_i}$, we have $I_2 = o(\frac{1}{\lambda^2})$. Hence, we have

$$
\frac{\partial J_\lambda}{\partial x_k} = \frac{-\alpha^p B}{2\lambda^2} \sum_{m=1}^{N} D_{k,m} Q(a) x_m + o\left(\frac{1}{\lambda^2}\right).
$$

From the equation above, equation (3_{k}) is equivalent to

$$
\frac{-\alpha^p B}{2\lambda^2} \sum_{m=1}^{N} D_{k,m} Q(a) x_m = \sum_{j=1}^{N} \beta_j \left\langle w, \frac{\partial^2 U_{\lambda a + x}}{\partial x_j \partial x_k} \right\rangle + o(\frac{1}{\lambda^2}),\ k = 1, \ldots, N.
$$

Due to Proposition 2.1, we have proved that $\left\langle w, \frac{\partial^2 U_{\lambda a+x}}{\partial x_j \partial x_k} \right\rangle = O(\frac{1}{\lambda^2})$. We claim that $\beta_j = O(\frac{1}{\lambda^2})$. Setting $\varphi = \frac{\partial U_{\lambda a+x}}{\partial x_k}$ in (3_{w}), we obtain

$$
\left\langle \frac{\partial J_\lambda}{\partial w}, \frac{\partial U_{\lambda a+x}}{\partial x_k} \right\rangle = \gamma \left\langle U_{\lambda a+x}, \frac{\partial U_{\lambda a+x}}{\partial x_k} \right\rangle + \sum_{j=1}^{N} \beta_j \left\langle \frac{\partial U_{\lambda a+x}}{\partial x_j}, \frac{\partial U_{\lambda a+x}}{\partial x_k} \right\rangle.
$$

By the above estimates, it follows that

$$
\left\langle U_{\lambda a+x}, \frac{\partial U_{\lambda a+x}}{\partial x_k} \right\rangle = O(\mathrm{e}^{-\tau_1 \lambda})
$$

and

$$
\left\langle \frac{\partial U_{\lambda a+x}}{\partial x_j}, \frac{\partial U_{\lambda a+x}}{\partial x_k} \right\rangle = O(\mathrm{e}^{-\tau_2 \lambda}),\ \text{for } k \neq j
$$

Therefore,

$$
\beta_k = \left(\left\langle \frac{\partial J_\lambda}{\partial w}, \frac{\partial U_{\lambda a+x}}{\partial x_k} \right\rangle + O(\mathrm{e}^{-\tau_1 \lambda}) \right) \Big/ \left(\| \frac{\partial U_{\lambda a+x}}{\partial x_k} \|^2 + O(\mathrm{e}^{-\tau_2 \lambda}) \right).
$$

Notice that $\|\frac{\partial U_{\lambda a+x}}{\partial x_k}\| \le C_0$, and by direct calculation we have

$$\left|\left\langle \frac{\partial J_\lambda}{\partial w}, \frac{\partial U_{\lambda a+x}}{\partial x_k}\right\rangle\right| = \left|\frac{\partial J_\lambda}{\partial x_k}\right| = O\left(\frac{1}{\lambda^2}\right).$$

Then (3_k) becomes

$$\frac{-B\alpha^p}{2\lambda^2}\sum_{m=1}^{N} D_{k,m}Q(a)x_m = o\left(\frac{1}{\lambda^2}\right),$$

and since $\det(D^2Q(a)) \neq 0$, for $x \in B_\rho(0)$, we have

$$x = W(x,\lambda),$$

in which $W(x,\lambda) = o(1)$, as $\lambda \to \infty$. This completes the proof of Proposition 2.3. □

By Proposition 2.3, we deduce that for large λ, $W(x,\lambda)$ maps $\bar{B}_\rho$ into itself. Thus the Brouwer fixed point theorem guarantees the existence of a fixed point in $\bar{B}_\rho$. We have

$$u_\lambda = \alpha_\lambda U_{\lambda a+x} + w_\lambda$$

is a solution of (1.3). By Proposition 2.1 and 2.2, $\alpha_\lambda \to 1$, $w_\lambda \to 0$ as $\lambda \to \infty$. Finally, we prove that u_λ is positive. Set $u_\lambda^- = \max\{0, -u_\lambda\}$. Then

$$S\left(\int_{\mathbb{R}^N_+}(u_\lambda^-)^p \mathrm{d}y\right)^{\frac{2}{p}} \le \|u_\lambda^-\|^2 = \int_{\mathbb{R}^N_+} Q\left(\frac{y}{\lambda}\right)(u_\lambda^-)^p \mathrm{d}y - \int_{\partial\mathbb{R}^N_+}\frac{\partial u_\lambda^-}{\partial x_n}u_\lambda^- \mathrm{d}\sigma,$$

and because of $u_\lambda^- \le \alpha_\lambda |w_\lambda|$,

$$\left|\int_{\partial\mathbb{R}^N_+}\frac{\partial u_\lambda^-}{\partial x_n}u_\lambda^- \mathrm{d}\sigma\right| \le \lambda^{-\frac{2}{p-2}}\int_{\partial\mathbb{R}^N_+}\left|f\left(\frac{y}{\lambda}, u_\lambda^-\right)\right| \cdot |u_\lambda^-|\mathrm{d}\sigma \le \lambda^{-\frac{2}{p-2}}C\|w_\lambda\| \to 0,$$

$$\int_{\mathbb{R}^N_+} Q\left(\frac{y}{\lambda}\right)(u_\lambda^-)^p\,\mathrm{d}y \le C\int_{\mathbb{R}^N_+}(u_\lambda^-)^p\,\mathrm{d}y.$$

From this, and if $u_\lambda^- \neq 0$, we deduce that

$$\int_{\mathbb{R}^N_+}(u_\lambda^-)^p\,\mathrm{d}y \ge C > 0,$$

but $\int_{\mathbb{R}^N_+}(u_\lambda^-)^p \mathrm{d}y \le \alpha_\lambda^p \int_{\mathbb{R}^N_+} w_\lambda^p \mathrm{d}y \to 0$. Hence, $u_\lambda^- = 0$.

3. Existence of multi-peak solutions

In this section, we will assume Q has two non-degenerate critical points and we construct solutions with two peaks for (1.3). We still follow the method in section 2, so we will be sketchy.

Theorem 3.1. *Assume that condition* (Q_2) *holds with* a^1, a^2. *Then there exists a constant* $\lambda_0 > 0$, *such that for each* $\lambda \in (\lambda_0, \infty)$, (1.3) *has a solution of the form*

$$u_\lambda(y) = \alpha_1 U(y - \lambda a^1 - x^1(\lambda)) + \alpha_2 U(y - \lambda a^2 - x^2(\lambda)) + w_\lambda,$$

and satisfying that for $\lambda \to \infty$,

$$\alpha_i \to (Q(a^i))^{-\frac{1}{p-2}}, \qquad x^i(\lambda) \to 0, \quad i = 1, 2;$$

$$\|w_\lambda\| \to 0, \quad w_\lambda \in E_{\lambda, x^1(\lambda), x^2(\lambda)}$$

in which

$$E_{\lambda,x^1,x^2} = \left\{ v \in H^1(\mathbb{R}^N_+) : \langle v, U_{\lambda a^j + x^j} \rangle = 0, \left\langle v, \frac{\partial U_{\lambda a^j + x^j}}{\partial x_i} \right\rangle = 0, \right.$$
$$\left. j = 1, 2;\ i = 1, \ldots, N \right\}.$$

Let $M^2_{\lambda,\eta},\ K_\lambda,\ J_\lambda$ be defined as in section 1. By Proposition 1.1, Theorem 3.1 will follow if we establish the existence of a critical point $(\alpha_1, \alpha_2, x^1, x^2, w) \in M^2_{\lambda,\eta}$ for J_λ. Thus we only need to get that, for small η and large λ, there exist scalars $\gamma_1, \gamma_2, \beta^1_j, \beta^2_j (j = 1, 2, \ldots, N), x_1, x_2 \in \bar{B}_\rho,\ w \in E_{\lambda, x_1, x_2}$ such that

$$\begin{cases} \dfrac{\partial J_\lambda}{\partial \alpha_i} = 0, \quad i = 1, 2, & (4_\alpha) \\ \dfrac{\partial J_\lambda}{\partial x^1_i} = \displaystyle\sum_{j=1}^N \beta^1_j \left\langle w, \frac{\partial^2 U_{\lambda^1 + x^1}}{\partial x_j \partial x_i} \right\rangle, \quad i = 1, \ldots, N, & (4_{1i}) \\ \dfrac{\partial J_\lambda}{\partial x^2_i} = \displaystyle\sum_{j=1}^N \beta^2_j \left\langle w, \frac{\partial^2 U_{\lambda a^2 + x^2}}{\partial x_j \partial x_i} \right\rangle, \quad i = 1, \ldots, N, & (4_{2i}) \\ \left\langle \dfrac{\partial J_\lambda}{\partial w}, \varphi \right\rangle = \gamma_1 \langle U_{\lambda a^1 + x^1}, \varphi \rangle & \\ \quad + \displaystyle\sum_{j=1}^N \beta^1_j \left\langle \frac{\partial U_{\lambda a^1 + x^1}}{\partial x^1_j}, \varphi \right\rangle + \gamma_2 \langle U_{\lambda a^2 + x^2}, \varphi \rangle & \\ \quad + \displaystyle\sum_{j=1}^N \beta^2_j \left\langle \frac{\partial U_{\lambda a^2 + x^2}}{\partial x^2_j}, \varphi \right\rangle, \quad \varphi \in H^1 & (4_w) \end{cases}$$

The proof of Theorem 3.1 is included in the following propositions.

PROPOSITION 3.1

Assume (Q_2) *holds, for* $\rho > 0$ *small enough. Then there exist a constant* $\lambda_0 > 0$ *and a* C^1 *map such that for any* $(\lambda, x^1, x^2) \in [\lambda_0, \infty) \times \bar{B}_\rho \times \bar{B}_\rho$, *associates* w_{λ,x^1,x^2},

$$(\lambda, x^1, x^2) \to w_{\lambda,x^1,x^2} : [\lambda_0, \infty) \times \bar{B}_\rho \times \bar{B}_\rho \to E_{\lambda,x^1,x^2}$$

such that (4_w) *is satisfied for some* $(\gamma_1, \gamma_2, \beta_1^1, \beta_2^1, \ldots, \beta_N^1, \beta_1^2, \ldots, \beta_N^2) \in \mathbb{R}^{2N+2}$. *And there exists* $C > 0$, *such that*

$$\|w_{\lambda,x^1,x^2}\| \le \frac{C}{\lambda^2}.$$

Proof. Like in Proposition 2.1, we start by Taylor expansion at $w = 0$. Write

$$J_\lambda(\alpha_1, \alpha_2, x^1, x^2, w) = J_\lambda(\alpha_1, \alpha_2, x^1, x^2, 0) + L_\lambda(w) + F_\lambda(w) + R(w).$$

$$L_\lambda(w) = \left\langle \sum_{i=1}^2 \alpha_i U_{\lambda a^i + x^i}, w \right\rangle - \int_{\mathbb{R}_+^N} Q\left(\frac{y}{\lambda}\right) \left(\sum_{i=1}^2 \alpha_i U_{\lambda a^i + x^i}\right)^{p-1} w \mathrm{d}y$$
$$- \lambda^{\frac{-p}{p-2}} \int_{\partial\mathbb{R}_+^N} f\left(\frac{y}{\lambda}, \lambda^{\frac{2}{p-2}} (\sum_{i=1}^2 \alpha_i U_{\lambda a^i + x^i})\right) w \mathrm{d}\sigma,$$

$$F_\lambda(w) = \frac{1}{2}\|w\|^2 - \frac{(p-1)}{2} \int_{\mathbb{R}_+^N} Q\left(\frac{y}{\lambda}\right) \left(\sum_{i=1}^2 \alpha_i U_{\lambda a^i + x^i}\right)^{p-2} w^2 \mathrm{d}y$$
$$- \lambda^{\frac{-p}{p-2}} \int_{\partial\mathbb{R}_+^N} f'\left(\frac{y}{\lambda}, \lambda^{\frac{2}{p-2}} \left(\sum_{i=1}^2 \alpha_i U_{\lambda a^i + x^i}\right)\right) w^2 \mathrm{d}\sigma$$

$$R(w) = O(\|w\|^{\overline{p}})$$

Claim 1. $\|L_\lambda\| = O\left(\frac{1}{\lambda^2}\right)$. Because of $(a+b)^p \le C_p(a^p + b^p)$ and U decays as $\mathrm{e}^{-|x|}$,

$$\int_{\mathbb{R}_+^N} Q\left(\frac{y}{\lambda}\right) \left(\sum_{i=1}^2 \alpha_i U_{\lambda a^i + x^i}\right)^{p-1} w \mathrm{d}y$$
$$\le C_p \int_{\mathbb{R}_+^N} Q\left(\frac{y}{\lambda}\right) [(\alpha_1 U_{\lambda a^1 + x^1})^{p-1} w + (\alpha_2 U_{\lambda a^2 + x^2})^{p-1} w] \mathrm{d}y.$$

By (2.4), we have

$$\left| \int_{\mathbb{R}_+^N} Q(y/\lambda)(\alpha_i U_{\lambda a^i + x^i})^{p-1} w \mathrm{d}y \right| \le \frac{C}{\lambda^2}\|w\|, \quad i = 1, 2.$$

By the same calculation as in (2.5),

$$\lambda^{\frac{-p}{p-2}} \int_{\partial\mathbb{R}_+^N} f\left(\frac{y}{\lambda}, \lambda^{\frac{2}{p-2}} \left(\sum_{i=1}^2 \alpha_i U_{\lambda a^i + x^i}\right)\right) w \mathrm{d}\sigma = O(\mathrm{e}^{-\tau\lambda})\|w\|.$$

Then we have Claim 1.

Claim 2. There exists ρ' such that $F_\lambda(w) \geq \rho'\|w\|^2$, for $w \in E_{\lambda,x^1,x^2}$ and λ is large enough. We first show

$$\begin{aligned}
&\int_{\mathbb{R}^N_+} \left(\sum_{i=1}^{2} \alpha_i U_{\lambda a^i + x^i}\right)^{p-2} w^2 \mathrm{d}y \\
&\quad = \int_{\mathbb{R}^N_+} \left[\sum_{i=1}^{2} \left(\alpha_i U_{\lambda a^i + x^i}\right)^{p-2} + O(\mathrm{e}^{-\lambda\tau})\right] w^2,
\end{aligned}$$

and by mean value theorem, as we do for (2.4), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N_+} Q\left(\frac{y}{\lambda}\right) \left(\sum_{i=1}^{2} \alpha_i U_{\lambda a^i + x^i}\right)^{p-2} w^2 \mathrm{d}y \\
&\quad = \int_{\mathbb{R}^N_+} Q\left(\frac{y}{\lambda}\right) \sum_{i=1}^{2} \left(\alpha_i U_{\lambda a^i + x^i}\right)^{p-2} w^2 \mathrm{d}y + O\left(\mathrm{e}^{-\lambda\tau}\right)\|w\|^2 \\
&\quad = \int_{\mathbb{R}^N_+} \sum_{i=1}^{2} \left(Q(a^i)\left(\alpha_i U_{\lambda a^i + x^i}\right)^{p-2}\right) w^2 \mathrm{d}y + O\left(\frac{1}{\lambda^2}\right)\|w\|^2.
\end{aligned}$$

For λ large enough and (f2),

$$\lambda^{\frac{-p}{p-2}} \int_{\partial\mathbb{R}^N_+} f'\left(\frac{y}{\lambda}, \lambda^{\frac{2}{p-2}}\left(\sum_{i=1}^{2} \alpha_i U_{\lambda a^i + x^i}\right)\right) w^2 \mathrm{d}\sigma = O(\mathrm{e}^{-\lambda\tau})\|w\|^2, \quad \text{for large } \lambda.$$

Here the parameter $\alpha_i \in (-\rho + Q(a^i)^{-\frac{1}{p-2}}, \rho + Q(a^i)^{-\frac{1}{p-2}})$, if we choose ρ small enough, by Proposition A2 and the above estimates, we have

$$\begin{aligned}
F_\lambda(w) &\geq \frac{1}{2}\|w\|^2 - (p-1)\int_{\mathbb{R}^N_+} Q\left(\frac{y}{\lambda}\right)\left(\sum_{i=1}^{2} \alpha_i U_{\lambda a^i + x^i}\right)^{p-2} w^2 \mathrm{d}y \\
&\quad + O(\mathrm{e}^{-\lambda\tau})\|w\|^2 \\
&\geq \frac{1}{2}\|w\|^2 - (p-1)\int_{\mathbb{R}^N_+} \left(\sum_{i=1}^{2} U_{\lambda a^i + x^i}\right)^{p-2} w^2 \mathrm{d}y \\
&\quad - (p-1)[Q(a^1)(\alpha_1)^{p-2} - 1]\int_{\mathbb{R}^N_+} U^{p-2}_{\lambda a^1 + x^1} w^2 \mathrm{d}y \\
&\quad - (p-1)[Q(a^2)(\alpha_2)^{p-2} - 1]\int_{\mathbb{R}^N_+} U^{p-2}_{\lambda a^2 + x^2} w^2 \mathrm{d}y + O\left(\frac{1}{\lambda^2}\right)\|w\|^2
\end{aligned}$$

$$\geq \frac{\delta}{2}\|w\|^2 - \frac{\varepsilon_\rho}{C}\int_{\mathbb{R}^N_+}\sum_{i=1}^{2} U^{p-2}_{\lambda a^i+x^i} w^2 \mathrm{d}y + O\left(\frac{1}{\lambda^2}\right)\|w\|^2$$
$$\geq \rho'\|w\|^2.$$

Then by Resize representation theorem, we can denote L_λ and F_λ in the form

$L_\lambda(w) = \langle L, w\rangle, L \in E_{\lambda,x^1,x^2},\ w \in E_{\lambda,x^1,x^2}.\ F_\lambda(w) = \langle Aw, w\rangle, w \in E_{\lambda,x^1,x^2}.$

A is a continuous and coercive symmetric on E_{λ,x^1,x^2}. Hence we have

$$\frac{\partial J_\lambda}{\partial w}\ |_{E_{\lambda,x^1,x^2}} = L + 2Aw + R'(w),$$

and (4_{w}) is equivalent to $L + 2Aw + R'(w) = 0$. Set $F_{\lambda,x}(L, w) = L + 2Aw + R'(w) :\ E_{\lambda,x^1,x^2} \times E_{\lambda,x^1,x^2} \to E_{\lambda,x^1,x^2}$. Then we have

$$F(0,0) = 0,\ \ \frac{\partial F}{\partial w}(L, w) = 2A + R''(w),\ \ R''(w) = O(\|w\|^{p-2}).$$

If $\|w\|$ is small, $\frac{\partial F}{\partial w}$ is invertible, and there exists $(L, w_{\lambda,x^1,x^2}(L))$, such that $F(L, w_{\lambda,x^1,x^2}) = 0$, that is, $\langle L, w\rangle + 2\langle Aw, w\rangle + \langle R'(w), w\rangle = 0$ and $\|w\| \leq C\|L\|$. Now we get a $w \in E_{\lambda,x_1,x_2}$ satisfying Proposition 3.1. □

PROPOSITION 3.2

If ρ is small enough and $\lambda \geq \lambda_0$, for $(\alpha_1, \alpha_2, x^1, x^2) \in \Sigma_\rho$ and $w = w_{\lambda,x^1,x^2}$ as in Proposition 3.1,

$$\frac{\partial J_\lambda}{\partial \alpha_i} = 0 \ \Leftrightarrow\ \alpha_i - Q(a^i)^{-\frac{1}{p-2}} = V_i(\alpha_i, x^1, x^2, \lambda),\quad i = 1, 2;$$

in which $V_i(\alpha_i, x^1, x^2, \lambda) = O(\frac{1}{\lambda^2})$.

Proof. Take the derivative of J_λ with respect to α_i,

$$\frac{\partial J_\lambda}{\partial \alpha_i} = \left\langle \sum_{i=1}^{2}\alpha_i U_{\lambda a^i+x^i}, U_{\lambda a^i+x^i}\right\rangle - \int_{\mathbb{R}^N_+} Q\left(\frac{y}{\lambda}\right)\left|\sum_{i=1}^{2}\alpha_i U_{\lambda a^i+x^i} + w\right|^{p-2}$$
$$\left(\sum_{i=1}^{2}\alpha_i U_{\lambda a^i+x^i} + w\right) U_{\lambda a^i+x^i}\mathrm{d}y$$
$$- \lambda^{\frac{-p}{p-2}}\int_{\partial\mathbb{R}^N_+} f\left(\frac{y}{\lambda}, \lambda^{\frac{2}{p-2}}\left(\sum_{i=1}^{2}\alpha_i U_{\lambda a^i+x^i}\right)\right) U_{\lambda a^i+x^i}\mathrm{d}\sigma$$

As in Proposition 2.2, by the same estimates, we have

$$\frac{\partial J_\lambda}{\partial \alpha_1} = \alpha_1\|U\|^2_{H^1(\mathbb{R}^N)} - \alpha_1^{p-1}\int_{\mathbb{R}^N} Q\left(\frac{y}{\lambda}\right) U^p_{\lambda a^1+x^1}\mathrm{d}y + O(\mathrm{e}^{-\lambda\tau})$$
$$= \alpha_1\|U\|^2_{\mathbb{R}^N} - \alpha_1^{p-1}\int_{\mathbb{R}^N} Q(a^1)U^p\mathrm{d}y + O\left(\frac{1}{\lambda^2}\right).$$

$$\frac{\partial J_\lambda}{\partial \alpha_1} = 0 \Leftrightarrow \alpha_1 A^2 - \alpha_1^{p-1} Q(a^1) A^2 + O\left(\frac{1}{\lambda^2}\right) = 0,$$

$$\alpha_1 = Q(a^1)^{-\frac{1}{p-2}} + O\left(\frac{1}{\lambda^2}\right).$$

The situation is similar to α_2. □

PROPOSITION 3.3

Assume that ρ is small enough and $\lambda \geq \lambda_0$. For $(\alpha_1, \alpha_2, x^1, x^2) \in \Sigma_\rho$ and $w = w_{\lambda, x^1, x^2}$ as in Proposition 3.1, $\forall\, k = 1, \ldots, N$,

$$\frac{\partial J_\lambda}{\partial x_k^i} = \frac{A_i}{\lambda^2} \sum_{j=1}^{N} D_{j,k} Q(a^i) x_j^i + o\left(\frac{1}{\lambda^2}\right),$$

$$(4_{1\mathrm{i}}),\ (4_{2\mathrm{i}}) \text{ holds} \Leftrightarrow x^i = W^i(x^i, \lambda),$$

in which $W^i(x^i, \lambda)$ is a continuous function $B_\rho(0) \to \mathbb{R}^N$, and $W^i(x^i, \lambda) = o(1),\ \ \lambda \to \infty$, uniformly for $x^i \in B_\rho(0)$.

The proof of this result is quite similar to that given earlier for Proposition 2.3, and so is omitted. The proof of Theorem 3.1 follows from Propositions 3.1, 3.2, 3.3 as in the proof of Theorem 2.1.

Appendix

PROPOSITION A1

There is a positive constant δ, independent of w, such that

$$\|w\|^2 - (p-1) \int_{\mathbf{R}_+^N} U_{\lambda a + x}^{p-2} w^2 \mathrm{d}y \geq \delta \|w\|^2,\ \forall w \in E_{\lambda, x}.$$

As the arguments in [4], we first consider the eigenvalue problem

$$\begin{cases} -\Delta\varphi + \varphi = \mu U_{\lambda a + x}^{p-2} \varphi & \text{in } \mathbb{R}_+^N, \\ \dfrac{\partial \varphi}{\partial n} = 0 & \text{on } \partial \mathbb{R}_+^N \end{cases} \tag{A1}$$

which is equivalent to

$$\begin{cases} -\Delta\varphi + \varphi = \mu U^{p-2} \varphi & \text{in } \Omega_\lambda = \{y \in \mathbb{R}^N : y_N > -\lambda a_N - x_N\}, \\ \dfrac{\partial \varphi}{\partial n} = 0 & \text{on } \partial \Omega_\lambda. \end{cases} \tag{A2}$$

Denote the j-th pair of eigenvalues and eigenfunctions of problem (A2) by $(\mu_{\lambda,j}, \phi_{\lambda,j})$ for $j = 1, \ldots, N+1$ with $\mu_{\lambda,1} \leq \cdots \leq \mu_{\lambda,N+1}$. As $\lambda \to \infty$, the limiting equation of (A2) is as follows:

$$-\Delta\varphi + \varphi = \mu U^{p-2} \varphi \quad \text{in } \mathbb{R}^N. \tag{A3}$$

Denote the j-th pair of eigenvalues and eigenfunctions of problem (A3) by (μ_j, ϕ_j) for $j = 1, \ldots, N+1$ with $\mu_1 \leq \cdots \leq \mu_{N+1}$. Then by [4], we have $\mu_1 = 1, \mu_2 = \mu_{N+1} = p-1$; $\phi_1 = U, \phi_j \in \text{span}\,\{\frac{\partial U_x}{\partial x_i} : i = 1, \ldots, N\}$ for $j = 2, \ldots, N+1$

Lemma A1. *Let* $\lambda \to \infty$. *We have the following facts*:

$$\mu_{\lambda,j} \to \mu_j, \quad j = 1, 2, \ldots, N+1; \quad \phi_{\lambda,j} \rightharpoonup \tilde{\phi_j} \ \ in\ H^1(\Sigma), \ \ \mathbb{R}^N_+ \subset \Sigma \Subset \mathbb{R}^N;$$

$$\tilde{\phi_j} \in \text{span}\left\{\frac{\partial U_x}{\partial x_i} : i = 1, \ldots, N\right\}, \quad j = 2, \ldots, N+1; \quad \tilde{\phi_1} = CU;$$

$$\int_{\Omega_\lambda} |\nabla \phi_{\lambda,j} - \nabla \tilde{\phi_j}|^2 + (\phi_{\lambda,j} - \tilde{\phi_j})^2 \to 0, \quad \lambda \to \infty, \quad j = 1, \ldots, N+1.$$

Proof. The first eigenvalue of problem (A2) is defined by $\mu_{\lambda,1} = \inf\{\int_{\Omega_\lambda} |\nabla u|^2 + u^2 : u \in H^1(\Omega_\lambda), \int_{\Omega_\lambda} U^{p-2}u^2 = 1\}$. Set $u = \frac{U(x)}{(\int_{\Omega_\lambda} U^p \mathrm{d}x)^{1/2}}$. Then

$$\int_{\Omega_\lambda} |\nabla u|^2 + u^2 \mathrm{d}x = \frac{\int_{\Omega_\lambda} |\nabla U|^2 + U^2 \mathrm{d}x}{\int_{\Omega_\lambda} U^p \mathrm{d}x} \to 1, \quad \lambda \to \infty.$$

Thus $\mu_{\lambda,1} \leq 1 + o(1)$, so we can assume that $\mu_{\lambda,1} \to \mu_1^*$ as $\lambda \to \infty$ and $\mu_1^* \leq 1$. First, $\mu_1^* \neq 0$, else we have $\int_{\Omega_\lambda} U^{p-2}\phi_{\lambda,1}^2 \to 0$ as $\lambda \to \infty$, which is a contradiction with $\int_{\Omega_\lambda} U^{p-2}\phi_{\lambda,1}^2 = 1$. Because $\phi_{\lambda,1}$ is bounded in $H^1(\Omega_\lambda)$ and

$$\begin{cases} -\Delta\phi_{\lambda,1} + \phi_{\lambda,1} = \mu_{\lambda,1}U^{p-2}\phi_{\lambda,1} & \text{in } \Omega_\lambda = \{y \in \mathbb{R}^N : y_N > -\lambda a_N - x_N\}, \\ \dfrac{\partial \phi_{\lambda,1}}{\partial n} = 0 & \text{on } \partial\Omega_\lambda, \end{cases}$$

$$\int_{\Omega_\lambda} \nabla\phi_{\lambda,1} \cdot \nabla\psi + \phi_{\lambda,1}\psi = \mu_{\lambda,1} \int_{\Omega_\lambda} U^{p-2}\phi_{\lambda,1}\psi, \quad \forall \psi \in H^1(\mathbb{R}^N),$$

we have that $\phi_{\lambda,1} \rightharpoonup \tilde{\phi_1}$ in $H^1(\Sigma)$, for $\mathbb{R}^N_+ \subset \Sigma \subset \mathbb{R}^N$. Let $\lambda \to \infty$,

$$\int_{\mathbb{R}^N} \nabla\tilde{\phi_1} \cdot \nabla\psi + \tilde{\phi_1}\psi \mathrm{d}x = \int_{\mathbb{R}^N} \mu_1^* U^{p-2}\tilde{\phi_1}\psi \mathrm{d}x,$$

thus

$$-\Delta\tilde{\phi_1} + \tilde{\phi_1} = \mu_1^* U^{p-2}\tilde{\phi_1} \quad \text{in } \mathbb{R}^N$$

and $(\mu_1^*, \tilde{\phi_1})$ is a pair of eigenvalue and eigenfunction of (A3) and $0 < \mu_1^* \leq 1$. Hence, $\mu_1^* = 1$ and $\tilde{\phi_1} = CU$.

By induction, we can get the rest of the conclusion. Suppose for $2 \leq k \leq N+1$, we have

$$\phi_{\lambda,j} \rightharpoonup \tilde{\phi}_j \quad \text{in } H^1(\Sigma), \quad \Sigma \subset \mathbb{R}^N, \quad \tilde{\phi}_j \in \text{span} \left\{ U, \frac{\partial U_x}{\partial x_i} : i = 1, \ldots, N \right\}. \quad j = 1, 2, \ldots, k-1.$$

The k-th eigenvalue of (A2) is characterized by

$$\mu_{\lambda,k} = \inf \left\{ \int_{\Omega_\lambda} |\nabla u|^2 + u^2 \mathrm{d}x : u \in H^1(\Omega_\lambda), \int_{\Omega_\lambda} U^{p-2} u^2 \mathrm{d}x = 1, \langle u, \phi_{\lambda,j} \rangle_{\Omega_\lambda} = 0, j = 1, \ldots, k-1 \right\}.$$

Claim 1. $\lim_{\lambda\to\infty} \mu_{\lambda,k} = p-1$. At first, let us prove $\mu_{\lambda,k} \le p-1+o(1)$. Choose $t_{\lambda,j} = \frac{\langle \phi_{\lambda,j}, \tilde{U} \rangle_{\Omega_\lambda}}{\mu_{\lambda,j}}$, $j = 1, \ldots, k-1, \tilde{U} \in \text{span}\{U, \frac{\partial U_x}{\partial x_i} : i = 1, \ldots, N\} \backslash \text{span}\{\tilde{\phi}_i, i = 1, \ldots, k-1\}$ and

$$\int_{\mathbb{R}^N} |\nabla \tilde{U}|^2 + |\tilde{U}|^2 \mathrm{d}x = p-1, \quad \int_{\Omega_\lambda} U^{p-2} \tilde{U}^2 \mathrm{d}x = 1.$$

Set $u_\lambda = \frac{\tilde{U} - \sum_{j=1}^{k-1} t_{\lambda,j} \phi_{\lambda,j}}{\left(\int_{\Omega_\lambda} U^{p-2} (\tilde{U} - \sum_{j=1}^{k-1} t_{\lambda,j} \phi_{\lambda,j})^2 \right)^{1/2}}$. Then u_λ satisfies

$$\int_{\Omega_\lambda} U^{p-2} u_\lambda^2 \mathrm{d}x = 1, \ \langle u_\lambda, \phi_{\lambda,j} \rangle_{\Omega_\lambda} = 0, \ j = 1, \ldots, k-1,$$

hence

$$\mu_{\lambda,k} \le \langle u_\lambda, u_\lambda \rangle_{\Omega_\lambda} \to p-1, \ j = 1, \ldots, k-1, \ \text{as } \lambda \to \infty.$$

As before, we have

$$\phi_{\lambda,k} \rightharpoonup \phi_k^* \quad \text{in } H^1(\Sigma)$$

and

$$\int_{\mathbb{R}^N} \nabla \phi_k^* \cdot \nabla \psi + \phi_k^* \psi = \int_{\mathbb{R}^N} \mu_k^* U^{p-2} \phi_k^* \psi, \quad \forall \psi \in H^1(\mathbb{R}^{\mathbb{N}}).$$

From $\langle \phi_{\lambda,k}, \phi_{\lambda,j} \rangle_{\Omega_\lambda} = 0$, $j = 1, \ldots, k-1$, we get that $\langle \phi_k^*, \tilde{\phi}_j \rangle = 0$. Thus

$$\mu_k^* = p-1, \quad \phi_k^* \in \text{span} \left\{ U, \frac{\partial U_x}{\partial x_i} : i = 1, \ldots, N \right\} \Big\backslash \text{span}\{\tilde{\phi}_j, j = 1, \ldots, k-1\}.$$

For $k = N+2$, by similar argument we can get $\lim_{\lambda\to\infty} \mu_{\lambda,N+2} = \mu_{N+2} > p-1$. □

*Proof of Proposition A*1. For all $w \in E_{\lambda,x}$, we set $\tilde{w}(y) = w(y + \lambda a + x)$, then $\tilde{w}(y)$ satisfies $\langle \tilde{w}(y), U\rangle_{\Omega_\lambda} = \langle \tilde{w}(y), \frac{\partial U}{\partial y_i}\rangle_{\Omega_\lambda} = 0, i = 1, \ldots, N$. Then the problem becomes

$$\|\tilde{w}\|^2_{\Omega_\lambda} - (p-1)\int_{\Omega_\lambda} U^{p-2}\tilde{w}^2 \mathrm{d}x \geq \rho'\|\tilde{w}\|^2_{\Omega_\lambda}, \ \forall w \in E_{\lambda,x}.$$

Let

$$\begin{aligned}\tilde{w} &= \sum_{i=1}^{N+1}\langle \tilde{w}, \phi_{\lambda,i}\rangle_{\Omega_\lambda}\phi_{\lambda,i} + R_\lambda,\\ \|\tilde{w}\|^2_{\Omega_\lambda} &= \sum_{i=1}^{N+1}\langle \tilde{w}, \phi_{\lambda,i}\rangle^2_{\Omega_\lambda}\|\phi_{\lambda,i}\|^2_{\Omega_\lambda} + \|R_\lambda\|^2_{\Omega_\lambda}.\end{aligned}$$

So, it is not hard to derive

$$\begin{aligned}\|R_\lambda\|^2_{\Omega_\lambda} &\geq \mu_{\lambda,N+2}\int_{\Omega_\lambda} U^{p-2}R_\lambda^2 \mathrm{d}x\\ &= \mu_{\lambda,N+2}\left[\int_{\Omega_\lambda} U^{p-2}\tilde{w}^2\mathrm{d}x - 2\sum_{i=1}^{N+1}\langle \tilde{w}, \phi_{\lambda,i}\rangle_{\Omega_\lambda}\right.\\ &\quad \left.\int_{\Omega_\lambda} U^{p-2}\tilde{w} + \sum_{i=1}^{N+1}\langle \tilde{w}, \phi_{\lambda,i}\rangle^2_{\Omega_\lambda}\right]\\ &= \mu_{\lambda,N+2}\left[\int_{\Omega_\lambda} U^{p-2}\tilde{w}^2\mathrm{d}x + \sum_{i=1}^{N+1}\langle \tilde{w}, \phi_{\lambda,i}\rangle^2_{\Omega_\lambda} + o(1)\|\tilde{w}\|_{\Omega_\lambda}\right].\end{aligned}$$

This implies

$$\begin{aligned}\|\tilde{w}\|^2_{\Omega_\lambda} &\geq \mu_{\lambda,N+2}\left[\int_{\Omega_\lambda} U^{p-2}\tilde{w}^2\mathrm{d}x + \sum_{i=1}^{N+1}\langle \tilde{w}, \phi_{\lambda,i}\rangle^2_{\Omega_\lambda}\right]\\ &\quad + \sum_{i=1}^{N+1}\langle \tilde{w}, \phi_{\lambda,i}\rangle^2_{\Omega_\lambda}\mu_{\lambda,i} + o(1).\end{aligned}$$

Because $\phi_{\lambda,i} \to \tilde{\phi}_i \in \mathrm{span}\{U, \frac{\partial U_x}{\partial x_i} : i = 1, \cdot, N\}$, $\langle \tilde{w}, \phi_{\lambda,i}\rangle \to 0$ as $\lambda \to \infty$;

$$\|\tilde{w}\|^2_{\Omega_\lambda} \geq \mu_{\lambda,N+2}\int_{\Omega_\lambda} U^{p-2}\tilde{w}^2\mathrm{d}x + o(1)\|\tilde{w}\|^2_{\Omega_\lambda}.$$

Hence, by $\mu_{\lambda,N+2} > p-1$ for large λ, there exists a small constant $\rho' > 0$ such that

$$\|\tilde{w}\|^2_{\Omega_\lambda} - (p-1)\int_{\Omega_\lambda} U^{p-2}\tilde{w}^2\mathrm{d}x \geq \rho'\|\tilde{w}\|^2_{\Omega_\lambda},$$

which is equivalent to

$$\|w\|^2 - (p-1)\int_{\mathbb{R}^N_+} U^{p-2}_{\lambda a+x} w^2 \mathrm{d}x \geq \rho' \|w\|^2.$$

PROPOSITION A2

There exists a positive constant δ *independent of* w, *such that for large* λ,

$$\|w\|^2 - (p-1)\int_{\mathbb{R}^N_+} (U_{\lambda a^1+x^1} + U_{\lambda a^2+x^2})^{p-2} w^2 \mathrm{d}y \geq \delta \|w\|^2,\ \forall w \in E_{\lambda,x^1,x^2}.$$

With the help of Proposition A1, we can show this result by a similar argument in [4].

Proof. Set

$$\Lambda_\lambda = \inf\left\{\int_{\mathbb{R}^N_+} |\nabla w|^2 + w^2 :\ w \in E_{\lambda,x^1,x^2},\right.$$
$$\left.\int_{\mathbb{R}^N_+} (U_{\lambda a^1+x^1} + U_{\lambda a^2+x^2})^{p-2} w^2 \mathrm{d}y = 1\right\}.$$

It suffices to show that $\Lambda_\lambda > p-1+c_0$, for some positive number c_0, as λ large enough. By contradiction, suppose there is a sequence $\lambda_n \to \infty$: $\lim_{n\to\infty} \Lambda_{\lambda_n} = \Lambda \leq p-1$. Namely, there exists $w_n \in H^1(\mathbb{R}^N_+)$:

$$\int_{\mathbb{R}^N_+} \nabla w_n \cdot \nabla U_{\lambda_n a^i+x^i} + w_n \cdot U_{\lambda_n a^i+x^i} \mathrm{d}y = 0, \quad i = 1, 2; \tag{A4}$$

$$\int_{\mathbb{R}^N_+} \nabla w_n \cdot \nabla \frac{\partial U_{\lambda_n a^i+x^i}}{\partial x^i_j} + w_n \cdot \frac{\partial U_{\lambda_n a^i+x^i}}{\partial x^i_j} \mathrm{d}y = 0,$$
$$i = 1, 2;\ j = 1, \ldots, N; \tag{A5}$$

$$\int_{\mathbb{R}^N_+} |\nabla w_n|^2 + w_n^2 \mathrm{d}y = \Lambda_{\lambda_n} \int_{\mathbb{R}^N_+} \left(U_{\lambda_n a^1+x^1} + U_{\lambda_n a^2+x^2}\right)^{p-2} w_n^2 \mathrm{d}y; \tag{A6}$$

$$\int_{\mathbb{R}^N_+} \nabla w_n \cdot \nabla \psi + w_n \psi \mathrm{d}y = \Lambda_{\lambda_n} \int_{\mathbb{R}^N_+} (U_{\lambda_n a^1+x^1} + U_{\lambda_n a^2+x^2})^{p-2} w_n \psi \mathrm{d}y,$$
$$\forall \psi \in E_{\lambda_n,x^1,x^2}. \tag{A7}$$

Set $\tilde{w}_n(x) = w_n(x + \lambda_n a^1 + x^1)$, $\Omega_{\lambda_n} = \mathbb{R}^N_+ - \{\lambda_n a^1 + x^1\}$. Then we have

$$\int_{\Omega_{\lambda_n}} |\nabla \tilde{w}_n|^2 + \tilde{w}_n^2 = \Lambda_{\lambda_n} \int_{\Omega_{\lambda_n}} (U + U_{\lambda_n a^2+x^2-\lambda_n a^1-x^1})^{p-2} \tilde{w}_n^2.$$

We can assume that for some $\tilde{w} \in H^1(\mathbb{R}^N)$, there holds

$$\lim_{n\to\infty} \tilde{w}_n = \tilde{w}, \quad \text{weakly in } H^1(\Sigma),\ \forall\ \mathbb{R}^N_+ \subset \Sigma \subset \mathbb{R}^N.$$

Then from (3) and (4) we have

$$\int_{\mathbb{R}^N} \nabla \tilde{w} \cdot \nabla U + \tilde{w} U \mathrm{d}y = 0, \tag{A8}$$

$$\int_{\mathbb{R}^N} \nabla \tilde{w} \cdot \nabla \frac{\partial U}{\partial x_j} + \tilde{w} \frac{\partial U}{\partial x_j} \mathrm{d}y = 0, \quad j = 1, \ldots, N. \tag{A9}$$

We claim that

$$\int_{\mathbb{R}^N} |\nabla \tilde{w}|^2 + |\tilde{w}|^2 = \Lambda \int_{\mathbb{R}^N} U^{p-2} \tilde{w}^2 \mathrm{d}y. \tag{A10}$$

Now we are to choose numbers $l_{\lambda_n,1}, l_{\lambda_n,2}, \ldots, l_{\lambda_n,2N+2}$ such that $v \in E_{\lambda_n,x^1,x^2}$, in which $w(x) = \tilde{w}(x - \lambda_n a^1 - x^1)$,

$$\begin{aligned} v = {} & w - l_{\lambda_n,1} U_{\lambda_n a^1 + x^1} - l_{\lambda_n,2} U_{\lambda_n a^2 + x^2} \\ & - \sum_{j=1}^{N} \left(l_{\lambda_n,j+2} \frac{\partial U_{\lambda_n a^1 + x^1}}{\partial x_j^1} + l_{\lambda_n,N+j+2} \frac{\partial U_{\lambda_n a^2 + x^2}}{\partial x_j^2} \right). \end{aligned}$$

This is equivalent to the following system of linear equations

$$\begin{aligned} \langle w, U_{\lambda_n a^1 + x^1} \rangle = {} & l_{\lambda_n,1} \| U_{\lambda_n a^1 + x^1} \|^2 + l_{\lambda_n,2} \langle U_{\lambda_n a^1 + x^1}, U_{\lambda_n a^2 + x^2} \rangle \\ & + \sum_{j=1}^{N} l_{\lambda_n,j+2} \left\langle \frac{\partial U_{\lambda_n a^1 + x^1}}{\partial x_j^1}, U_{\lambda_n a^1 + x^1} \right\rangle \\ & + \sum_{j=1}^{N} l_{\lambda_n,N+j+2} \left\langle \frac{\partial U_{\lambda_n a^2 + x^2}}{\partial x_j^2}, U_{\lambda_n a^1 + x^1} \right\rangle, \end{aligned} \tag{A11}$$

$$\begin{aligned} \langle w, U_{\lambda_n a^2 + x^2} \rangle = {} & l_{\lambda_n,2} \| U_{\lambda_n a^2 + x^2} \|^2 + l_{\lambda_n,1} \langle U_{\lambda_n a^1 + x^1}, U_{\lambda_n a^2 + x^2} \rangle \\ & + \sum_{j=1}^{N} l_{\lambda_n,j+2} \left\langle \frac{\partial U_{\lambda_n a^1 + x^1}}{\partial x_j^1}, U_{\lambda_n a^2 + x^2} \right\rangle \\ & + \sum_{j=1}^{N} l_{\lambda_n,N+j+2} \left\langle \frac{\partial U_{\lambda_n a^2 + x^2}}{\partial x_j^2}, U_{\lambda_n a^2 + x^2} \right\rangle, \end{aligned} \tag{A12}$$

$$\begin{aligned} \left\langle w, \frac{\partial U_{\lambda_n a^1 + x^1}}{\partial x_i^1} \right\rangle = {} & l_{\lambda_n,1} \left\langle U_{\lambda_n a^1 + x^1}, \frac{\partial U_{\lambda_n a^1 + x^1}}{\partial x_i^1} \right\rangle \\ & + l_{\lambda_n,2} \left\langle \frac{\partial U_{\lambda_n a^1 + x^1}}{\partial x_i^1}, U_{\lambda_n a^2 + x^2} \right\rangle \end{aligned}$$

$$+\sum_{j=1}^{N} l_{\lambda_n,j+2}\left\langle \frac{\partial U_{\lambda_n a^1+x^1}}{\partial x_j^1}, \frac{\partial U_{\lambda_n a^1+x^1}}{\partial x_i^1}\right\rangle$$
$$+\sum_{j=1}^{N} l_{\lambda_n,N+j+2}\left\langle \frac{\partial U_{\lambda_n a^2+x^2}}{\partial x_j^2}, \frac{\partial U_{\lambda_n a^1+x^1}}{\partial x_i^1}\right\rangle, \tag{A13}$$

$$\left\langle w, \frac{\partial U_{\lambda_n a^2+x^2}}{\partial x_i^2}\right\rangle = l_{\lambda_n,1}\left\langle U_{\lambda_n a^1+x^1}, \frac{\partial U_{\lambda_n a^2+x^2}}{\partial x_i^2}\right\rangle$$
$$+ l_{\lambda_n,2}\left\langle \frac{\partial U_{\lambda_n a^2+x^2}}{\partial x_i^2}, U_{\lambda_n a^2+x^2}\right\rangle$$
$$+\sum_{j=1}^{N} l_{\lambda_n,j+2}\left\langle \frac{\partial U_{\lambda_n a^1+x^1}}{\partial x_j^1}, \frac{\partial U_{\lambda_n a^2+x^2}}{\partial x_i^2}\right\rangle$$
$$+\sum_{j=1}^{N} l_{\lambda_n,N+j+2}\left\langle \frac{\partial U_{\lambda_n a^2+x^2}}{\partial x_j^2}, \frac{\partial U_{\lambda_n a^2+x^2}}{\partial x_i^2}\right\rangle. \tag{A14}$$

To solve the system of linear equations, we need to establish estimates of the coefficients. We know that

$$\langle U_{\lambda a^1+x^1}, U_{\lambda a^2+x^2}\rangle = o(1), \quad \lambda \to \infty;$$
$$\left\langle \frac{\partial U_{\lambda a^1+x^1}}{\partial x_j^1}, U_{\lambda a^1+x^1}\right\rangle = \int_{\Omega_\lambda} \nabla \frac{\partial U}{\partial x_j}\cdot \nabla U + \frac{\partial U}{\partial x_j}\cdot U = o(1), \quad \lambda \to \infty.$$

Similarly,

$$\left\langle \frac{\partial U_{\lambda a^i+x^i}}{\partial x_j^i}, U_{\lambda a^k+x^k}\right\rangle = o(1), \;\; \lambda \to \infty, \;\; i = 1, 2; \;\; k = 1, 2;$$
$$\|U_{\lambda a^i+x^i}\|^2 \to A^2, \;\; \lambda \to \infty, \;\; i = 1, 2; \;\; A^2 = \|U\|^2_{H^1(\mathbb{R}^N)},$$
$$\left\| \frac{\partial U_{\lambda a^i+x^i}}{\partial x_j^i}\right\|^2 \to B^2, \;\; \lambda \to \infty, \;\; i = 1, 2; \quad j = 1, \ldots, N;$$
$$B^2 = \|\nabla U\|^2_{H^1(\mathbb{R}^N)}.$$

By (6) we have

$$\langle w, U_{\lambda_n a^1+x^1}\rangle = \int_{\mathbb{R}_+^N} \nabla w \cdot \nabla U_{\lambda_n a^1+x^1} + wU_{\lambda_n a^1+x^1}\,\mathrm{d}y$$
$$= \int_{\mathbb{R}_+^N-\{\lambda_n a^1+x^1\}} \nabla w(x+\lambda_n a^1+x^1)\cdot \nabla U + w(x+\lambda_n a^1+x^1)U\,\mathrm{d}y$$
$$\to \int_{\mathbb{R}^N} \nabla \tilde{w}\cdot \nabla U + \tilde{w}U\,\mathrm{d}y = 0, \;\; \lambda_n \to \infty.$$

Similarly,

$$\left\langle w, \frac{\partial U_{\lambda_n a^1+x^1}}{\partial x_j^1} \right\rangle = o(1), \quad \lambda_n \to \infty, \quad j = 1, \ldots, N;$$

$$\langle w, U_{\lambda_n a^2+x^2} \rangle = \int_{\mathbb{R}_+^N - \lambda a^2 - x^2} \nabla \tilde{w} \cdot \nabla U(x + \lambda_n a^1 + x^1 - \lambda_n a^2 - x^2)$$
$$+ \tilde{w} U(x + \lambda_n a^1 + x^1 - \lambda_n a^2 - x^2) \mathrm{d}y.$$

By (3) and $\tilde{w}_n \rightharpoonup \tilde{w}$, we have

$$\langle w, U_{\lambda_n a^2+x^2} \rangle = o(1).$$

Similarly, $\langle w, \frac{\partial U_{\lambda_n a^2+x^2}}{\partial x_j^2} \rangle = o(1)$, $\lambda_n \to \infty$, $j = 1, \ldots, N$. Owing to the above estimates, we know that the system of linear equations is equivalent to

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,2N+2} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,2N+2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{2N+2,1} & a_{2N+2,2} & \cdots & a_{2N+2,2N+2} \end{pmatrix} \cdot \begin{pmatrix} l_{\lambda,1} \\ l_{\lambda,2} \\ \vdots \\ l_{\lambda,2N+2} \end{pmatrix} = \begin{pmatrix} o(1) \\ o(1) \\ \vdots \\ o(1) \end{pmatrix},$$

in which $a_{ij} = o(1)$ if $i \neq j$, $a_{ii} > a_0 > 0$, for $i, j = 1, \ldots, 2N+2$. Therefore, (10)–(13) has a solution and $l_{\lambda,i} = o(1)$, $i = 1, \ldots, 2N+2$. Next, we will give the proof of (9). Set

$$v_n = w - l_{\lambda_n,1} U_{\lambda_n a^1+x^1} - l_{\lambda_n,2} U_{\lambda_n a^2+x^2}$$
$$- \sum_{j=1}^{N} \left(l_{\lambda_n,j+2} \frac{\partial U_{\lambda_n a^1+x^1}}{\partial x_j^1} + l_{\lambda_n,N+j+2} \frac{\partial U_{\lambda_n a^2+x^2}}{\partial x_j^2} \right),$$

such that $v_n \in E_{\lambda_n,x^1,x^2}$ and substitute $\psi = v_n$ into (6). We have

$$\int_{\mathbb{R}_+^N} \nabla w_n \cdot \nabla w + w_n w \mathrm{d}y = \Lambda_{\lambda_n} \int_{\mathbb{R}_+^N} (U_{\lambda_n a_1+x^1} + U_{\lambda_n a_2+x^2})^{p-2} w_n w \mathrm{d}y$$
$$- \Lambda_{\lambda_n} \Bigg[l_{\lambda_n,1} \int_{\mathbb{R}_+^N} (U_{\lambda_n a_1+x^1} + U_{\lambda_n a_2+x^2})^{p-2} w_n U_{\lambda_n a_1+x^1} \mathrm{d}y$$
$$+ l_{\lambda_n,2} \int_{\mathbb{R}_+^N} (U_{\lambda_n a_1+x^1} + U_{\lambda_n a_2+x^2})^{p-2} w_n U_{\lambda_n a_2+x^2} \mathrm{d}y$$
$$+ \sum_{j=1}^{N} \Bigg(l_{\lambda_n,j+2} \int_{\mathbb{R}_+^N} (U_{\lambda_n a_1+x^1} + U_{\lambda_n a_2+x^2})^{p-2} w_n \frac{\partial U_{\lambda_n a^1+x^1}}{\partial x_j^1} \mathrm{d}y$$
$$+ l_{\lambda_n,j+2+N} \int_{\mathbb{R}_+^N} (U_{\lambda_n a_1+x^1} + U_{\lambda_n a_2+x^2})^{p-2} w_n \frac{\partial U_{\lambda_n a^2+x^2}}{\partial x_j^2} \Bigg) \mathrm{d}y \Bigg].$$

Thus $\int_{\mathbb{R}^N_+-\lambda_n a_1-x^1} \nabla\tilde{w}_n \cdot \nabla\tilde{w} + \tilde{w}_n\tilde{w}\mathrm{d}y \to \int_{\mathbb{R}^N} |\nabla\tilde{w}|^2 + \tilde{w}^2\mathrm{d}y$, and the right-hand side of the equation $\to \Lambda\int_{\mathbb{R}^N} U^{p-2}\tilde{w}^2$. If $\tilde{w} \not\equiv 0$, then $\Lambda = p-1$. In fact, by (7) and (8),

$$p-1 \le \inf\left\{\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 : u \in H^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N} U^{p-2}u^2 = 1,\right.$$
$$\left.\langle u, U\rangle_{\mathbb{R}^N} = \left\langle u, \frac{\partial U}{\partial x_i}\right\rangle_{\mathbb{R}^N} = 0\right\} \le \Lambda \le p-1.$$

Therefore $\tilde{w}$ is an eigenfunction with eigenvalue $p-1$, $\tilde{w} \in \mathrm{Span}\{\frac{\partial U}{\partial x_i}, i = 1, \ldots, N\}$. This is a contradiction with (8). We have $\tilde{w} = 0$. So, $\widetilde{w_n} \rightharpoonup 0$ in $H^1(\Sigma)$, $\forall\,\Sigma : \mathbb{R}^N_+ \subset \Sigma \subset \mathbb{R}^N$. Similarly, $\hat{w}_n = w_n(x + \lambda_n a^2 + x^2)$, $\hat{w}_n \rightharpoonup 0$, in $H^1(\Sigma)$. Then we have

$$\begin{aligned}1 &= \int_{\mathbb{R}^N_+} (U_{\lambda_n a^1+x^1} + U_{\lambda_n a^2+x^2})^{p-2} w_n^2\\ &= \int_{\mathbb{R}^N_+} U^{p-2}_{\lambda_n a^1+x^1} w_n^2 + \int_{\mathbb{R}^N_+} U^{p-2}_{\lambda_n a^2+x^2} w_n^2 + o(1)\\ &= \int_{\mathbb{R}^N_+-\lambda_n a^1-x^1} U^{p-2}\widetilde{w_n}^2 + \int_{\mathbb{R}^N_+-\lambda_n a^2-x^2} U^{p-2}\widehat{w_n}^2 + o(1)\\ &\to 0.\end{aligned}$$

Hence it is a contradiction. Thus $\Lambda_\lambda > p-1+c_0$, for λ large enough. □

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COMMUNICATING EDITOR: S Kesavan