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A generalization of total graphs

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Abstract. Let R be a commutative ring with nonzero identity, $L_n(R)$ be the set of all lower triangular $n \times n$ matrices, and U be a triangular subset of R^n, i.e., the product of any lower triangular matrix with the transpose of any element of U belongs to U. The graph $GT^n_U(R^n)$ is a simple graph whose vertices consists of all elements of R^n, and two distinct vertices $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are adjacent if and only if $(x_1 + y_1, \ldots, x_n + y_n) \in U$. The graph $GT^n_U(R^n)$ is a generalization for total graphs. In this paper, we investigate the basic properties of $GT^n_U(R^n)$. Moreover, we study the planarity of the graphs $GT^n_U(U)$, $GT^n_U(R^n \backslash U)$ and $GT^n_U(R^n)$.

Keywords. Total graph; triangular subset; planarity; girth; diameter.

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1. Introduction

Let R be a commutative ring with nonzero identity and $Z(R)$ be the set of zero divisors of R. The concept of the total graph was first introduced by Anderson and Badawi [6]. The total graph of R, which is denoted by $T(\Gamma(R))$, is a simple graph with all elements of R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in Z(R)$. The total graph has been considered and investigated in [2,7,14,15]. Recently in [4], the generalized total graph was introduced and investigated. The generalized total graph of R is the simple graph $GT_H(R)$ with all elements of R as vertices, and two distinct vertices x and y are adjacent if and only if $x + y \in H$, where H is a nonempty proper subset of R with the following properties, which is called a multiplicative prime subset of R:

(i) $ab \in H$ for every $a \in H$ and $b \in R$;
(ii) if $ab \in H$ for $a, b \in R$, then either $a \in H$ or $b \in H$.

There are many papers which interlink graph theory and ring theory. Several classes of graphs associated with algebraic structures have been actively investigated (see for example, [1,3,5,8–10,12,13]).

Let R be a commutative ring with nonzero identity, $L_n(R)$ be the set of all lower triangular $n \times n$ matrices, and U be a subset of R^n, where n is a positive integer. We say that U is a triangular subset of R^n if the following condition holds:

For all $(u_1, \ldots, u_n) \in U$, $A \in L_n(R)$ and $(w_1, \ldots, w_n) \in R^n$, if $A[(u_1, \ldots, u_n)]^T = [w_1, \ldots, w_n]^T$, then $(w_1, \ldots, w_n) \in U$.

In this paper, for a triangular subset U of R^n, we introduce the graph $GT_U^n(R^n)$ as a generalization of total graphs. $GT_U^n(R^n)$ is a simple graph whose vertices consists of all elements of R^n, and two distinct vertices $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are adjacent if and only if $(x_1 + y_1, \ldots, x_n + y_n) \in U$. In the case that $n = 1$ and $U = Z(R)$, then $GT_U^1(R)$ is the usual total graph of R as introduced in [6]. In the case that $U = H$ is a multiplicative prime subset of R, then U is a triangular subset of R, and so, $GT_H(R)$ is a spaning subgraph of $GT_U^1(R)$. For $A \subseteq R^n$, let $GT_U^n(A)$ be the induced subgraph of $GT_U^n(R^n)$ with all elements of A as the vertices.

In section 2 of this paper, we investigate some basic properties of the graphs $GT_U^n(R^n)$, $GT_U^n(U)$ and $GT_U^n(R^n \backslash U)$. In section 3, we study the planarity of the graphs $GT_U^n(U)$, $GT_U^n(R^n \backslash U)$ and $GT_U^n(R^n)$.

Now, we recall some definitions and notations on graphs. We use the standard terminology of graphs in [11]. Let G be a simple graph. We say that G is *connected* if there is a path between any two distinct vertices of G, otherwise G is *disconnected*. Also, we say that G is *totally disconnected* if no two vertices of G are adjacent. For vertices x and y of G, we use the notation $x \sim y$ to denote that x and y are adjacent. Also, the length of a shortest path from x to y is denoted by $d(x, y)$ if a path from x to y exists. Also we define $d(x, y) = 0$ and $d(x, y) = \infty$ if there is no path between x and y. The *diameter* of G is $\text{diam}(G) = \sup\{d(x, y) : x, y \in V(G)\}$. The *girth* of G, denoted by $\text{gr}(G)$, is the length of a smallest cycle in G (if G contains no cycles, then $\text{gr}(G) = \infty$). A graph G is said to be *complete bipartite* if the vertices of G can be partitioned into two disjoint sets V_1, V_2 such that no two vertices in any V_1 or V_2 are adjacent, but for every $u \in V_1$, $v \in V_2$, the vertices u and v are adjacent. Then we use the symbol $K_{m,n}$ for the complete bipartite graph where the cardinal numbers of V_1 and V_2 are m, n, respectively. A graph with n vertices in which each pair of distinct vertices is joined by an edge is called a *complete graph*, and is denoted by K_n. A graph G is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends. A *subdivision* of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A remarkable simple characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.

2. Basic properties

Let U be a triangular subset of R^n. Clearly $(0, \ldots, 0) \in U$. Also, if $(1, \ldots, 1) \in U$, then, for any $(r_1, \ldots, r_n) \in R^n$, we have

$$\begin{pmatrix} r_1 & & & \\ & r_2 & & \\ & & \ddots & \\ & & & r_n \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}$$

which implies that $(r_1, \ldots, r_n) \in U$, and so $U = R^n$. Therefore, throughout the paper, we assume that $(1, \ldots, 1) \notin U$.

Now if $(a_1, \ldots, a_n) \in U$, we have $(a_1, a_1 + a_2, \ldots, a_1 + a_2 + \cdots + a_n) \in U$, since we have the following equality:

$$\begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & & & \\ 1 & 1 & \ldots & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_1 + a_2 \\ \vdots \\ a_1 + a_2 + \cdots + a_n \end{pmatrix}.$$

In the case that R is a field and U is a triangular subset of R^n, we have $U = (R_1, \ldots, R_n)$, where

$$R_i = \begin{cases} R & i \geq m \\ 0 & i < m \end{cases}$$

for some $2 \leq m \leq n$.

In this section, we determine the diameter and girth of the graphs $GT_U^n(R^n \backslash U)$, $GT_U^n(U)$ and $GT_U^n(R^n)$. Clearly $GT_U^n(U)$ is a connected graph because all vertices are adjacent to $(0, \ldots, 0)$.

Theorem 2.1. *Let R be a commutative ring and U be a triangular subset of R^n. Then*

(1) *U is closed under addition if and only if $U = I_1 \times \cdots \times I_n$, where each I_i is an ideal of R and $I_j \subseteq I_k$ for every j, k, $1 \leq j \leq k \leq n$.*
(2) *U is not closed under addition if and only if $U = \cup_{l \in A}(I_{l1} \times \cdots \times I_{ln})$, where I_{li} is an ideal of R and $I_{l1} \subseteq I_{l2} \subseteq \cdots \subseteq I_{ln}$ for every $l \in A$, $i \in \{1, 2, \ldots, n\}$ and $|A| \geq 2$.*
(3) *If V, W are triangular subsets of R^n, then $V \cup W$ is a triangular subset of R^n.*

Proof. (1) Let U be closed under addition. First we show that U is an ideal of R^n. Assume that $r = (r_1, \ldots, r_n)$ and $x = (x_1, \ldots, x_n)$. The relation

$$\begin{pmatrix} rx = r_1x_1 \\ r_2x_2 \\ \cdot \\ \cdot \\ \cdot \\ r_nx_n \end{pmatrix} = \begin{pmatrix} r_1 & & & \\ & r_2 & & \\ & & \ddots & \\ & & & r_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \\ \in U \end{pmatrix}$$

holds by triangularity of U. Thus U is an ideal by closedness under addition. Since U is an ideal of R, we have $U = I_1 \times \cdots \times I_n$, where each I_i is an ideal of R.
Now, assume that $x \in I_j$ and $j \leq k$ for $j, k \in \{1, \ldots, n\}$. Let $A(jk) = [a_{rs}]$ with

$$a_{rs} = \begin{cases} 1 & if\ r = k, s = j \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Since $j \leq k$, we have that $A[jk]$ is a lower triangular matrix. It is easy to show that $A[jk]x^j = x^k$, where $x^r := (\overbrace{0, \ldots, 0}^{r-1}, x, 0, \ldots, 0)^T$ for $r \in \{1, \ldots, n\}$. Hence $x \in I_k$.
The converse statement is clear.

(2) Suppose that U is not closed under addition. Let $x = (x_1, \dots, x_n) \in U$. Then $R^n x = R^n(x_1, \dots, x_n) = (Rx_1, \dots, Rx_n) = Rx_1 \times, \dots, \times Rx_n \subseteq U$ is an ideal of R^n. One can easily see that $U = \cup_{x \in U} R^n x$. Since U is a triangular subset of R^n and is not closed under addition, we have $U = \cup_{l \in A}(I_{l1} \times \cdots \times I_{ln})$, where I_{li} is an ideal of R for every $l \in A$, $i \in \{1, 2, \dots, n\}$ and $|A| \geq 2$. By a similar method that we used in part (1), we can prove that $I_{l1} \subseteq I_{l2} \subseteq \cdots \subseteq I_{ln}$, for every $l \in A$, $i \in \{1, 2, \dots, n\}$ and $|A| \geq 2$.

One can easily check that the converse statement holds.

(3) The proof is clear. □

In view of Theorem 2.1, if $R = \mathbb{Z} \times \mathbb{Z}$, then $U = 8\mathbb{Z} \times 4\mathbb{Z}$ and $U = (8\mathbb{Z} \times 4\mathbb{Z}) \cup (9\mathbb{Z} \times 3\mathbb{Z})$ are triangular subsets of R that are not multiplicative prime subsets of R.

The following result is an analogue of [4, Theorem 2.1].

Theorem 2.2. *Let R be a commutative ring, U be a triangular subset of R^n, which is closed under addition. Then $GT^n_U(U)$ is a complete subgraph of $GT^n_U(R^n)$, and $GT^n_U(U)$ is disjoint from $GT^n_U(R^n \backslash U)$. In particular, $GT^n_U(U)$ is connected and $GT^n_U(R^n)$ is never connected.*

The following result is an analogue of [4, Theorem 2.2]. The proof is similar to that in [4], and hence we omit the proof.

Theorem 2.3. *Let R be a commutative ring, U be a triangular subset of R^n and $(x_1, \dots, x_n) \in R^n \backslash U$. If U is closed under addition and $|U| = \alpha$, then the following statements hold*:

(1) *If $(2x_1, \dots, 2x_n) \in U$, then the coset $(x_1, \dots, x_n) + U$ is a component of $GT^n_U(R^n \backslash U)$ which is isomorphic to K_α.*
(2) *If $(2x_1, \dots, 2x_n) \notin U$, then $((x_1, \dots, x_n) + U) \cup ((-x_1, \dots, -x_n) + U)$ is a complete bipartite component of $GT^n_U(R^n \backslash U)$.*

The following corollary immediately follows from Theorem 2.3 and it is an analogue of [4, Theorem 2.5].

COROLLARY 2.4

Let R be a commutative ring and $U \subseteq R^n$ be a triangular subset and closed under addition. Then the following statements hold:

(1) $\mathrm{diam}(GT^n_U(R^n \backslash U)) = 1, 2$ *or* ∞.
(2) $\mathrm{diam}(GT^n_U(U)) = 1$.
(3) $\mathrm{gr}(GT^n_U(R^n \backslash U)) = 3, 4$ *or* ∞.
(4) $\mathrm{gr}(GT^n_U(U)) = 3$ *or* ∞.

In the following, we consider the case that U is not closed under addition. The proof is similar to the proof of [4, Theorem 3.1] and hence we omit the proof.

Theorem 2.5. *Let R be a commutative ring and U be a triangular subset of R^n which is not closed under addition. Then the following statements hold*:

(1) *$GT^n_U(U)$ is connected and* $\mathrm{diam}(GT^n_U(U)) = 2$.
(2) *The graphs $GT^n_U(U)$ and $GT^n_U(R^n\backslash U)$ are not disjoint.*
(3) *If $GT^n_U(R^n\backslash U)$ is connected, then so is $GT^n_U(R^n)$.*

The following result is an analogue of [4, Theorem 3.3]. Hence we omit the proof.

Theorem 2.6. *Let R be a commutative ring and $U \subseteq R^n$ be a triangular subset which is not closed under addition. Then $GT^n_U(R^n)$ is connected if and only if $\langle U\rangle = R^n$.*

The following result is an analogue of [4, Theorem 3.3]. Again, the proof is similar to that in [4] and hence we omit the proof.

Theorem 2.7. *Let R be a commutative ring, U be a triangular subset of R^n which is not closed under addition such that $\langle U\rangle = R^n$. Let $m \geq 2$ be the least integer that $R = \langle(a_{1,1},\ldots,a_{1,n}),\ldots,(a_{m,1},\ldots,a_{m,n})\rangle$, for some distinct elements $(a_{1,1},\ldots,a_{1,n}),\ldots,(a_{m,1},\ldots,a_{m,n}) \in U$. Then* $\mathrm{diam}(GT^n_U(R^n)) = m$.

The following result is an analogue of [4, Corollary 3.5]. For a proof see [4, Corollary 3.5].

COROLLARY 2.8

Let R be a commutative ring and U be a triangular subset of R^n which is not closed under addition and $GT^n_U(R^n)$ is connected. Then the following statements hold:

(1) $\mathrm{diam}(GT^n_U(R^n)) = d((0,\ldots,0),(1,\ldots,1))$.
(2) *If* $\mathrm{diam}(GT^n_U(R^n)) = m$, *then* $\mathrm{diam}(GT^n_U(R^n\backslash U)) \geq m-2$.

COROLLARY 2.9

Let R be a commutative ring and U be a triangular subset of R^n which is not closed under addition. If U contains two comaximal ideals of R^n, then $GT^n_U(R^n)$ is connected with $\mathrm{diam}(GT^n_U(R^n)) = 2$.

Proof. Suppose that $U_1, U_2 \subseteq U$ be two comaximal ideals of R^n. Then $R^n = U_1 + U_2$ and we have $R^n = \langle(a_1,\ldots,a_n),(b_1,\ldots,b_n)\rangle$, for some $(a_1,\ldots,a_n) \in U_1$ and $(b_1,\ldots,b_n) \in U_2$. Thus $GT^n_U(R^n)$ is connected with $\mathrm{diam}(GT^n_U(R^n)) = 2$. □

Theorem 2.10. *Let R be a commutative ring, U be a triangular subset of R^n which is not closed under addition. Then* $\mathrm{gr}(GT^n_U(U)) = 3$ *or* ∞.

Proof. Since $\mathrm{diam}(GT^n_U(U)) = 2$ and every vertex in U is adjacent to $(0,\ldots,0)$, we have $\mathrm{gr}(GT^n_U(U)) = 3$ or ∞. □

The proof of the following result is similar to that in [4, Theorem 3.14(5)].

Theorem 2.11. *Let R be a commutative ring, U be a triangular subset of R^n which is not closed under addition. Then* $\mathrm{gr}(GT^n_U(R^n\backslash U)) \in \{3, 4, \infty\}$.

3. Planarity

Let U be a triangular subset of R^n. For each $X \in U$, let i_X be a positive integer that the first nonzero component of X is in the i_X-th place. Now set

$$m := \min\{i_X | X \in U\}.$$

In this section, we study the planarity of the graphs $GT_U^n(U)$, $GT_U^n(R^n \backslash U)$ and $GT_U^n(R^n)$. In the rest of the paper, we always assume that $m \leq n-1$.

Theorem 3.1. *Let R be a commutative ring, U be a triangular subset of R^n, $4 \leq n$ and $m \leq n-2$. Then $GT_U^n(U)$ and $GT_U^n(R^n \backslash U)$ are not planar.*

Proof. Since $m \leq n-2$, without loss of generality, we may assume that there exists $(0, \ldots, 0, a, 0, 0) \in U$, where $a \neq 0$. If $\mathrm{char}(R) = 2$, then the vertices of the set

$$\{(1, \ldots, 1, a, 0, 0), (1, \ldots, 1, a, 0, a), (1, \ldots, 1, 0, a, 0), (1, \ldots, 1, a, a, a), (1, \ldots, 1, 0, 0, a)\}$$

form the complete graph K_5 in $GT_U^n(R^n \backslash U)$, and so, by Kuratowski's theorem, $GT_U^n(R^n \backslash U)$ is not planar. If $\mathrm{char}(R) \neq 2$, then the vertices of the set $\{(1, \ldots, 1, a, 0, 0), (1, \ldots, 1, 0, a, 0), (1, \ldots, 1, 0, 0, a)\}$ are adjacent to the vertices of the set $\{(-1, \ldots, -1, a, 0, 0), (-1, \ldots, -1, 0, a, 0), (-1, \ldots, -1, 0, 0, a)\}$, and so $K_{3,3}$ is a subgraph of $GT_U^n(R^n \backslash U)$. Hence, the graph $GT_U^n(R^n \backslash U)$ is not planar. Also, the set of vertices

$$\{(0, \ldots, 0, a), (0, \ldots, 0, a, 0), (0, \ldots, 0, a, a, a), (0, \ldots, 0, a, 0, a), (0, \ldots, 0)\}$$

form the complete graph K_5 in $GT_U^n(U)$ which implies that it is not planar. □

Theorem 3.2. *Let R be a commutative ring, U be a triangular subset of R^n, $n \geq 3$ and $m = n-1$. Then the following statements hold:*

(1) *If* $\mathrm{char}(R) \neq 2$, *then* $GT_U^n(R^n \backslash U)$ *is not planar.*
(2) *If* $\mathrm{char}(R) = 2$, *then* $GT_U^n(R^n \backslash U)$ *is planar if and only if*

$$U = \{(0, \ldots, 0), (0, \ldots, 0, a, 0), (0, \ldots, 0, a), (0, \ldots, 0, a, a)\},$$

for some $a \in R$.

Proof.

(1) Since $m = n-1$, there exists an element $(0, \ldots, 0, a, 0) \in U$, where $a \neq 0$. Now, one can easily see that the induced subgraph of $GT_U^n(R^n \backslash U)$ with vertex set

$$\{(1, \ldots, 1, a, 0), (1, \ldots, 1, 0, a), (1, \ldots, 1, a, a)\} \cup \{(-1, \ldots, -1, -a, 0), (-1, \ldots, -1, 0, -a), (-1, \ldots, -1, -a, -a)\}$$

contains a copy of $K_{3,3}$. So $GT_U^n(R^n \backslash U)$ is not planar.
(2) First assume that $(0, \ldots, 0, a, b) \in U$, where $a, b \neq 0$ and $a \neq b$. Then the set of vertices

$$\{(1, \ldots, 1, a, b), (1, \ldots, 1, a, 0), (1, \ldots, 1, 0, a), (1, \ldots, 1, a, a), (1, \ldots, 1, 0, 0)\}$$

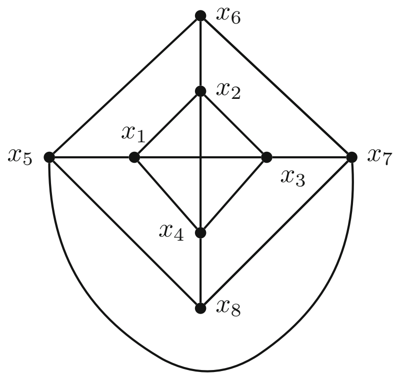

Figure 1. A subdivision of K_5.

forms the graph K_5, and so $GT_U^n(R^n\backslash U)$ is not planar. If there is $(0, \ldots, 0, a, 0) \in U$ such that $Ra \neq \{0, a\}$, then it is easy to see that $(0, \ldots, 0, a, b) \in U$, where $b \in Ra\backslash\{0, a\}$, and so $GT_U^n(R^n\backslash U)$ is not planar. If $(0, \ldots, 0, b), (0, \ldots, 0, a, 0) \in U$, where $a \neq b$, then by figure 1, $GT_U^n(R^n\backslash U)$ contains a subdivision of K_5, where $x_1 = (1, \ldots, 1, 0, 0), x_2 = (1, \ldots, 1, 0, a), x_3 = (1, \ldots, 1, a, a), x_1 = (1, \ldots, 1, a, 0), x_5 = (1, \ldots, 1, 0, b), x_6 = (1, \ldots, 1, 0, a + b), x_7 = (1, \ldots, 1, a, a + b), x_8 = (1, \ldots, 1, a, b)$. Therefore $GT_U^n(R^n\backslash U)$ is not planar. Now, the only remaining case for U is that $U = \{(0, \ldots, 0), (0, \ldots, 0, a, 0), (0, \ldots, 0, a), (0, \ldots, 0, a, a)\}$. In this situation, for each vertex $(x_1, \ldots, x_n)$ in $GT_U^n(R^n\backslash U)$, the set of distinct vertices

$$\begin{aligned} A = \{&(x_1, \ldots, x_n), (x_1, \ldots, x_{n-2}, x_{n-1} + a, x_n),\\ &(x_1, \ldots, x_{n-2}, x_{n-1} + a, x_n + a),\\ &(x_1, \ldots, x_{n-1}, x_n + a)\} \end{aligned}$$

forms the graph K_4, and any vertex which is adjacent to $(x_1, \ldots, x_n)$ belongs to $A\backslash\{(x_1, \ldots, x_n)\}$. Hence in this situation, $GT_U^n(R^n\backslash U)$ is the union of some copies of the graph K_4.

Now, by the above discussion, we can see that $GT_U^n(R^n\backslash U)$ is planar if and only if $U = \{(0, \ldots, 0), (0, \ldots, 0, a, 0), (0, \ldots, 0, a), (0, \ldots, 0, a, a)\}$, for some nonzero element $a \in R$. □

Theorem 3.3. *Let R be a commutative ring, $U \subseteq R^n$ be a triangular subset of R^n, $n \geq 3$ and $m = 1$. Then $GT_U^n(R^n\backslash U)$ and $GT_U^n(U)$ are not planar.*

Proof. If $(a, 0, \ldots, 0) \in U$, where $a \neq 0$, then the set of vertices

$$\{(0, \ldots, 0), (0, \ldots, 0, a, 0, 0,), (0, \ldots, 0, a, 0), (0, \ldots, 0, a), (0, \ldots, 0, a, a)\}$$

forms tha graph K_5 in $GT_U^n(U)$. Hence $GT_U^n(U)$ is not planar, which implies that $GT_U^n(R^n)$ is not planar. If $\mathrm{char}(R) \neq 2$, then the vertices of the set $\{(1, \ldots, 1, a, 0), (1, \ldots, 1, 0, a), (1, \ldots, 1, a, a)\}$ are adjacent to the vertices of the set $\{(-1, \ldots, -1, a, 0), (-1, \ldots, -1, 0,$

$a), (-1, \ldots, -1, a, a)\}$, and so $K_{3,3}$ is a subgraph of $GT_U^n(R^n \backslash U)$. Therefore $GT_U^n(R^n \backslash U)$ is not planar. If $\mathrm{char}(R) = 2$, then $(1, \ldots, 1, 0, 0)$, $(1, \ldots, 1, 0, a)$, $(1, \ldots, 1, a, 0)$, $(1, \ldots, 1, a, a)$,$(1, \ldots, 1, a+1, 0, 0)$ form the graph K_5 in $GT_U^n(R^n \backslash U)$. Hence $GT_U^n(R^n \backslash U)$ is not planar. □

Lemma 3.4. *Let R be commutative ring, U be a triangular subset of R^2, $m = 1$ and $(a, b) \in U$ such that $a \neq b$ are nonzero elements in R. Then the graph $GT_U^2(R^2 \backslash U)$ is not planar.*

Proof. In order to investigate the planarity of $GT_U^2(R^2 \backslash U)$, we consider the following two cases:

Case 1. $\mathrm{char}(R) \neq 2$. Then the set of vertices

$$\{(1, a), (1, b), (1, 0)\} \cup \{(-1, a), (-1, b), (-1, 0)\}$$

forms the bipartite graph $K_{3,3}$, and so, by Kuratowski's theorem, $GT_U^2(R^2 \backslash U)$ is not planar.
Case 2. $\mathrm{char}(R) = 2$. First assume that $(a + 1, a) \notin U$. Then $(a + 1, a)$, $(1, a + b)$, $(1, 0)$, $(1, a)$ and $(1, b)$ are in $R^2 \backslash U$ and they are adjacent. So K_5 is a subgraph of $GT_U^2(R^2 \backslash U)$ which implies that $GT_U^2(R^2 \backslash U)$ is not planar. If $(a+1, a) \in U$ and $|R| \geq 5$, then $(a + 1, 1) \in U$ because $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a+1 \\ a \end{pmatrix} = \begin{pmatrix} a+1 \\ 1 \end{pmatrix}$. Hence $(a + 1, 1) \in U$, which implies that, for every $r \in R$, $(a + 1, r) \in U$. Suppose that x_1, x_2, x_3, x_4 and x_5 are distinict elements of R. Thus $(1, x_1)$, $(1, x_2)$, $(1, x_3)$, $(1, x_4)$ and $(1, x_5)$ are distinct vertices in $GT_U^2(R^2 \backslash U)$ which are adjacent, and so K_5 is a subgraph of $GT_U^2(R^2 \backslash U)$. Hence $GT_U^2(R^2 \backslash U)$ is not planar. Now if $(a + 1, a) \in U$ and $|R| < 5$, then $R \cong \mathbb{Z}_2$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. If $R \cong \mathbb{Z}_2$, then one can easily see that there exists no nonzero element x and y with $(x, y) \in U$ such that $x \neq y$, hence by our assumption R is not $\mathbb{Z}_2$. Now assume that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Let $\mathfrak{o} = (0, 0)$, $\mathfrak{a} = (1, 0)$, $\mathfrak{b} = (0, 1)$ and $\mathfrak{c} = (1, 1)$. Then we have one of the following situations for U:

(i) $U = \{(\mathfrak{o}, \mathfrak{o}), (\mathfrak{a}, \mathfrak{o}), (\mathfrak{a}, \mathfrak{a}), (\mathfrak{o}, \mathfrak{a}), (\mathfrak{a}, \mathfrak{b}), (\mathfrak{o}, \mathfrak{b}), (\mathfrak{a}, \mathfrak{c}), (\mathfrak{o}, \mathfrak{c})\}$. In this situation, we find a copy of $K_{3,3}$ with vertex set $\{(\mathfrak{b}.\mathfrak{o}), (\mathfrak{b}, \mathfrak{a}), (\mathfrak{b}, \mathfrak{b})\} \cup \{(\mathfrak{c}.\mathfrak{o}), (\mathfrak{c}, \mathfrak{a}), (\mathfrak{c}, \mathfrak{b})\}$, which means that $GT_U^2(R^2 \backslash U)$ is not planar.
(ii) $U = \{(\mathfrak{o}, \mathfrak{o}), (\mathfrak{b}, \mathfrak{o}), (\mathfrak{b}, \mathfrak{b}), (\mathfrak{o}, \mathfrak{b}), (\mathfrak{b}, \mathfrak{a}), (\mathfrak{o}, \mathfrak{a}), (\mathfrak{b}, \mathfrak{c}), (\mathfrak{o}, \mathfrak{c})\}$. In this situation, we find a copy of $K_{3,3}$ with vertex set $\{(\mathfrak{a}.\mathfrak{o}), (\mathfrak{a}, \mathfrak{b}), (\mathfrak{a}, \mathfrak{a})\} \cup \{(\mathfrak{c}.\mathfrak{o}), (\mathfrak{c}, \mathfrak{a}), (\mathfrak{c}, \mathfrak{b})\}$, which means that $GT_U^2(R^2 \backslash U)$ is not planar.

Lemma 3.5. *Let R be a commutative ring and U be a triangular subset of R^2. If $(a, 0), (0, b) \in U$ such that $a \neq b$ are nonzero elements in R, then the graph $GT_U^2(R^2 \backslash U)$ is not planar.*

Proof. If $\mathrm{char}(R) = 2$, then we have a subdivision of K_5 in $GT_U^2(R^2 \backslash U)$, which is isomorphic to figure 1 , where $x_1 = (1, 1), x_2 = (1, a + 1), x_3 = (a + 1, a + 1), x_4 = (a + 1, 1), x_5 = (1, b + 1), x_6 = (1, a + b + 1), x_7 = (a + 1, a + b + 1), x_8 = (a + 1, b + 1)$, and hence $GT_U^2(R^2 \backslash U)$ is not planar. Now assume that $\mathrm{char}(R) \neq 2$ and $a \neq 2$. If one of the vertices $(1 - a, a)$ or $(a - 1, -a)$ belongs to U, then, by Lemma 3.4, $GT_U^2(R^2 \backslash U)$ is not planar. If $(1-a, a)$ and $(a-1, -a)$ belong to $R^2 \backslash U$, then $GT_U^2(R^2 \backslash U)$

contains a subgraph isomorphic to $K_{3,3}$ with vertex set $\{(1,0), (1,a), (1-a,-a)\} \cup \{(a-1,a), (-1,a), (-1,0)\}$, and so $GT_U^2(R^2 \backslash U)$ is not planar. If $\mathrm{char}(R) \neq 2$ and $a = 2$, then we have either $2R = \{0,2\}$ or $2R \neq \{0,2\}$. In the situation that $2R \neq \{0,2\}$, there exists $x \in 2R \backslash \{0,2\}$ such that $(2,x) \in U$. Then, by Lemma 3.4, $GT_U^2(R^2 \backslash U)$ is not planar. Now if $2R = \{0,2\}$, then we have $4 = 0$ which implies that $\mathrm{char}(R) \leq 4$. Since the characteristic of R is not 2 and also it is not 3, we have $\mathrm{char}(R) = 4$. It is easy to see that $\mathbb{Z}_4$ is the only ring with an ideal $\{0,2\}$ and characteristic 4. Hence $U = \{(0,0), (2,0), (0,2), (2,2), (0,1), (0,3)\}$. Thus the set of vertices $\{(1,0), (1,1), (1,2)\} \cup \{(3,0), (3,1), (3,2)\}$ forms the graph $K_{3,3}$, and so $GT_U^2(R^2 \backslash U)$ is not planar. □

Theorem 3.6. *Let R be a commutative ring and $U = \{(a,0), (a,a), (0,a), (0,0)\}$ be a triangular subset of R^2 such that $a \neq 0$. Then the following statements hold*:

(1) *If* $\mathrm{char}(R) \neq 2$, *then the graph $GT_U^2(R^2 \backslash U)$ is planar if and only if $a = 2$ and $R \cong \mathbb{Z}_4$*
(2) *If* $\mathrm{char}(R) = 2$, *then the graph $GT_U^2(R^2 \backslash U)$ is planar.*

Proof.

(1) First assume that $GT_U^2(R^2 \backslash U)$ is planar. Let $a \neq 2$. Then the vertices of the set $\{(1,0), (1,a), (1-a,0)\}$ are adjacent to the vertices of set $\{(a-1,a), (-1,a), (-1,0)\}$, and so $K_{3,3}$ is a subgraph of $GT_U^2(R^2 \backslash U)$. Hence the graph $GT_U^2(R^2 \backslash U)$ is not planar, which is a contradication. So we have $a = 2$. Also, $\{0,2\}$ is an ideal of R, and since $\mathrm{char}(R) \neq 2$, we have $R \cong \mathbb{Z}_4$. Conversly, the statement is clear.
(2) Suppose that (x,y) is a vertex in $GT_U^2(R^2 \backslash U)$. Then the set of distinct vertices $A = \{(x,y), (x+a,y), (x,y+a), (x+a,y+a)\}$ forms the graph K_4, and any vertex which is adjacent to (x,y) belongs to $A \backslash \{(x,y)\}$. Hence $GT_U^2(R^2 \backslash U)$ is the union of some copies of K_4, and so the graph $GT_U^2(R^2 \backslash U)$ is planar. □

Now, we have the following corollary.

COROLLARY 3.7

Let R be a commutative ring and U be a triangular subset of R^n. Then $GT_U^n(R^n \backslash U)$ is planar if and only if one of the following statements hold:

(i) $n = 2$, $R \cong \mathbb{Z}_4$ *and* $U = \{(0,0), (2,2), (0,2), (2,0)\}$.
(ii) $n = 2$, $\mathrm{char}(R) = 2$ *and* $U = \{(a,0), (a,a), (0,a), (0,0)\}$, *for some* $a \in R$.
(iii) $n \geq 3$, $\mathrm{char}(R) = 2$ *and* $U = \{(0,\ldots,0), (0,\ldots,0,a,0), (0,\ldots,0,a), (0,\ldots,0,a,a)\}$, *for some* $a \in R$.

The only remaining case for investigating the planarity of $GT_U^2(U)$, is the case that $m = n-1$. In the following two theorems, we give partial answers for this case.

Theorem 3.8. *Let R be a commutative ring*, *U be a triangular subset of R^n*, *$m = n-1$ and $(0,\ldots,0,a,b) \in U$ such that $a \neq b$ are nonzero elements in R. Then the graph $GT_U^2(U)$ is not planar.*

Proof. The vertices $(0, \ldots, 0, a, 0)$, $(0, \ldots, 0, 0, a)$, $(0, \ldots, 0, a, a)$, $(0, \ldots, 0, a, b)$ and $(0, \ldots, 0, 0, b)$ belong to U that are adjacent. So K_5 is a subgraph of $GT_U^2(U)$, which implies that $GT_U^2(U)$ is not planar. □

We say that U has the Property $(\star)$, if $(0, \ldots, 0, x) \in U$ implies that $(0, \ldots, 0, x, 0) \in U$.

Theorem 3.9. *Assume that $m = n - 1$ and U has the Property $(\star)$. Then $GT_U^n(U)$ is planar if and only if for each two nonzero elements $(0, \ldots, 0, x, 0)$ and $(0, \ldots, 0, y, 0)$ in U, we have $(0, \ldots, 0, x + y, 0) \notin U$.*

Proof. First assume that $GT_U^n(U)$ is planar. Suppose on the contrary that $(0, \ldots 0, a, 0)$, $(0, \ldots, 0, b, 0)$, $(0, \ldots, 0, c, 0)$ belong to U, where $a + b = c$, for some $a, b \in R$. Then the vertices of the set

$$\{(0, \ldots, 0, a, 0), (0, \ldots, 0, a, a), (0, \ldots, 0, a), (0, \ldots, 0), (0, \ldots, 0, b, 0), (0, \ldots, 0, b, b), (0, \ldots, 0, b)\}$$

forms a subdivision of K_5, and so $GT_U^n(U)$ is not planar.

Conversly, suppose that $(0, \ldots, 0, a, b) \in U$ where $a \neq b$. Then it is easy to see that the vertices $(0, \ldots, 0, a + b)$, $(0, \ldots, 0, b)$ are in U. Now by Property $(\star)$, we have $(0, \ldots, 0, a + b, 0), (0, \ldots, 0, b, 0) \in U$, which is a contradication. So for each nonzero element $a \in R$, where $(0, \ldots, 0, a, 0) \in U$, the induced subgraph of $GT_U^n(U)$ with vertex set $(0, \ldots, 0)$, $(0, \ldots, 0, a)$, $(0, \ldots, 0, a, 0)$, $(0, \ldots, 0, a, a)$ forms the graph K_4. Now, one can easily see that $GT_U^n(U)$ is the union of some copies of K_4, that are adjacent in $(0, \ldots, 0)$. Therefore $GT_U^n(U)$ is planar. □

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