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A note on generalized skew derivations on Lie ideals

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Abstract. Let $\mathcal{R}$ be a prime ring, $\mathcal{Z}(\mathcal{R})$ its center, $\mathcal{C}$ its extended centroid, $\mathcal{L}$ a Lie ideal of $\mathcal{R}$, $\mathcal{F}$ a generalized skew derivation associated with a skew derivation d and automorphism α. Assume that there exist $t \geq 1$ and $m, n \geq 0$ fixed integers such that $vu = u^m \mathcal{F}(uv)^t u^n$ for all $u, v \in \mathcal{L}$. Then it is shown that either $\mathcal{L}$ is central or char$(\mathcal{R}) = 2$, $\mathcal{R} \subseteq \mathcal{M}_2(\mathcal{C})$, the ring of 2×2 matrices over $\mathcal{C}$, $\mathcal{L}$ is commutative and $u^2 \in \mathcal{Z}(\mathcal{R})$, for all $u \in \mathcal{L}$. In particular, if $\mathcal{L} = [\mathcal{R}, \mathcal{R}]$, then $\mathcal{R}$ is commutative.

Keywords. Generalized skew derivation; automorphism; (semi-)prime ring; Lie ideal; extended centroid.

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1. Introduction

Throughout this paper, $\mathcal{R}$ will denote a prime ring (unless otherwise mentioned) with center $\mathcal{Z}(\mathcal{R})$, extended centroid $\mathcal{C}$, and right Martindale quotient ring $\mathcal{Q}_r$. We refer the reader to [2] for the definitions and the related properties of these objects. For any $x, y \in \mathcal{R}$, the symbol $[x, y]$ will denote the commutator $xy - yx$. An additive subgroup $\mathcal{L}$ of $\mathcal{R}$ is said to be a Lie ideal of $\mathcal{R}$ if for all $u \in \mathcal{L}$ and $r \in \mathcal{R}$, $[u, r] \in \mathcal{L}$. Suppose that α is an automorphism of R. A ring $\mathcal{R}$ is said to be prime if $a\mathcal{R}b = (0)$ implies that $a = 0$ or $b = 0$. By a skew derivation of $\mathcal{R}$ we mean an additive map d from $\mathcal{R}$ into itself which satisfies the rule $d(xy) = d(x)y + \alpha(x)d(y)$ for all $x, y \in \mathcal{R}$. For $\alpha = 1$ is the identity automorphism of $\mathcal{R}$, d is known as a derivation of $\mathcal{R}$. In particular, for a fixed $a \in \mathcal{R}$, the mapping $I_a : \mathcal{R} \to \mathcal{R}$ given by $I_a(x) = [x, a]$ is a derivation called an inner derivation of $\mathcal{R}$. An additive function $\mathcal{G} : \mathcal{R} \longrightarrow \mathcal{R}$ is called a generalized inner derivation if $\mathcal{G}(x) = ax + xb$, for fixed $a, b \in \mathcal{R}$. For such a mapping, it is easy to see that

$$\mathcal{G}(xy) = \mathcal{G}(x)y + x[y, b] = \mathcal{G}(x)y + xI_b(y), \quad \text{for all } x, y \in \mathcal{R}.$$

Motivated by the above observation, Bresar [4] introduced the concept of generalized derivation on $\mathcal{R}$. An additive map $\mathcal{G} : \mathcal{R} \longrightarrow \mathcal{R}$ is called a generalized derivation of $\mathcal{R}$ if there exists a derivation d of $\mathcal{R}$ such that $\mathcal{G}(xy) = \mathcal{G}(x)y + xd(y)$, for all $x, y \in \mathcal{R}$. Familiar examples of generalized derivations are derivations of $\mathcal{R}$. Moreover, generalized derivation with $d = 0$ covers the concept of left multipliers, that is, an additive map $\mathcal{G} : \mathcal{R} \longrightarrow \mathcal{R}$ satisfying $\mathcal{G}(xy) = \mathcal{G}(x)y$, for all $x, y \in \mathcal{R}$. An additive mapping $\mathcal{F} : \mathcal{R} \longrightarrow \mathcal{R}$ is said to be a (right) *generalized skew derivation* of $\mathcal{R}$ if there exists a skew derivation d of $\mathcal{R}$ with associated automorphism α such that

$$\mathcal{F}(xy) = \mathcal{F}(x)y + \alpha(x)d(y)$$

holds for all $x, y \in \mathcal{R}$, d is called an *associated skew derivation* of $\mathcal{F}$ and α is called an *associated automorphism* of $\mathcal{F}$.

The property $x^n = x$ has been one among the favourites of many ring theorists over the last many decades since Jacobson [13] first studied the commutativity of rings satisfying this condition in order to generalize the classical Wedderburn theorem [21]. This result was further generalized by Sercoid and MacHale [20] who proved that commutativity of an arbitrary ring $\mathcal{R}$ (not necessarily prime) follows even if the above condition is weakened as $(xy)^n = xy$, for all $x, y \in \mathcal{R}$ and integer $n = n(x, y) > 1$. Further, using this result, Ligh and Luh [17] pointed out that such rings are direct sum of J-rings (rings satisfying the property $x^n = x$) and zero rings. Later, Bell and Ligh [3] obtained direct sum decomposition of rings satisfying the property $xy = (xy)^2 f(x, y)$, where $f(X, Y) \in \mathbb{Z} < X, Y >$, the ring of polynomial in two non-commuting indeterminates. Further, Ashraf [1] established, a decomposition theorem for rings satisfying $yx = x^m f(xy)x^n$ or $xy = x^m f(xy)x^n$, where m, n are non-negative integers and $f(X) \in X^2\mathbb{Z}[X]$, which in turn allows us to determine the commutativity of $\mathcal{R}$. Now in this perspective, one can consider the following related ring properties:

(i) Let $m \geq 0, n \geq 0, p > 1$ be fixed integers and $\mathcal{L}$ a Lie ideal of a prime ring $\mathcal{R}$ which admits a generalized skew derivation $\mathcal{F}$ such that $vu = u^m \mathcal{F}(uv)^p u^n$ for all $u, v \in \mathcal{L}$

(ii) Let $m \geq 0, n \geq 0, p > 1$ be fixed integers and $\mathcal{L}$ a Lie ideal of a prime ring $\mathcal{R}$ which admits a generalized skew derivation $\mathcal{F}$ such that $uv = u^m \mathcal{F}(uv)^p u^n$ for all $u, v \in \mathcal{L}$.

In the present paper, under rather a weak assumption, it is shown that if $\mathcal{R}$ admits a generalized skew derivation $\mathcal{F}$ satisfying either of the above conditions, then either $\mathcal{L}$ is central or char$(\mathcal{R}) = 2$, $\mathcal{R} \subseteq \mathcal{M}_2(\mathcal{C})$, the ring of 2×2 matrices over $\mathcal{C}$, $\mathcal{L}$ is commutative and $u^2 \in \mathcal{Z}(\mathcal{R})$ for all $u \in \mathcal{L}$. We confine our investigation to the condition (i) only, similar result can also be established for the ring satisfying the property (ii). In fact, we prove the following theorems:

Theorem 1. *Let $\mathcal{R}$ be a prime ring, $\mathcal{F}$ a generalized skew derivation associated with a skew derivation d and an automorphism α. Assume that there exist $t \geq 1$ and $m, n \geq 0$ fixed integers such that*

$$[y_1, y_2][x_1, x_2] = [x_1, x_2]^m \mathcal{F}([x_1, x_2][y_1, y_2])^t [x_1, x_2]^n$$

holds for all $x_1, x_2, y_1, y_2 \in \mathcal{R}$. Then $\mathcal{R}$ is commutative.

Theorem 2. *Let $\mathcal{R}$ be a prime ring, $\mathcal{Z}(\mathcal{R})$ its center, $\mathcal{C}$ its extended centroid, $\mathcal{L}$ a Lie ideal of $\mathcal{R}$, $\mathcal{F}$ a generalized skew derivation associated with a skew derivation d and an automorphism α. Assume that there exist $t \geq 1$ and $m, n \geq 0$ fixed integers such that*

$$vu = u^m \mathcal{F}(uv)^t u^n$$

for all $u, v \in \mathcal{L}$. Then either $\mathcal{L}$ is central or $\mathrm{char}(\mathcal{R}) = 2$, $\mathcal{R} \subseteq \mathcal{M}_2(\mathcal{C})$, *the ring of* 2×2 *matrices over* $\mathcal{C}$, *$\mathcal{L}$ is commutative and $u^2 \in \mathcal{Z}(\mathcal{R})$, for all $u \in \mathcal{L}$.*

In what follows, let $\mathcal{Q}_r$ be the right Martindale quotient ring and $\mathcal{C} = \mathcal{Z}(\mathcal{Q}_r)$ the center of $\mathcal{Q}_r$.

2. Preliminaries

In the present section, we collect some famous results which will be used as tools to prove our main theorems.

Fact 3. Following [10], if d is a non-zero skew-derivation of $\mathcal{R}$ and

$$\Phi\big(x_1, \ldots, x_n, d(x_1), \ldots, d(x_n)\big)$$

is a skew-differential identity of $\mathcal{R}$, then one of the following statements holds:

(a) either d is an inner skew derivation of $\mathcal{R}$;
(b) or $\mathcal{R}$ satisfies the generalized polynomial identity

$$\Phi(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n).$$

Fact 4. Let $\mathcal{R}$ be a prime ring and $\mathcal{I}$ a two-sided ideal of $\mathcal{R}$. Then $\mathcal{I}$, $\mathcal{R}$ and $\mathcal{Q}_r$ satisfy the same generalized polynomial identities with coefficients in $\mathcal{Q}_r$ (see [6]). Furthermore, $\mathcal{I}$, $\mathcal{R}$, and $\mathcal{Q}_r$ satisfy the same generalized polynomial identities with automorphisms (see [8, Theorem 1]).

Fact 5. Let $\mathcal{R}$ be a ring with $\mathrm{char}(\mathcal{R}) = 0$. Then an automorphism α of $\mathcal{Q}_r$ is called *Frobenius* if $\alpha(x) = x$ for all $x \in \mathcal{C}$. On the other hand, if $\mathrm{char}(\mathcal{R}) = p \geq 2$, an automorphism α is *Frobenius* if there exists a fixed integer t such that $\alpha(x) = x^{p^t}$ for all $x \in \mathcal{C}$. In [8, Theorem 2], Chuang proved that if $\Phi(x_i, \alpha(x_i))$ is a generalized polynomial identity for $\mathcal{R}$, where $\mathcal{R}$ is a prime ring and $\alpha \in \mathrm{Aut}(\mathcal{R})$ is an automorphism of $\mathcal{R}$ which is not Frobenius, then $\mathcal{R}$ also satisfies the non-trivial generalized polynomial identity $\Phi(x_i, y_i)$, where x_i and y_i are distinct indeterminates.

Fact 6. Let $\mathcal{R}$ be a prime ring and $\alpha \in \mathrm{Aut}(\mathcal{R})$ be an automorphism of $\mathcal{R}$ which is outer. In [7, Main Theorem] it is proved that if $\Phi(x_i, \alpha(x_i))$ is a generalized polynomial identity for $\mathcal{R}$, then $\mathcal{R}$ also satisfies a non-trivial generalized polynomial identity ($\mathcal{R}$ is a GPI-ring).

Fact 7. Finally, let us also mention that if $\mathcal{R}$ is a prime ring satisfying a non-trivial generalized polynomial identity and α is an automorphism of $\mathcal{R}$ such that $\alpha(x) = x$ for all $x \in \mathcal{C}$, then α is an inner automorphism of $\mathcal{R}$ [2, Theorem 4.7.4].

3. Results

In the sequel, we will make a frequent use of the following easy results:

Lemma 8. *Let $\mathcal{R}$ be a prime ring such that*

$$[x_1, x_2][x_3, x_4] = 0$$

for all $x_1, x_2, x_3, x_4 \in \mathcal{R}$. Then $\mathcal{R}$ is commutative.

Proof. It is an easy reduction of [9, Main Theorem']. □

Lemma 9. *Let $\mathcal{R}$ be a prime ring, $\mathcal{Q}_r$ its right Martindale quotient ring, $\mathcal{C}$ its extended centroid, $f(x_1, \ldots, x_n)$ a multilinear polynomial over $\mathcal{C}$ and $0 \neq a \in \mathcal{R}$ such that*

$$\big[af(x_1, \ldots, x_n), f(x_1, \ldots, x_n)\big] = 0$$

for all $x_1, \ldots, x_n \in \mathcal{R}$. Then either $a \in \mathcal{Z}(\mathcal{R})$ or $f(x_1, \ldots, x_n)$ is central-valued on $\mathcal{R}$.

Proof. Since

$$\big[a, f(x_1, \ldots, x_n)\big] f(x_1, \ldots, x_n) = 0 \tag{3.1}$$

for all $x_1, \ldots, x_n \in \mathcal{R}$, by [22, Theorem 1], either $a \in \mathcal{Z}(\mathcal{R})$ or $f(x_1, \ldots, x_n)$ is central valued on $\mathcal{R}$ or $\mathcal{R} \subseteq \mathcal{M}_2(\mathcal{C})$, the ring of 2×2 matrices over $\mathcal{C}$. Here if we suppose that $a \notin \mathcal{Z}(\mathcal{R})$, then $\mathcal{R}$ satisfies a polynomial identity. By [19, Theorem 2.3.29, p. 131] (see also [14, Lemma 2]), there exists a field $\mathcal{K}$ such that $\mathcal{Q} \subseteq \mathcal{M}_l(\mathcal{K})$ and $\mathcal{M}_l(\mathcal{K})$ satisfies (3.1). We assume that $f(x_1, \ldots, x_n)$ is not central valued on $\mathcal{R}$ and prove that a contradiction follows. Let e_{ij} be the usual matrix unit, with 1 in the (i, j)-entry and zero elsewhere. By [16], there exist $u_1, \ldots, u_n \in \mathcal{M}_l(\mathcal{K})$ and $\lambda \in \mathcal{K} - \{0\}$, such that $f(u_1, \ldots, u_n) = \lambda e_{ij}$, with $i \neq j$. Moreover, since the set $\{f(v_1, \ldots, v_n) \mid v_1, \ldots, v_n \in \mathcal{M}_l(\mathcal{K})\}$ is invariant under the action of all F-automorphisms of $\mathcal{M}_t(F)$, for any $i \neq j$, there exist $r_1, \ldots, r_n \in \mathcal{M}_l(\mathcal{K})$ such that $f(r_1, \ldots, r_n) = \lambda e_{ij} \neq 0$. Therefore, for $f(r_1, \ldots, r_n) = \lambda e_{ij} \neq 0$ in (3.1), it follows that $\lambda^2 e_{ij} a e_{ij} = 0$ for all $i \neq j$, i.e., a is a diagonal matrix in $\mathcal{M}_l(\mathcal{K})$. In this case, by using a standard argument, it is easy to see that a must be central, which is a contradiction. □

Analogously one may prove the following (we omit the proof for brevity):

Lemma 10. *Let $\mathcal{R}$ be a prime ring, $\mathcal{Q}_r$ its right Martindale quotient ring, $\mathcal{C}$ its extended centroid, $f(x_1, \ldots, x_n)$ a multilinear polynomial over $\mathcal{C}$ and $0 \neq a \in \mathcal{R}$ such that*

$$\big[f(x_1, \ldots, x_n)a, f(x_1, \ldots, x_n)\big] = 0$$

for all $x_1, \ldots, x_n \in \mathcal{R}$. Then either $a \in \mathcal{Z}(\mathcal{R})$ or $f(x_1, \ldots, x_n)$ is central-valued on $\mathcal{R}$.

As a consequence of the previous results, we also get the following:

COROLLARY 11

If $\mathcal{R}$ is a non-commutative prime ring and $0 \neq a \in \mathcal{R}$ such that

$$\big[a[x_1, x_2], [x_1, x_2]\big] = 0$$

for all $x_1, x_2 \in \mathcal{R}$, then $a \in \mathcal{Z}(\mathcal{R})$.

COROLLARY 12

If $\mathcal{R}$ is a prime ring and $0 \neq a \in \mathcal{R}$ such that

$$\big[[x_1, x_2][x_3, x_4]a, [x_5, x_6][x_7, x_8]\big] = 0$$

for all $x_1, \ldots, x_8 \in \mathcal{R}$, then $\mathcal{R}$ is commutative.

Remark 13. Firstly, we notice that if $\mathcal{F} = 0$, then $[y_1, y_2][x_1, x_2] = 0$, for any $x_1, x_2, y_1, y_2 \in \mathcal{R}$. Hence by Lemma 8, $\mathcal{R}$ is commutative and we are done. For this reason, in all that follows we will assume $\mathcal{F} \neq 0$.

Lemma 14. *Let $\mathcal{R}$ be a prime ring, $\mathcal{F}$ a non-zero generalized skew derivation associated with a skew derivation d and automorphism α. Assume that there exist $p, q \in \mathcal{Q}_r$, where p is an invertible element in $\mathcal{Q}_r$, such that $d(x) = \alpha(x)q - qx$ and $\alpha(x) = pxp^{-1}$, for all $x \in \mathcal{R}$. If there exist $t \geq 1$ and $m, n \geq 0$ fixed integers such that*

$$[y_1, y_2][x_1, x_2] = [x_1, x_2]^m \mathcal{F}([x_1, x_2][y_1, y_2])^t [x_1, x_2]^n \tag{3.2}$$

for all $x_1, x_2, y_1, y_2 \in \mathcal{R}$, then $\mathcal{R}$ is commutative.

Proof. By our assumption $\mathcal{R}$ satisfies

$$\begin{aligned}&[y_1, y_2][x_1, x_2] - [x_1, x_2]^m \big((a - q)[x_1, x_2][y_1, y_2] \\ &\quad + p[x_1, x_2][y_1, y_2]p^{-1}q\big)^t [x_1, x_2]^n.\end{aligned} \tag{3.3}$$

Since $\mathcal{R}$ and $\mathcal{Q}_r$ satisfy the same generalized polynomial identities with coefficients in $\mathcal{Q}_r$ (see [6]), it follows that (3.3) is satisfied by $\mathcal{Q}_r$. If $p^{-1}q \in \mathcal{C} = \mathcal{Z}(\mathcal{Q}_r)$, then $\mathcal{Q}_r$ satisfies the generalized polynomial identity

$$[y_1, y_2][x_1, x_2] - [x_1, x_2]^m \big(a[x_1, x_2][y_1, y_2]\big)^t [x_1, x_2]^n. \tag{3.4}$$

If $a \notin \mathcal{C}$, then (3.4) is a non-trivial generalized polynomial identity for $\mathcal{Q}_r$, i.e., (3.2) is a non-trivial generalized polynomial identity for $\mathcal{Q}_r$, as well as for $\mathcal{R}$, as required.

On the other hand, if $a \in \mathcal{C}$, then (3.4) is a polynomial identity for $\mathcal{Q}_r$. In this case it is well known that $\mathcal{Q}_r$ is a finite-dimensional central simple algebra over $\mathcal{C}$. Moreover, if we assume that $p^{-1}q \notin \mathcal{C}$ and (3.3) is a non-trivial generalized polynomial identity for $\mathcal{Q}_r$. Therefore, in any case, $\mathcal{Q}_r$ is a GPI-ring. Hence, by [18], it follows that $\mathcal{S} = \mathcal{R}\mathcal{C}$ is a primitive ring with soc$(\mathcal{R}) = \mathcal{H} \neq 0$ and $e\mathcal{H}e$ is a simple central algebra finite dimensional over $\mathcal{C}$, for any minimal idempotent element $e \in \mathcal{S}$. Moreover, we may assume $\mathcal{H}$

non-commutative, otherwise also $\mathcal{R}$ must be commutative. Notice that $\mathcal{H}$ satisfies (3.3) (see, for example, [14, proof of Theorem 1]).

Since $\mathcal{H}$ is a simple ring, one of the following holds: either $\mathcal{H}$ does not contain any non-trivial idempotent element or $\mathcal{H}$ is generated by its idempotents. In this last case, suppose that $\mathcal{H}$ contains three minimal orthogonal idempotent elements e, f, g. Let

$$[x_1, x_2] = [ex, f] = exf \quad [y_1, y_2] = [gy, e] = gye, \quad x, y \in \mathcal{H}.$$

By (3.3) and since $[x_1, x_2][y_1, y_2] = 0$, we get $0 = [y_1, y_2][x_1, x_2] = gyexf$, for any $x, y \in \mathcal{H}$. Therefore, the primeness of $\mathcal{H}$ forces either $e = 0$ or $f = 0$ or $g = 0$, in any case, a contradiction.

Therefore $\mathcal{H}$ cannot contain three minimal orthogonal idempotent elements and so $\mathcal{H} = \mathcal{M}_2(\mathcal{D})$, for a suitable division ring $\mathcal{D}$ finite dimensional over its center. This implies that $\mathcal{Q}_r = \mathcal{H}$ and $a, p, q, p^{-1}q \in \mathcal{H}$. By [19, Theorem 2.3.29, p. 131] (see also [14, Lemma 2]), there exists a field $\mathcal{K}$ such that $\mathcal{H} \subseteq \mathcal{M}_l(\mathcal{K})$ and $\mathcal{M}_l(\mathcal{K})$ satisfies (3.3). As we have just seen, if $l \geq 3$, then a contradiction follows. On the other hand, if $l = 1$ then $\mathcal{R} \subseteq \mathcal{K}$ and we are also done, thus we say $\mathcal{H} \subseteq \mathcal{M}_2(\mathcal{K})$. We want to prove that, in this case, we have a contradiction.

As above, let e_{ij} the usual matrix unit. If either $m \geq 2$ or $n \geq 2$, let $[x_1, x_2] = [e_{12}, e_{22}] = e_{12}$ and $[y_1, y_2] = [e_{22}, e_{21}] = e_{21}$ in (3.3), so that it follows the contradiction $e_{22} = 0$.

Moreover, if $m = 1$ and $n \in \{0, 1\}$ for $[x_1, x_2] = [e_{12}, e_{22}] = e_{12}$ and $[y_1, y_2] = [e_{22}, e_{21}] = e_{21}$ in (3.3) and left-multiplying by e_{22}, we obtain again the contradiction that $e_{22} = 0$.

We consider now the case $m = 0$ and $n \in \{0, 1\}$. More precisely, we start with $n = 1$. We recall that, since $\mathcal{H}$ is a central simple algebra of dimension 4 over its center, $[x, y]^2 \in C$, for any $x, y \in \mathcal{H}$. Therefore, for any $x, y \in \mathcal{H}$, either $[x, y]$ is invertible or $[x, y]^2 = 0$.

In case if $[x_1, x_2]$ is not invertible for all $x_1, x_2 \in \mathcal{H}$, then $[x_1, x_2]^2 = 0$ for all $x_1, x_2 \in \mathcal{H}$ and by Lemma 8, we have that $\mathcal{H}$ is commutative, a contradiction. Thus we assume that there exist $x_1, x_2 \in \mathcal{H}$ such that $u = [x_1, x_2]$ is invertible. Then, by (3.3) $\mathcal{H}$ satisfies

$$[y_1, y_2]u - \big((a - q)u[y_1, y_2] + pu[y_1, y_2]p^{-1}q\big)^t u$$

and right-multiplying by u^{-1}, it follows that

$$[y_1, y_2] - \big((a - q)u[y_1, y_2] + pu[y_1, y_2]p^{-1}q\big)^t$$

is satisfied by $\mathcal{H}$. By commuting this last relation with $(a-q)u[y_1, y_2]+pu[y_1, y_2]p^{-1}q$, we have that $\mathcal{H}$ satisfies

$$\big[(a - q)u[y_1, y_2] + pu[y_1, y_2]p^{-1}q, [y_1, y_2]\big] = 0. \tag{3.5}$$

Replace $[y_1, y_2]$ with $e_{ij}, i \neq j$ in (3.5). Left-multiplying by e_{ij}, it follows that

$$e_{ij}(pu)e_{ij}(p^{-1}q)e_{ij} = 0.$$

If we assume that the matrix $(p^{-1}q)$ is not diagonal, then there exists $i \neq j \in \{1, 2\}$, such that the (j, i)-entry of (pu) is zero, for any invertible element $u \in [\mathcal{R}, \mathcal{R}]$. Here we write $p = \sum_{l,m=1}^{2} e_{lm} p_{lm}$, $p^{-1}q = \sum_{l,m=1}^{2} e_{lm} q_{lm}$, for $p_{lm}, q_{lm} \in \mathcal{Z}(\mathcal{R})$. Without loss of generality, we suppose that

$$q_{21} \neq 0. \tag{3.6}$$

Therefore, in light of our last argument, since $e_{11} - e_{22}$ and $e_{12} + e_{21}$ are invertible elements of $[\mathcal{H}, \mathcal{H}]$, both the (2, 1)-entry of the matrix $p(e_{11} - e_{22})$ is zero and the (2, 1)-entry of the matrix $p(e_{12} + e_{21})$ is zero. By computations it follows that $p_{21} = 0$ and $p_{22} = 0$, which is a contradiction since p is an invertible matrix. Hence the matrix $(p^{-1}q)$ is diagonal, and by using a standard argument, one may prove that $(p^{-1}q)$ is central in $\mathcal{M}_2(\mathcal{K})$. By (3.5) it follows that

$$\big[au[y_1, y_2], [y_1, y_2]\big] = 0 \tag{3.7}$$

which implies $0 \neq au \in \mathcal{C}$ (see Corollary 11), for any invertible element $u \in [\mathcal{H}, \mathcal{H}]$. By using easy computations, one may prove that $a \in \mathcal{C}$, and it follows the contradiction that any invertible element $u \in [\mathcal{H}, \mathcal{H}]$ is central.

Let $m = 0$ and $n = 0$. Then, by (3.3) $\mathcal{H}$ satisfies

$$[y_1, y_2][x_1, x_2] - \big((a - q)[x_1, x_2][y_1, y_2] + p[x_1, x_2][y_1, y_2]p^{-1}q\big)^t$$

and by commuting with $(a - q)[x_1, x_2][y_1, y_2] + p[x_1, x_2][y_1, y_2]p^{-1}q$, we get

$$\big[(a - q)[x_1, x_2][y_1, y_2] + p[x_1, x_2][y_1, y_2]p^{-1}q, [y_1, y_2][x_1, x_2]\big] = 0. \tag{3.8}$$

Denote $v = [x_1, x_2]$ an invertible element and replace $[y_1, y_2]$ with e_{ij}, $i \neq j$ in (3.8). Left-multiplying by e_{ij}, it follows that

$$e_{ij}(pv)e_{ij}(p^{-1}q)e_{ij}v = 0.$$

If we assume that the matrix $(p^{-1}q)$ is not diagonal, then there exists $i \neq j \in \{1, 2\}$, such that the (j, i)-entry of (pv) is zero, for any invertible element $v \in [\mathcal{R}, \mathcal{R}]$. As in relation (3.6), we assume that the (2, 1)-entry of the matrix $(p^{-1}q)$ is not zero, and by using the same above computations, it follows a contradiction.

The previous argument says that $\mathcal{H}$ does not contain any non-trivial idempotent element, then $\mathcal{H}$ is a finite dimensional division algebra over $\mathcal{C}$ and $\mathcal{H} = \mathcal{R}\mathcal{C} = \mathcal{Q}$. If $\mathcal{C}$ is finite then $\mathcal{H}$ is a finite division ring, i.e., $\mathcal{H}$ is a commutative field and so $\mathcal{R}$ is commutative too.

If $\mathcal{C}$ is infinite then $\mathcal{H} \otimes_{\mathcal{C}} \mathcal{K} \cong \mathcal{M}_r(\mathcal{K})$, where $\mathcal{K}$ is a splitting field of $\mathcal{H}$. In this case, a Vandermonde determinant argument shows that (3.3) is still an identity in $\mathcal{M}_r(\mathcal{K})$. As above one can see that if $r \geq 2$, then a number of contradictions follow. Then $r = 1$ and $\mathcal{H}$ is commutative as well as $\mathcal{R}$. □

COROLLARY 15

Let $\mathcal{R}$ be a prime ring, $a, b \in \mathcal{R}$ and assume that there exist $t \geq 1$ and $m, n \geq 0$ fixed integers such that

$$[y_1, y_2][x_1, x_2] = [x_1, x_2]^m \left(a[x_1, x_2][y_1, y_2] + [x_1, x_2][y_1, y_2]b\right)^t [x_1, x_2]^n$$

for all $x_1, x_2, y_1, y_2 \in \mathcal{R}$, then $\mathcal{R}$ is commutative.

Lemma 16. *Let $\mathcal{R}$ be a prime ring, $\mathcal{F}$ a non-zero generalized skew derivations associated with a skew derivation d and automorphism α. Assume that d is an outer skew derivation of $\mathcal{R}$. If there exist $t \geq 1$ and $m, n \geq 0$ fixed integers such that*

$$[y_1, y_2][x_1, x_2] = [x_1, x_2]^m \mathcal{F}([x_1, x_2][y_1, y_2])^t [x_1, x_2]^n \tag{3.9}$$

for all $x_1, x_2, y_1, y_2 \in \mathcal{R}$, then $\mathcal{R}$ is commutative.

Proof. In [5, Lemma 2], Chang proved that there exists $a \in \mathcal{Q}_r$ such that $\mathcal{F}(x) = ax + d(x)$, for all $x \in \mathcal{R}$.

Notice that in case $d = 0$, then $\mathcal{F}(x) = ax$ for any $x \in \mathcal{R}$ and the commutativity of $\mathcal{R}$ follow from Lemma 14. Here we consider the case $d \neq 0$. Thus $\mathcal{R}$ satisfies

$$\begin{aligned}
&[y_1, y_2][x_1, x_2] - [x_1, x_2]^m \big(a[x_1, x_2][y_1, y_2] + d(x_1)x_2[y_1, y_2] \\
&+ \alpha(x_1)d(x_2)[y_1, y_2] - d(x_2)x_1[y_1, y_2] - \alpha(x_2)d(x_1)[y_1, y_2] \\
&+ \alpha([x_1, x_2])d(y_1)y_2 + \alpha([x_1, x_2])\alpha(y_1)d(y_2) \\
&- \alpha([x_1, x_2])d(y_2)y_1 - \alpha([x_1, x_2])\alpha(y_2)d(y_1)\big)^t [x_1, x_2]^n.
\end{aligned}$$

Since d is outer, $\mathcal{R}$ satisfies

$$\begin{aligned}
&[y_1, y_2][x_1, x_2] - [x_1, x_2]^m \big(a[x_1, x_2][y_1, y_2] + z_1x_2[y_1, y_2] + \alpha(x_1)z_2[y_1, y_2] \\
&- z_2x_1[y_1, y_2] - \alpha(x_2)z_1[y_1, y_2] + \alpha([x_1, x_2])z_3y_2 + \alpha([x_1, x_2])\alpha(y_1)z_4 \\
&- \alpha([x_1, x_2])z_4y_1 - \alpha([x_1, x_2])\alpha(y_2)z_3\big)^t [x_1, x_2]^n
\end{aligned}$$

and, in particular, R satisfies the non-trivial generalized polynomial identity

$$[y_1, y_2][x_1, x_2] - [x_1, x_2]^m \left(a[x_1, x_2][y_1, y_2]\right)^t [x_1, x_2]^n.$$

Also in this case, by Lemma 14, $\mathcal{R}$ must be commutative. □

4. Proof of main results

We facilitate our discussion with the following lemma:

Lemma 17. *Let $\mathcal{R}$ be a division ring, $\mathcal{F}$ a non-zero generalized skew derivations associated with a skew derivation d and automorphism α. Assume that there exists an element $q \in R$*

such that $d(x) = \alpha(x)q - qx$ and α is an outer automorphism of $\mathcal{R}$. If there exist $t \geq 1$ and $m, n \geq 0$ fixed integers such that

$$[y_1, y_2][x_1, x_2] = [x_1, x_2]^m \mathcal{F}([x_1, x_2][y_1, y_2])^t [x_1, x_2]^n$$

for all $x_1, x_2, y_1, y_2 \in \mathcal{R}$, then $\mathcal{R}$ is commutative.

Proof. We assume that $\mathcal{R}$ is not commutative and prove that a number of contradictions follows. Since $\mathcal{R}$ satisfies

$$[y_1, y_2][x_1, x_2] - [x_1, x_2]^m \big((a - q)[x_1, x_2][y_1, y_2] + \alpha([x_1, x_2][y_1, y_2])q\big)^t [x_1, x_2]^n, \tag{4.1}$$

by Fact 6, $\mathcal{R}$ is a GPI-ring. Thus $\mathcal{Q}_r$ is a primitive ring having nonzero socle and its associated division ring $\mathcal{D}$ is finite-dimensional over $\mathcal{C}$. If C is finite, then it follows that $\mathcal{Q}_r$ is also finite. By Wedderburn's theorem, $\mathcal{Q}_r$ is a field, then $\mathcal{R}$ is commutative, which is a contradiction. Hence from now on, we assume that C is infinite. If α is not Frobenius, then, by Fact 5, $\mathcal{R}$ also satisfies

$$[y_1, y_2][x_1, x_2] - [x_1, x_2]^m \big((a - q)[x_1, x_2][y_1, y_2] + [z_1, z_2][z_3, z_4]q\big)^t [x_1, x_2]^n \tag{4.2}$$

and, in particular,

$$[x_1, x_2]^m \big([z_1, z_2][z_3, z_4]q\big)^t [x_1, x_2]^n \tag{4.3}$$

is a generalized polynomial identity for $\mathcal{R}$. Since $\mathcal{R}$ is not commutative, there exist $x_1, x_2 \in \mathcal{R}$ such that $[x_1, x_2] \neq 0$. Thus, since $\mathcal{R}$ is a division ring, (4.3) implies that $\mathcal{R}$ satisfies $[z_1, z_2][z_3, z_4]q$, that is, $q = 0$. Hence $a \neq 0$ and (4.1) reduces to

$$[y_1, y_2][x_1, x_2] - [x_1, x_2]^m \big(a[x_1, x_2][y_1, y_2]\big)^t [x_1, x_2]^n. \tag{4.4}$$

Therefore, by Corollary 15, $\mathcal{R}$ is commutative, which is a contradiction again. Hence in all that follows we assume that α is Frobenius. Thus $\mathrm{char}(\mathcal{R}) = p > 0$ (if not $\alpha(\lambda) = \lambda$ for all $\lambda \in \mathcal{C}$ and α must be X-inner). Moreover $\alpha(\lambda) = \lambda^{p^h}$ for all $\lambda \in C$, where h is some fixed integer. Notice that, since C is infinite, there exist infinitely many $\lambda \in \mathcal{C}$ such that $\lambda^k \neq 1$ for $k = 1, \ldots, m+n$. Now replace y_1 by λy_1 in (4.1), where $0 \neq \lambda \in \mathcal{C}$. Thus there exists a suitable integer $h \geq 1$ such that $\mathcal{R}$ satisfies

$$[y_1, y_2][x_1, x_2] - \lambda^{t-1}[x_1, x_2]^m \big((a - q)[x_1, x_2][y_1, y_2] + \lambda^{p^h - 1}\alpha([x_1, x_2][y_1, y_2])q\big)^t [x_1, x_2]^n. \tag{4.5}$$

On the other hand, replacing x_1 by λx_1 in (4.1) it follows that $\mathcal{R}$ satisfies

$$[y_1, y_2][x_1, x_2] - \lambda^{m+n+t-1}[x_1, x_2]^m \big((a - q)[x_1, x_2][y_1, y_2] + \lambda^{p^h - 1}\alpha([x_1, x_2][y_1, y_2])q\big)^t [x_1, x_2]^n. \tag{4.6}$$

Hence, multiplying (4.5) by λ^{m+n} and comparing with (4.6), one has that $(\lambda^{m+n}-1)[y_1, y_2][x_1, x_2]$ is satisfied by $\mathcal{R}$, which is again a contradiction, since $\lambda^{m+n} \neq 1$ and $\mathcal{R}$ is not commutative. □

Proof of Theorem 1. We assume that

$$[y_1, y_2][x_1, x_2] = [x_1, x_2]^m \mathcal{F}([x_1, x_2][y_1, y_2])^t [x_1, x_2]^n \tag{4.7}$$

for all $x_1, x_2, y_1, y_2 \in \mathcal{R}$. As above write $\mathcal{F}(x) = ax + d(x)$ for any $x \in \mathcal{R}$, with $a \in \mathcal{Q}_r$. In case d is outer then by Lemma 16, $\mathcal{R}$ is commutative and we are done.

Let d be an inner skew derivation of $\mathcal{R}$, that is, there exists an element $q \in \mathcal{R}$ such that $d(x) = \alpha(x)q - qx$ for all $x \in \mathcal{R}$. Thus $\mathcal{F}(x) = (a-q)x + \alpha(x)q$. By our assumption, $\mathcal{R}$ satisfies

$$\begin{aligned}&[y_1, y_2][x_1, x_2] - [x_1, x_2]^m \big((a-q)[x_1, x_2][y_1, y_2]\\&\quad + \alpha([x_1, x_2][y_1, y_2])q\big)^t [x_1, x_2]^n.\end{aligned} \tag{4.8}$$

In light of Lemma 14, if α is an inner automorphism of $\mathcal{R}$, then we get the required conclusion. Hence, here we assume that α is outer. Then by Theorem 1 in [7], $\mathcal{Q}_r$ satisfies (4.8), moreover by the main theorem in [7], $\mathcal{Q}_r$ is a GPI-ring.

Since $\mathcal{Q}_r$ is a GPI-ring, by [18], $\mathcal{Q}_r$ is a primitive ring, which is isomorphic to a dense subring of the ring of linear transformations of a vector space $\mathcal{V}$ over a division ring $\mathcal{D}$.

In case $\dim_{\mathcal{D}} \mathcal{V} = 1$, then $\mathcal{Q}_r$ is a division ring and, by Lemma 17, we are done. Thus, we may assume that $\dim_{\mathcal{D}} \mathcal{V} \geq 2$ and $\mathcal{R}$ is not commutative. We prove that a contradiction follows.

First we recall that, under our last assumption, $\mathcal{Q}_r$ contains some non-trivial idempotent elements.

If either $m > 1$ or $n > 1$, we choose $e^2 = e \in \mathcal{Q}_r$ and $[x_1, x_2] = [er, (1-e)]$, for any $r \in \mathcal{Q}_r$. Therefore, by (4.8) we have that $[y_1, y_2]er(1-e) = 0$, which implies $[y_1, y_2] = 0$, for any $y_1, y_2 \in \mathcal{Q}_r$, a contradiction.

In case $m = 1$, let $e^2 = e \in \mathcal{Q}_r$. In relation (4.8), choose $[x_1, x_2] = [er, (1-e)]$ for any $r \in \mathcal{Q}_r$. Left multiplying by $(1-e)$, it follows that $(1-e)[y_1, y_2]er(1-e) = 0$, for any $y_1, y_2 \in \mathcal{Q}_r$, which implies the contradiction $[y_1, y_2] = 0$, for all $y_1, y_2 \in \mathcal{Q}_r$.

Hence we may assume $m = 0$ and $n \leq 1$.

Moreover, by [12, p. 79], there exists a semi-linear automorphism $\mathcal{T} \in \mathrm{End}(\mathcal{V})$ such that $\alpha(x) = \mathcal{T}x\mathcal{T}^{-1}$ for all $x \in \mathcal{Q}_r$. We notice that, if for any $v \in \mathcal{V}$ there exists $\lambda_v \in \mathcal{D}$ such that $\mathcal{T}^{-1}qv = v\lambda_v$, then, by a standard argument it follows that there exists a unique $\lambda \in \mathcal{D}$ such that $\mathcal{T}^{-1}qv = v\lambda$, for all $v \in \mathcal{V}$. In this case

$$\begin{aligned}\big((a-q)x + \alpha(x)q\big)v &= \big((a-q)x + \mathcal{T}x\mathcal{T}^{-1}q\big)v = (a-q)xv + \mathcal{T}(xv\lambda)\\&= (a-q)xv + \mathcal{T}((xv)\lambda) = (a-q)xv + \mathcal{T}(\mathcal{T}^{-1}qxv)\\&= (a-q)xv + qxv = axv.\end{aligned}$$

Hence, for all $v \in \mathcal{V}$,

$$\big(\mathcal{F}(x) - ax\big)v = 0$$

which implies $\mathcal{F}(x) = ax$ for all $x \in \mathcal{Q}_r$, since $\mathcal{V}$ is faithful. In this case, application of Corollary 15 implies the commutativity of $\mathcal{R}$, a contradiction. Therefore, there exists $v \in \mathcal{V}$ such that $\{v, \mathcal{T}^{-1}qv\}$ are linearly $\mathcal{D}$-independent. Now we sperate the proof in two different cases:

Case 1. $m = 0, n = 1$ In this case, $\mathcal{Q}_r$ satisfies

$$[y_1, y_2][x_1, x_2] - \big((a-q)[x_1, x_2][y_1, y_2] + \alpha([x_1, x_2][y_1, y_2])q\big)^t[x_1, x_2]. \tag{4.9}$$

If $\dim_{\mathcal{D}} \mathcal{V} \geq 3$ then there exists $w \in \mathcal{V}$ such that $\{w, v, \mathcal{T}^{-1}qv\}$ are linearly $\mathcal{D}$-independent. By the density of $\mathcal{Q}_r$, there exist $r_1, r_2, s_1, s_2 \in \mathcal{Q}_r$ such that

$$\begin{aligned}
&r_1 v = 0, \quad r_1\mathcal{T}^{-1}qv = v, \quad r_2 v = \mathcal{T}^{-1}qv, \quad r_2\mathcal{T}^{-1}qv = w\\
&s_1 v = 0, \quad s_1\mathcal{T}^{-1}qv = \mathcal{T}^{-1}qv, \quad s_2 v = \mathcal{T}^{-1}qv, \quad s_2\mathcal{T}^{-1}qv = 0.
\end{aligned}$$

Thus we get the following contradiction:

$$\begin{aligned}
0 &= \big([s_1, s_2][r_1, r_2] - \big((a-q)[r_1, r_2][s_1, s_2] + \alpha([r_1, r_2][s_1, s_2])q\big)^t[r_1, r_2]\big)v\\
&= \mathcal{T}^{-1}qv \neq 0.
\end{aligned}$$

Therefore $\dim_{\mathcal{D}} \mathcal{V} = 2$ and $\mathcal{Q}_r = \mathcal{M}_2(\mathcal{D})$, the ring of 2×2 matrices over $\mathcal{D}$. Let $u = [x_1, x_2]$ be an invertible matrix, so that, right-multiplying (4.9) by u^{-1}, we find that $\mathcal{Q}_r$ satisfies

$$[y_1, y_2] - \big((a-q)u[y_1, y_2] + \alpha(u[y_1, y_2])q\big)^t \tag{4.10}$$

and hence commuting (4.10) with $(a-q)u[y_1, y_2] + \alpha(u[y_1, y_2])q$, one has that

$$\big[[y_1, y_2], (a-q)u[y_1, y_2] + \alpha(u)\alpha([y_1, y_2])q\big] \tag{4.11}$$

is satisfied by $\mathcal{Q}_r$. Since α is outer and $\alpha(y_i)$-word degree is 1, by (4.11), $\mathcal{Q}_r$ satisfies

$$\big[[y_1, y_2], (a-q)u[y_1, y_2] + \alpha(u)[z_1, z_2]q\big] \tag{4.12}$$

and hence, in particular,

$$\big[[y_1, y_2], \alpha(u)[z_1, z_2]q\big] = 0 \tag{4.13}$$

and

$$\big[[y_1, y_2], (a-q)u[y_1, y_2]\big] = 0. \tag{4.14}$$

By (4.13), $\alpha(u)[\mathcal{Q}_r, \mathcal{Q}_r]q \subseteq \mathcal{C}$ and, since $\mathcal{Q}_r$ is not commutative, it is easy to see that it follows $q = 0$. On the other hand, by (4.14) and Corollary 11, we have that $(a-q)u \in \mathcal{C}$, for any $u = [x_1, x_2]$ invertible matrix. As in Lemma 14, this implies $a - q = 0$, that is, $a = q = 0$, which is a contradiction.

Case 2. $m = 0, n = 0$ In this case, $\mathcal{Q}_r$ satisfies

$$[y_1, y_2][x_1, x_2] - \big((a-q)[x_1, x_2][y_1, y_2] + \alpha([x_1, x_2][y_1, y_2])q\big)^t. \tag{4.15}$$

If $\dim_{\mathcal{D}} \mathcal{V} \geq 3$ then there exists $w \in \mathcal{V}$ such that $\{w, v, T^{-1}qv\}$ are linearly $\mathcal{D}$-independent. By the density of $\mathcal{Q}_r$ there exist $r_1, r_2, s_1, s_2 \in \mathcal{Q}_r$ such that

$$\begin{aligned} &r_1 v = r_2 v = v, \quad r_1 \mathcal{T}^{-1} qv = 0, \quad r_2 \mathcal{T}^{-1} qv = w, \quad r_1 w = \mathcal{T}^{-1} v \\ &s_1 v = s_2 v = 0, \quad s_1 \mathcal{T}^{-1} qv = 0, \quad s_2 \mathcal{T}^{-1} qv = w, \quad s_1 w = \mathcal{T}^{-1} qv. \end{aligned}$$

Thus we get the following contradiction:

$$0 = \big([s_1, s_2][r_1, r_2] - \big((a-q)[r_1, r_2][s_1, s_2] + \alpha([r_1, r_2][s_1, s_2])q\big)^t\big)v = v \neq 0.$$

Therefore $\dim_{\mathcal{D}} \mathcal{V} = 2$ and $\mathcal{Q}_r = \mathcal{M}_2(\mathcal{D})$. Starting from (4.15) and commuting with $(a-q)[x_1, x_2][y_1, y_2] + \alpha([x_1, x_2][y_1, y_2])q$, it follows that

$$\big[(a-q)[x_1, x_2][y_1, y_2] + \alpha([x_1, x_2][y_1, y_2])q, [y_1, y_2][x_1, x_2]\big] \tag{4.16}$$

is satisfied by $\mathcal{Q}_r$. Since α is outer and $\alpha(y_i)$-word degree is 1, by (4.17), $\mathcal{Q}_r$ satisfies also

$$\big[(a-q)[x_1, x_2][y_1, y_2] + [z_1, z_2][z_3, z_4]q, [y_1, y_2][x_1, x_2]\big]$$

and so Q_r satisfies both

$$\big[(a-q)[x_1, x_2][y_1, y_2], [y_1, y_2][x_1, x_2]\big] \tag{4.17}$$

and

$$\big[[z_1, z_2][z_3, z_4]q, [y_1, y_2][x_1, x_2]\big]. \tag{4.18}$$

Relation (4.18) and Corollary 12 imply $q = 0$, since $\mathcal{Q}_r$ is not commutative. Thus by (4.17),

$$\big[a[x_1, x_2][y_1, y_2], [y_1, y_2][x_1, x_2]\big] \tag{4.19}$$

is satisfied by $\mathcal{Q}_r = \mathcal{M}_2(\mathcal{D})$. For $i \neq j$ and $[x_1, x_2] = e_{ij}$ and $[y_1, y_2] = e_{ji}$ in (4.19), one has that the matrix a has all off-diagonal entries zero, that is a is a diagonal matrix. Standard argument shows that a is a central matrix in $\mathcal{Q}_r$, moreover $a \neq 0$ since $\mathcal{F} \neq 0$. Thus, from (4.19), $\mathcal{Q}_r$ satisfies the polynomial identity

$$\big[[x_1, x_2][y_1, y_2], [y_1, y_2][x_1, x_2]\big]$$

and by using the same above choices (for instance, $[x_1, x_2] = e_{12}$ and $[y_1, y_2] = e_{21}$) we get a contradiction. □

Proof of Theorem 2. Let $\mathcal{L}$ be a Lie ideal of $\mathcal{R}$ and assume $\mathcal{L}$ is not central. In case $\mathcal{L}$ is not commutative, then $0 \neq \mathcal{I} = \mathcal{R}[\mathcal{L}, \mathcal{L}]\mathcal{R}$ is a two-sided ideal of $\mathcal{R}$. Then $0 \neq [\mathcal{I}, \mathcal{R}] \subseteq \mathcal{L}$. According to our hypothesis,

$$vu = u^m \mathcal{F}(uv)^t u^n$$

holds for all $u \in [\mathcal{I}, \mathcal{R}]$. Since $\mathcal{I}$, $\mathcal{R}$ and $\mathcal{Q}_r$ satisfy the same generalized polynomial identities with automorphisms, $\mathcal{Q}_r$ satisfies

$$[y_1, y_2][x_1, x_2] = [x_1, x_2]^m \mathcal{F}([x_1, x_2][y_1, y_2])^t [x_1, x_2]^n,$$

and by Theorem 1, it follows the contradiction that $\mathcal{R}$ is commutative. Thus $\mathcal{L}$ is a commutative Lie ideal of $\mathcal{R}$ and, by [15, Lemma 2, Theorem 4], it follows that $\mathrm{char}(\mathcal{R}) = 2$, $\mathcal{R} \subseteq \mathcal{M}_2(\mathcal{C})$ and $u^2 \in \mathcal{Z}(\mathcal{R})$ for any $u \in \mathcal{L}$, as required. □

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