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Global weighted estimates for second-order nondivergence elliptic and parabolic equations

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Abstract. In this paper, we obtain the global weighted L^p estimates for second-order nondivergence elliptic and parabolic equations with small BMO coefficients in the whole space. As a corollary, we obtain L^p-type regularity estimates for such equations.

Keywords. Weighted; L^p estimates; second-order; nondivergence; small BMO; elliptic; parabolic; the whole space.

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1. Introduction

One of the key *a priori* estimates in the theory of second-order elliptic and parabolic equations is L^p regularity theory, which was found by Calderón–Zygmund [10]. The L^p-type regularity is the fundamental theory for elliptic and parabolic equations, which plays an important role in the theory of partial differential equations, and is the basis for the existence and uniqueness of solutions. Furthermore, many authors [2,6,7,9,11–13,24,25,29,30,32] generalized such estimates for second-order elliptic and parabolic equations under a very weak assumption on the coefficients and a mild geometric condition on the boundary. More recently, some authors [14,20–22] obtained the global L^p-type regularity in the whole space for second-order divergence and nondivergence elliptic and parabolic equations since for issues related to stochastic processes, it is enough to consider the corresponding PDEs in the whole space.

In this paper, we are mainly concerned with global weighted L^p estimates for the following linear parabolic equations of nondivergence form

$$\mathcal{L}u =: u_t - a_{ij}(x,t)u_{x_i x_j} + b_i(x,t)u_{x_i} + c(x,t)u = f, \quad (x,t) \in \mathbb{R}^n_T, \tag{1.1}$$

$$u(x,0) = 0, \tag{1.2}$$

where $\mathbb{R}^n_T =: \mathbb{R}^n \times (0,T)$, $x = (x_1, x_2, \ldots, x_n)$, $i, j = 1, 2, \ldots, n$, and the summation convention is understood. Moreover, the coefficients satisfy

$$a_{ij} = a_{ji},\ |b_i| + |c| \le \Lambda,\ \ \Lambda^{-1}|\xi|^2 \le a_{ij}\xi_i\xi_j \le \Lambda|\xi|^2 \tag{1.3}$$

for all $i, j = 1, 2, \ldots, n$, $\xi \in \mathbb{R}^n$ and positive constant Λ. In fact, as a corollary of the parabolic case, we can obtain the corresponding result in the elliptic case

$$\mathcal{M}u =: -a_{ij}(x)u_{x_i x_j} + b_i(x)u_{x_i} + c(x)u = f, \quad x \in \mathbb{R}^n,$$

where the coefficients satisfy (1.3).

Throughout this paper, we assume that the coefficients of $A = \{a_{ij}\}$ are in BMO space and their semi-norms are small enough. More precisely, we introduce the following definition.

DEFINITION 1.1 (Small BMO condition)

(i) *Elliptic case.* We say that the matrix $A = \{a_{ij}(x)\}_{n\times n}$ of coefficients is (δ, R)-vanishing if for a given $\delta > 0$, there exists $R > 0$ such that

$$\sup_{0<r\le R}\ \sup_{x\in\mathbb{R}^n} \fint_{B_r(x)} \left|A(y) - \bar{A}_{B_r(x)}\right| \mathrm{d}y \le \delta, \tag{1.4}$$

where

$$\bar{A}_{B_r(x)} = \fint_{B_r(x)} A(y)\mathrm{d}y.$$

(ii) *Parabolic case.* We say that the matrix $A = \{a_{ij}(x,t)\}_{n\times n}$ is (δ, R)-vanishing if

$$\sup_{0<r\le R}\ \sup_{z\in\mathbb{R}^n\times\mathbb{R}} \fint_{Q_r(z)} \left|A(y,s) - \bar{A}_{Q_r(z)}\right| \mathrm{d}y\mathrm{d}s \le \delta,$$

where $z = (x, t)$ and $Q_r(z) = B_r(x) \times (t - r^2, t + r^2)$, and

$$\bar{A}_{Q_r(z)} = \fint_{Q_r(z)} A(y,s)\mathrm{d}y\mathrm{d}s.$$

When A is (δ, R)-vanishing and Ω is a time-independent/time-dependent Reifenberg flat domain, Byun and Wang [7,8] studied L^p estimates

$$\int_\Omega |\nabla u|^p \mathrm{d}z \le C \int_\Omega |\mathbf{f}|^p \mathrm{d}z \quad \text{for any} \quad p > 1$$

for weak solutions of

$$u_t - \mathrm{div}(A\nabla u) = \mathrm{div}\,\mathbf{f} \quad \text{in } \Omega \times (0, T]. \tag{1.5}$$

Furthermore, Dong *et al.* [15,21] obtained the global L^p estimates

$$\int_{\mathbb{R}^n_T} |\nabla u|^p \mathrm{d}z \le C \int_{\mathbb{R}^n_T} |\mathbf{f}|^p \mathrm{d}z \tag{1.6}$$

for weak solutions of the divergence parabolic equation (1.5) in $\mathbb{R}^n_T$ with different coefficients. Furthermore, they studied the global L^p estimates [16,20]

$$\int_{\mathbb{R}^n_T} |D^2 u|^p \mathrm{d}z \leq C \int_{\mathbb{R}^n_T} |f|^p \mathrm{d}z \tag{1.7}$$

for (1.1) in $\mathbb{R}^n_T$ with different coefficients.

We now introduce the weighted Lebesgue spaces (see [19,23,27,28,31,33,34]).

DEFINITION 1.2

A_p for some $p > 1$ is the class of the Muckenhoupt weights $w \in A_p$ if $w \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$, $w > 0$ almost everywhere and there exists a constant C such that for any cylinder Q_r in $\mathbb{R}^{n+1}$,

$$\left(\fint_{Q_r} w(z)\mathrm{d}z\right)\left(\fint_{Q_r} w(z)^{\frac{-1}{p-1}}\mathrm{d}z\right)^{p-1} \leq C.$$

Moreover, we denote

$$A_\infty = \bigcup_{1<p<\infty} A_p \quad \text{and} \quad w(Q_r) = \int_{Q_r} w(z)\mathrm{d}z.$$

Furthermore, the corresponding weighted Lebesgue space $L^p_w(Q_r)$ consists of all functions h which satisfy

$$\|h\|_{L^p_w(Q_r)} =: \left(\int_{Q_r} |h|^p\, w(z)\mathrm{d}z\right)^{1/p} < \infty.$$

Moreover, the weighted Sobolev space $W^{2,1}_{p,w}(Q_r)$ consists of all functions $h \in L^p_w(Q_r)$ for which the norm

$$\|h\|_{W^{2,1}_{p,w}(Q_r)} =: \left(\sum_{0\leq 2r+s\leq 2} \int_{Q_r} \left|D^r_t D^s_x h\right|^p w(z)\mathrm{d}z\right)^{1/p} < \infty.$$

In this work, we shall use the Hardy–Littlewood maximal function which controls the local behavior of a function.

DEFINITION 1.3

Let v be a locally integrable function. The Hardy–Littlewood maximal function $\mathcal{M}v(z)$ is defined as

$$\mathcal{M}v(z) = \sup \fint_Q |v(y,s)|\mathrm{d}y\mathrm{d}s,$$

where the sup is taken over all square cubes Q in $\mathbb{R}^{n+1}$ containing $z = (x,t)$.

It is well known that the maximal functions satisfy strong p–p estimate for any $1 < p < \infty$ and weak 1–1 estimate (see [33]).

Lemma 1.4 (*see* [4,5,18,23,27,28,33,34]). *Assume that* $w(z) \in A_p$ *for some* $p > 1$ *and* $g \in L^p_w(\mathbb{R}^{n+1})$. *Then we have*

(1)

$$\|\mathcal{M}g\|_{L^p_w(\mathbb{R}^{n+1})} \leq C\|g\|_{L^p_w(\mathbb{R}^{n+1})}.$$

(2)

$$w\left(\left\{z \in \mathbb{R}^{n+1} : \mathcal{M}g(z) > \mu\right\}\right) \leq \frac{C}{\mu^p}\int_{\mathbb{R}^{n+1}} |g|^p\, w(z)\mathrm{d}z.$$

(3)

$$\int_{\mathbb{R}^{n+1}} |g|^p w(z)\mathrm{d}z = p\int_0^\infty \mu^{p-1} w\left(\left\{z \in \mathbb{R}^{n+1} : |g| > \mu\right\}\right)\mathrm{d}\mu.$$

Lemma 1.5 (*see* [4,5,18,23,27,28,33,34]). *Assume that* $w \in A_p$ *for some* $p > 1$.

(1) $A_{p_1} \subset A_p$ *for any* $1 < p_1 \leq p < \infty$.
(2) *There exist a small positive constant* $\epsilon_0 < 1$ *and a constant* $C_1 > 1$ *such that*

$$\left(\fint_{Q_r} w(z)^{1+\epsilon_0} dz\right)^{\frac{1}{1+\epsilon_0}} \leq C_1 \fint_{Q_r} w(z)\mathrm{d}z.$$

(3) *There exists a small positive constant* $\sigma > 0$ *such that*

$$C_3\left(\frac{|Q_r|}{|Q_R|}\right)^p \leq \frac{w(Q_r)}{w(Q_R)} \leq C_2\left(\frac{|Q_r|}{|Q_R|}\right)^\sigma,$$

for any cylinders $Q_r \subset Q_R \subset \mathbb{R}^{n+1}$, *where* $C_2 > 1$ *and* $C_3 > 0$.

Furthermore, we have the following results.

Lemma 1.6. *Assume that* $w \in A_p$ *for some* $p > 1$.

(1) *There exists a positive constant* $p_2 \in (1, p)$ *such that*

$$w \in A_{p_2}.$$

(2) *For any cylinder* $Q_r \subset \mathbb{R}^{n+1}$, *we have*

$$L^p_w(Q_r) \subset L^q(Q_r) \quad \text{for } q =: \frac{p}{p_2} \in (1, p).$$

Proof. Since $w \in A_p$, then from Definition 1.2, we have

$$\begin{aligned}
&\left(\fint_{Q_r} w(z)\mathrm{d}z\right)\left(\fint_{Q_r} w(z)^{\frac{-1}{p-1}}\mathrm{d}z\right)^{p-1}\\
&= \left(\fint_{Q_r}\left(w(z)^{\frac{-1}{p-1}}\right)^{1-p}\mathrm{d}z\right)\left(\fint_{Q_r} w(z)^{\frac{-1}{p-1}}\mathrm{d}z\right)^{p-1}\\
&= \left[\left(\fint_{Q_r}\left(w(z)^{\frac{-1}{p-1}}\right)^{-\frac{1}{\frac{p}{p-1}-1}}\mathrm{d}z\right)^{\frac{p}{p-1}-1}\left(\fint_{Q_r} w(z)^{\frac{-1}{p-1}}\mathrm{d}z\right)\right]^{p-1} \leq C
\end{aligned}$$

for any cylinder Q_r in $\mathbb{R}^{n+1}$, which implies that $w(z)^{\frac{-1}{p-1}} \in A_{\frac{p}{p-1}}$. Therefore, from Lemma 1.5(2), we have

$$\left(\fint_{Q_r} w(z)^{-\frac{1+\epsilon_0'}{p-1}} \mathrm{d}z\right)^{\frac{1}{1+\epsilon_0'}} \leq C \fint_{Q_r} w(z)^{\frac{-1}{p-1}} \mathrm{d}z \tag{1.8}$$

for some $\epsilon_0' \in (0, 1)$. Let

$$p_2 = 1 + \frac{p-1}{1+\epsilon_0'} \in (1, p).$$

Then from (1.8) and the fact that $w \in A_p$, we find that

$$\begin{aligned}
&\left(\fint_{Q_r} w(z)\mathrm{d}z\right)\left(\fint_{Q_r} w(z)^{\frac{-1}{p_2-1}} \mathrm{d}z\right)^{p_2-1} \\
&= \left(\fint_{Q_r} w(z)\mathrm{d}z\right)\left(\fint_{Q_r} w(z)^{-\frac{1+\epsilon_0'}{p-1}} \mathrm{d}z\right)^{\frac{p-1}{1+\epsilon_0'}} \\
&\leq C\left(\fint_{Q_r} w(z)\mathrm{d}z\right)\left(\fint_{Q_r} w(z)^{-\frac{1}{p-1}} \mathrm{d}z\right)^{p-1} \leq C,
\end{aligned} \tag{1.9}$$

which implies that $w \in A_{p_2}$. Thus, (1) is true. Furthermore, if $f \in L^p_w(Q_r)$, then from Hölder's inequality and (1.9), we have

$$\begin{aligned}
\int_{Q_r} |f|^{\frac{p}{p_2}} \mathrm{d}z &= \int_{Q_r} |f|^{\frac{p}{p_2}} w(z)^{\frac{1}{p_2}} w(z)^{-\frac{1}{p_2}} \mathrm{d}z \\
&\leq \left(\int_{Q_r} |f|^p w(z)\mathrm{d}z\right)^{\frac{1}{p_2}} \left(\int_{Q_r} w(z)^{-\frac{1}{p_2-1}} \mathrm{d}z\right)^{1-\frac{1}{p_2}} \\
&\leq C\left(\int_{Q_r} |f|^p w(z)\mathrm{d}z\right)^{\frac{1}{p_2}} \left(\frac{|Q_r|}{w(Q_r)}\right)^{\frac{1}{p_2}} \leq C,
\end{aligned}$$

since $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $w > 0$ almost everywhere. This finishes our proof. □

Recently, Byun *et al.* [3] obtained the global $W^{2,p}_w(\Omega)$ estimates for viscosity solutions to the Dirichlet problem for fully nonlinear elliptic equations, where Ω is a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in C^{1,1}$. In this paper, we shall prove the global $W^{2,p}_w(\mathbb{R}^n)$ and $W^{2,1}_{p,w}(\mathbb{R}^n_T)$ estimates for linear second-order nondivergence elliptic and parabolic equations with small BMO coefficients. Now let us state the main results of this work.

Theorem 1.7. *Assume that $w(x) \in A_p$ for some $p > 1$. The coefficients satisfy* (1.3) *and the small BMO condition, and $f \in L^p_w(\mathbb{R}^n)$. Then there exist positive constants δ, λ_0, depending only on n, w, p, Λ, R such that if $\delta < \delta_0$ and $u \in W^{2,p}(\mathbb{R}^n)$ is the solution of*

$$\mathcal{M}u + \lambda u = f, \quad x \in \mathbb{R}^n \tag{1.10}$$

for any $\lambda \geq \lambda_0$, *we have*

$$\int_{\mathbb{R}^n} |u|^p w(x)\mathrm{d}x + \int_{\mathbb{R}^n} |Du|^p w(x)\mathrm{d}x + \int_{\mathbb{R}^n} |D^2u|^p w(x)\mathrm{d}x \\ \leq C \int_{\mathbb{R}^n} |f|^p w(x)\mathrm{d}x, \tag{1.11}$$

where the constant C *does not depend on* u, f.

Theorem 1.8. *Assume that* $w(z) \in A_p$ *for some* $p > 1$, *the coefficients satisfy* (1.3) *and the small BMO condition, and* $f \in L^p_w(\mathbb{R}^n_T)$. *Let* $u \in W^{2,1}_p(\mathbb{R}^n_T)$ *be the solution of* (1.1)–(1.2). *Then there exists a positive constant* δ_0, *depending only on* n, w, p, Λ, R *such that if* $\delta < \delta_0$, *we have*

$$\int_{\mathbb{R}^n_T} |u_t|^p w(z)+|u|^p w(z)+|Du|^p w(z)+|D^2u|^p w(z)\mathrm{d}z \leq C \int_{\mathbb{R}^n_T} |f|^p w(z)\mathrm{d}z, \tag{1.12}$$

where the constant C *does not depend on* u, f.

Remark 1.9. Indeed, if $w \equiv 1$, (1.11) and (1.12) are reduced to the classical L^p estimates.

2. Proof of the main result

In this section, we shall complete the proofs of the main results, Theorems 1.7 and 1.8. Our work was mainly motivated by [4,5,27,28], where the authors studied the local/global weighted gradient estimates for linear/nonlinear elliptic and parabolic equations in the bounded Reifenberg flat domain. The harmonic analysis free approach in [4] is based on the covering/iteration argument which was first developed by Acerbi and Mingione [1,29]. The main approach in [5,27,28] was to make use of harmonic analysis tools such as the maximal function operator which was first developed by Caffarelli and Peral [9].

This section is organized as follows: In § 2.1, we first give an auxiliary result, Theorem 2.1. In § 2.2, motivated by [15,16,20,21], we shall remove the artificial assumption $u \in C_0^\infty(Q_{R/2})$ in the statement of Theorem 2.1 and then complete the proof of the main results.

2.1 *Auxiliary result*

Here we prove the following auxiliary result, which will be a crucial ingredient in the proof of Theorem 1.8. Assume that $Q_{R/2} \equiv Q_{R/2}(x_0, t_0) \subset \mathbb{R}^n_T$, where $R > 0$ is defined in Definition 1.1. In fact, we may as well assume that u and f are vanishing in $\{t < 0\}$ since we only consider the space $\mathbb{R}^n_T$. Let $u \in C_0^\infty(Q_{R/2})$ be the solution of

$$\mathcal{L}_1 u =: u_t - a_{ij}(x,t)u_{x_i x_j} = f, \quad (x,t) \in \mathbb{R}^n_T, \tag{2.1}$$

$$u(x,0) = 0. \tag{2.2}$$

Theorem 2.1. *Under the same assumptions on w, A, f as those in Theorem 1.8, let $u \in C_0^\infty(Q_{R/2})$ be the solution of (2.1)–(2.2). Then there exists a small constant $\delta = \delta(n, w, p, \Lambda, R) > 0$ such that*

$$\int_{\mathbb{R}^n_T} |u_t|^p w(z) + |D^2 u|^p w(z)\mathrm{d}z \le C \int_{\mathbb{R}^n_T} |f|^p w(z)\mathrm{d}z, \tag{2.3}$$

where the constant C does not depend on u, f.

We first give the following Calderón–Zygmund decomposition, which is much influenced by [26].

Lemma 2.2. *Let D be a square cube in $\mathbb{R}^{n+1}$ and $A, B \subset D$ be measurable sets. Assume that $0 < w(A) < \mu w(D)$ for $0 < \mu < 1$. Then there exists a sequence of disjoint square cubes $\{Q_k\}_{k\in\mathbb{N}}$ satisfying*

(1) $w\left(A \setminus \bigcup_{k\in\mathbb{N}} Q_k\right) = 0$,
(2) $w\left(A \cap Q_k\right) > \mu w\left(Q_k\right)$,
(3) $w\left(A \cap \widetilde{Q}_k\right) \le \mu w\left(\widetilde{Q}_k\right)$ if *$\widetilde{Q}_k$ is the predecessor (father) of Q_k.*

Furthermore, if for any Q_k, its predecessor $\widetilde{Q}_k$ satisfies

$$w\left(B \cap \widetilde{Q}_k\right) > \alpha w\left(\widetilde{Q}_k\right) \quad \text{for } 0 < \alpha < 1, \tag{2.4}$$

then we have

$$w(A) \le \frac{\mu}{\alpha} w(B).$$

Proof.

(1) We first divide D into 2^{n+1} (denote by $\{Q_1^{j_1}\}_{j_1=1}^{2^n}$) disjoint square cubes (daughters) with the same size. Choose those square cubes satisfying $w(A \cap Q_1^{j_1}) > \mu w(Q_1^{j_1})$ and continue to divide every remaining square cube $Q_1^{j_1}$ into 2^{n+1} (denote by $\{Q_2^{j_1,j_2}\}_{j_2=1}^{2^n}$) disjoint square cubes with the same size. Therefore, we obtain a sequence of disjoint square cubes $\{Q_k\}_{k\in\mathbb{N}}$ which satisfy (2)–(3) by repeating the above process. If $z \in D \setminus \bigcup_{k\in\mathbb{N}} Q_k$, then there is a sequence of square cubes P_i containing z with the diameters of P_i converging to 0 and

$$w\left(A \cap P_i\right) \le \mu w\left(P_i\right).$$

That is to say,

$$\int_{A\cap P_i} w(z)\mathrm{d}z \le \mu \int_{P_i} w(z)\mathrm{d}z.$$

From the elementary measure theory and the fact that $w(z) > 0$ almost everywhere, we can conclude that $z \in D \setminus A$ for almost every $z \in D \setminus \bigcup_{k\in\mathbb{N}} Q_k$, which implies that

$$\left| A \setminus \bigcup_{k\in\mathbb{N}} Q_k \right| = 0.$$

Thus, from Lemma 1.5(3), we conclude that (1) is true.
(2) Let $\widetilde{Q_k}$ be the predecessor (father) of Q_k. Now we choose a disjoint predecessor subsequence $\{\widetilde{Q_{k_j}}\}$ (still denote by $\{\widetilde{Q_k}\}$) such that

$$\bigcup_{k\in\mathbb{N}} Q_k \subset \bigcup_{k\in\mathbb{N}} \widetilde{Q_k}.$$

Thus, from (1), (3) and the hypothesis (2.4), we deduce that

$$w(A) = \sum_k w(A \cap \widetilde{Q_k}) \leq \mu \sum_k w(\widetilde{Q_k}) < \frac{\mu}{\alpha} \sum_k w(B \cap \widetilde{Q_k}) \leq \frac{\mu}{\alpha} w(B),$$

which completes our proof. □

Next, we shall prove the following important lemma.

Lemma 2.3. *Assume that $w \in A_p$ with $p > 1$. Let $u \in C_0^\infty(Q_{R/2})$ be the solution of (2.1)–(2.2). For any $\mu \in (0, 1)$, there exist $N = N(n, \Lambda, T, q) > 1$ and $\delta = \delta(n, \sigma, \mu) \in (0, 1)$ such that if*

$$\begin{aligned} &w(\{z \in \tilde{Q} : \mathcal{M}(|D^2u|^q)(z) \leq 1\} \cap \{z \in \tilde{Q} : \mathcal{M}(|f|^q)(z) \leq \delta^q\}) \\ &\quad > \frac{1}{2} w(\tilde{Q}), \end{aligned} \tag{2.5}$$

where $\tilde{Q}$ is the so-called predecessor (father) of the square cube Q with $|\tilde{Q}| = |2Q|$ and q is defined in Lemma 1.6, then we have

$$w(\{z \in Q : \mathcal{M}(|D^2u|^q)(z) > N^q\}) \leq \mu w(Q). \tag{2.6}$$

Proof. From Lemma 1.5(3) and (2.5), we find that

$$\begin{aligned} &\frac{|\{z \in \tilde{Q} : \mathcal{M}(|D^2u|^q)(z) \leq 1\} \cap \{z \in \tilde{Q} : \mathcal{M}(|f|^q)(z) \leq \delta^q\}|}{|\tilde{Q}|} \\ &\quad \geq \left[\frac{w(\{z \in \tilde{Q} : \mathcal{M}(|D^2u|^q)(z) \leq 1\} \cap \{z \in \tilde{Q} : \mathcal{M}(|f|^q)(z) \leq \delta^q\})}{C_2 w(\tilde{Q})} \right]^{\frac{1}{\sigma}} \\ &\quad \geq (2C_2)^{-\frac{1}{\sigma}} \in (0, 1), \end{aligned}$$

since $C_2 > 1$ and $\sigma > 0$. That is to say,

$$|\{z \in \tilde{Q} : \mathcal{M}(|D^2u|^q)(z) \leq 1\} \cap \{z \in \tilde{Q} : \mathcal{M}(|f|^q)(z) \leq \delta^q\}| \geq (2C_2)^{-\frac{1}{\sigma}} |\tilde{Q}|.$$

Therefore, there exists $z_0 \in \tilde{Q}$ satisfying

$$\mathcal{M}(|D^2u|^q)(z_0) \leq 1 \quad \text{and} \quad \mathcal{M}(|f|^q)(z_0) \leq \delta^q. \tag{2.7}$$

Since $z_0 \in \tilde{Q} \subset 3Q$, we conclude that

$$⨍_{8\sqrt{n}Q} |D^2u|^q \mathrm{d}z \leq 1 \quad \text{and} \quad ⨍_{8\sqrt{n}Q} |f|^q \mathrm{d}z \leq \delta^q. \tag{2.8}$$

Using the well-known self-improving regularity (see Chapter 5 of [17]), we have

$$⨍_{7\sqrt{n}Q} |D^2u|^{q+\epsilon_0} \mathrm{d}z \leq C, \tag{2.9}$$

where $\epsilon_0 = \epsilon_0(n, q)$.

Case 1: $Q_{R/2} \cap Q = \phi$. Then from the fact that $u \in C_0^\infty(Q_{R/2})$ and the weak 1–1 estimate, we have

$$|\{z \in Q : \mathcal{M}(|D^2u|^q)(z) \geq N^q\}| \leq \frac{C}{N^q} \int_Q |D^2u|^q \mathrm{d}z = 0 \quad \text{for any} \quad N > 1,$$

which implies that (2.6) is true for any $\mu > 0$ by using Lemma 1.5(3).

Case 2: $Q_{R/2} \cap Q \neq \phi$. We first assume that the center and length of the square cube $Q \equiv Q_k$ are $z_i = (x_i, t_i)$ and $2l_i$, respectively. We divide into two cases.
Case 2.1: $l_i \geq \frac{R}{2\sqrt{n}}$. Then $8\sqrt{n}Q \supset Q_{8\sqrt{n}l_i}(z_i) \supset Q_{R/2}$ and $8\sqrt{n}Q \supset Q_{8\sqrt{n}l_i}(z_i) \supset 8Q$. Let v be the solution of

$$\begin{cases} v_t - \overline{a_{ij}}_{Q_R} v_{x_i x_j} = 0 & \text{in } Q_{8\sqrt{n}l_i}(z_i), \\ \qquad\qquad v = u & \text{on } \partial_p Q_{8\sqrt{n}l_i}(z_i). \end{cases}$$

Then from (2.1), $u - v$ satisfies

$$ll(u-v)_t - \overline{a_{ij}}_{Q_R}(u-v)_{x_i x_j} = f + (a_{ij} - \overline{a_{ij}}_{Q_R})u_{x_i x_j} \quad \text{in } Q_{8\sqrt{n}l_i}(z_i),$$
$$u - v = 0 \quad \text{on } \partial_p Q_{8\sqrt{n}l_i}(z_i).$$

Since $u \in C_0^\infty(Q_{R/2})$ with $Q_{R/2} \subset Q_{8\sqrt{n}l_i}(z_i)$, from the elementary L^p-type estimates, we have

$$\begin{aligned} &\int_{Q_{8\sqrt{n}l_i}(z_i)} |D^2(u-v)|^q \mathrm{d}z \\ &\leq C \int_{Q_{8\sqrt{n}l_i}(z_i)} |f + (a_{ij} - (a_{ij})_{Q_R})u_{x_i x_j}|^q \mathrm{d}z \\ &\leq C \int_{8\sqrt{n}Q} |f|^q \, dz + C \int_{Q_{R/2}} |(a_{ij} - (a_{ij})_{Q_R})u_{x_i x_j}|^q \mathrm{d}z, \end{aligned}$$

which implies that

$$\begin{aligned}
&\int_{Q_{8\sqrt{n}l_i}(z_i)} |D^2(u-v)|^q \mathrm{d}z \\
&\leq C\left\{|8\sqrt{n}Q|\delta^q + \left(\int_{Q_R} |a_{ij} - \overline{a_{ij}}_{Q_R}|^{\frac{q(q+\epsilon_0)}{\epsilon_0}} \mathrm{d}z\right)^{\frac{\epsilon_0}{q+\epsilon_0}} \left(\int_{Q_{R/2}} |D^2 u|^{q+\epsilon_0} \mathrm{d}z\right)^{\frac{q}{q+\epsilon_0}}\right\} \\
&\leq C\{|8\sqrt{n}Q|\delta^q + |Q_R|^{\frac{\epsilon_0}{q+\epsilon_0}} \delta^{\frac{\epsilon_0}{q+\epsilon_0}} |7\sqrt{n}Q|^{\frac{q}{q+\epsilon_0}}\} \\
&\leq C|8\sqrt{n}Q|(\delta^q + \delta^{\frac{\epsilon_0}{q+\epsilon_0}}) \leq C|8\sqrt{n}Q|\delta^{\frac{\epsilon_0}{q+\epsilon_0}} \leq C|Q_{8\sqrt{n}l_i}(z_i)|\delta^{\frac{\epsilon_0}{q+\epsilon_0}},
\end{aligned} \tag{2.10}$$

by using the Hölder's inequalities (2.8) and (2.9). Here we have used the fact that

$$\begin{aligned}
&\left(\fint_{Q_R} |a_{ij} - (a_{ij})_{Q_R}|^{\frac{q(q+\epsilon_0)}{\epsilon_0}} \mathrm{d}z\right)^{\frac{\epsilon_0}{q+\epsilon_0}} \\
&\quad \leq (2\Lambda)^{\frac{q(q+\epsilon_0)-\epsilon_0}{q+\epsilon_0}} \left(\fint_{Q_R} |a_{ij} - (a_{ij})_{Q_R}| \mathrm{d}z\right)^{\frac{\epsilon_0}{q+\epsilon_0}} \leq C\delta^{\frac{\epsilon_0}{q+\epsilon_0}}
\end{aligned} \tag{2.11}$$

as a consequence of Definition 1.1 and (1.3). Actually, (2.10) implies that

$$\fint_{Q_{8\sqrt{n}l_i}(z_i)} |D^2(u-v)|^q \mathrm{d}z \leq C\delta^{\frac{\epsilon_0}{q+\epsilon_0}}. \tag{2.12}$$

Moreover, from (2.7) we know that

$$\left(\fint_{Q_{8\sqrt{n}l_i}(z_i)} |D^2 u|^q \mathrm{d}z\right)^{\frac{1}{q}} \leq C\left(\fint_{8\sqrt{n}Q} |D^2 u|^q \mathrm{d}z\right)^{\frac{1}{q}} \leq C. \tag{2.13}$$

Furthermore, from (2.12), (2.13) and the elementary L^∞_{loc} estimate, we find that

$$\sup_{8Q} |D^2 v|^q \leq N_1^q, \tag{2.14}$$

where $N_1 = N_1(n, \Lambda, T, q) > 1$. Next, we shall prove that

$$\{z \in Q : \mathcal{M}(|D^2 u|^q)(z) > N_2^q\} \subset \{z \in Q : \mathcal{M}(|D^2(u-v)|^q)(z) > N_1^q\}, \tag{2.15}$$

where $N_2^q =: \max\{2^q N_1^q, 6^{n+1}\}$. Actually, from (2.14) we find that

$$|D^2 u|^q \leq 2^{q-1}(|D^2 v|^q + |D^2(u-v)|^q) \leq 2^{q-1}(N_1^q + |D^2(u-v)|^q) \quad \text{for any } z \in 8Q.$$

Let z be a point in $\{z \in Q : \mathcal{M}(|D^2(u-v)|^q)(z) \leq N_1^q\}$. Assume that Q_1 is a square with $z \in Q_1$. If $Q_1 \subset 8Q$, then we have

$$\fint_{Q_1} |D^2 u|^q \mathrm{d}z \leq 2^{q-1}\left(N_1^q + \fint_{Q_1} |D^2(u-v)|^q \mathrm{d}z\right) \leq 2^q N_1^q. \tag{2.16}$$

Moreover, if $Q_1 \not\subset 8Q$, then we have $z \in Q \subset 2Q_1$ and $z_0 \in \tilde{Q} \subset 3Q \subset 6Q_1$. Therefore, from (2.8), we find that

$$\fint_{Q_1} |D^2u|^q \mathrm{d}z \le 6^{n+1} \fint_{6Q_1} |D^2u|^q \mathrm{d}z \le 6^{n+1}. \tag{2.17}$$

Thus, it follows from (2.16) to (2.17) that $\mathcal{M}(|D^2u|^q)(z) \le N_2^q$, which implies that (2.15) is true. So, from (2.12) to (2.15), we conclude that

$$\begin{aligned} &|\{z \in Q : \mathcal{M}(|D^2u|^q)(z) > N_2^q\}| \\ &\quad \le |\{z \in Q : \mathcal{M}(|D^2(u-v)|^q)(z) > N_1^q\}| \\ &\quad \le \frac{1}{N_1^q} \int_Q |D^2(u-v)|^q \mathrm{d}z \le C\delta^{\frac{\epsilon_0}{q+\epsilon_0}} |Q|, \end{aligned}$$

which implies that

$$w(\{z \in Q : \mathcal{M}(|D^2u|^q)(z) > N_2^q\}) \le C\delta^{\frac{\sigma\epsilon_0}{q+\epsilon_0}} w(Q) \le \mu w(Q)$$

in view of Lemma 1.5(3), by choosing δ small enough satisfying the last inequality.

Case 2.2: $l_i < \frac{R}{2\sqrt{n}}$. Then $2\sqrt{n}l_i < R$ and $2Q \subset Q_{2\sqrt{n}l_i}(z_i) \subset 2\sqrt{n}Q$. Let v be the solution of

$$\begin{cases} v_t - \overline{a_{ij}}_{Q_{2\sqrt{n}l_i}(z_i)} v_{x_ix_j} = 0 & \text{in } Q_{2\sqrt{n}l_i}(z_i), \\ \qquad\qquad v - u = 0 & \text{on } \partial_p Q_{2\sqrt{n}l_i}(z_i). \end{cases}$$

Then from (2.1), $u-v$ satisfies

$$\begin{aligned} (u-v)_t - \overline{a_{ij}}_{Q_{2\sqrt{n}l_i}(z_i)}(u-v)_{x_ix_j} &= f + (a_{ij} - \overline{a_{ij}}_{Q_{2\sqrt{n}l_i}(z_i)})u_{x_ix_j} \quad \text{in } Q_{2\sqrt{n}l_i}(z_i), \\ u - v &= 0 \quad \text{on } \partial_p Q_{2\sqrt{n}l_i}(z_i). \end{aligned}$$

From the elementary L^p-type estimates, we have

$$\begin{aligned} &\left(\fint_{Q_{2\sqrt{n}l_i}(z_i)} |D^2(u-v)|^q \mathrm{d}z\right)^{\frac{1}{q}} \\ &\le C\left(\fint_{Q_{2\sqrt{n}l_i}(z_i)} |f + (a_{ij} - (a_{ij})_{Q_{2\sqrt{n}l_i}(z_i)})u_{x_ix_j}|^q \mathrm{d}z\right)^{\frac{1}{q}} \\ &\le C\left(\fint_{Q_{2\sqrt{n}l_i}(z_i)} |(a_{ij} - (a_{ij})_{Q_{2\sqrt{n}l_i}(z_i)})u_{x_ix_j}|^q \mathrm{d}z\right)^{\frac{1}{q}} + C\left(\fint_{8\sqrt{n}Q} |f|^q \mathrm{d}z\right)^{\frac{1}{q}}, \end{aligned}$$

which implies that

$$\left(\fint_{Q_{2\sqrt{n}l_i}(z_i)} |D^2(u-v)|^q \mathrm{d}z\right)^{\frac{1}{q}}$$
$$\leq C\left(\fint_{Q_{2\sqrt{n}l_i}(z_i)} |a_{ij} - \overline{a_{ij}}_{Q_{2\sqrt{n}l_i}(z_i)}|^{\frac{q(q+\epsilon_0)}{\epsilon_0}} \mathrm{d}z\right)^{\frac{\epsilon_0}{q(q+\epsilon_0)}} \left(\fint_{Q_{2\sqrt{n}l_i}(z_i)} |D^2u|^{q+\epsilon_0} \mathrm{d}z\right)^{\frac{1}{q+\epsilon_0}}$$
$$+C\delta$$
$$\leq C\left\{\delta + \delta^{\frac{\epsilon_0}{q(q+\epsilon_0)}} \left(\fint_{7\sqrt{n}Q} |D^2u|^{q+\epsilon_0} \mathrm{d}z\right)^{\frac{1}{q+\epsilon_0}}\right\} \leq C\left(\delta + \delta^{\frac{\epsilon_0}{q(q+\epsilon_0)}}\right) \leq C\delta^{\frac{\epsilon_0}{q(q+\epsilon_0)}},$$

by using Hölder's inequalities, (2.8), (2.9) and (2.11). Moreover, from (2.8), we know that

$$\left(\fint_{Q_{2\sqrt{n}l_i}(z_i)} |D^2u|^q \mathrm{d}z\right)^{\frac{1}{q}} \leq C\left(\fint_{8\sqrt{n}Q} |D^2u|^q \mathrm{d}z\right)^{\frac{1}{q}} \leq C.$$

Furthermore, from the two inequalities above and the elementary L^∞_{loc} estimate, we find that

$$\sup_{2Q} |D^2v|^q \leq N_3^q, \tag{2.18}$$

where $N_3 = N_3(n, \Lambda, T, q) > 1$. Since the rest of the proof is totally similar to Case 2.1, we omit the details. Thus we complete the proof. □

Furthermore, we can obtain the following result.

Lemma 2.4. *Assume that $\mu \in (0,1)$ with $C_2\mu^\sigma < 1$ and w, δ, N satisfy the same conditions as those in Lemma 2.3. For any $\lambda > 0$, we have*

$$\begin{aligned} &w(\{z \in \mathbb{R}^{n+1} : \mathcal{M}(|D^2u|^q)(z) \geq \lambda N^q\}) \\ &\quad\leq 2C_2\mu^\sigma[w(\{z \in \mathbb{R}^{n+1} : \mathcal{M}(|D^2u|^q)(z) > \lambda\}) \\ &\qquad+ w(\{z \in \mathbb{R}^{n+1} : \mathcal{M}(|f|^q)(z) > \lambda\delta^q\})]. \end{aligned} \tag{2.19}$$

Proof. Without loss of generality, we may as well assume that $\lambda = 1$. Let

$$\mathbb{R}^{n+1} = \bigcup_{i=1}^{\infty} \overline{D_i},$$

where $\{D_i\}$ is a sequence of disjoint square cubes. Moreover, from weak 1–1 estimate and L^p-type estimates for (2.1)–(2.2), we conclude that

$$|\{z \in \mathbb{R}^{n+1} : \mathcal{M}(|D^2u|^q)(z) \geq N^q\}| \leq \frac{C}{N^q}\|D^2u\|_{L^q(\mathbb{R}^{n+1})} \leq C\|f\|_{L^q(\mathbb{R}^{n+1})}.$$

Since supp $f \subset Q_{R/2}$, we have

$$|\{z \in D_i : \mathcal{M}(|D^2u|^q)(z) \geq N^q\}| \leq \mu|D_i|$$

by selecting $|D_i|$ large enough for $i \in \mathbb{N}$. Furthermore, from Lemma 1.5(3), we have

$$w(\{z \in D_i : \mathcal{M}(|D^2u|^q)(z) \geq N^q\}) \leq C_2\mu^\sigma w(D_i).$$

We denote

$$A = \{z \in D_i : \mathcal{M}(|D^2u|^q)(z) \geq N^q\}$$

and

$$B = \{z \in D_i : \mathcal{M}(|D^2u|^q)(z) > 1\} \cup \{z \in D_i : \mathcal{M}(|f|^q)(z) > \delta^q\}.$$

Then $A, B \subset D_i$ and $w(A) \leq C_2\mu^\sigma w(D_i)$ with $C_2\mu^\sigma < 1$. Therefore, it follows from Lemma 2.2 that there exists a sequence of disjoint square cubes $\{Q_k\}$ satisfying

(1) $w(A \setminus \bigcup_{k\in\mathbb{N}} Q_k) = 0$,
(2) $w(A \cap Q_k) > C_2\mu^\sigma w(Q_k)$,
(3) $w(A \cap \widetilde{Q_k}) \leq C_2\mu^\sigma w(\widetilde{Q_k})$, if $\widetilde{Q_k}$ is the predecessor(father) of Q_k.

If $w(\widetilde{Q_k} \cap B) \leq \frac{1}{2}w(\widetilde{Q_k})$, where $\widetilde{Q_k}$ is the predecessor of Q_k, then we obtain

$$w(\{z \in \widetilde{Q_k} : \mathcal{M}(|D^2u|^q)(z) \leq 1\} \cap \{z \in \widetilde{Q_k} : \mathcal{M}(|f|^q)(z) \leq \delta^q\}) > \frac{1}{2}w(\widetilde{Q_k}).$$

Furthermore, it follows from Lemma 2.3 that

$$w(A \cap Q_k) \leq w(\{z \in Q_k : \mathcal{M}(|D^2u|^q)(z) \geq N^q\}) \leq C_2\mu^\sigma w(Q_k).$$

So, we get a contradiction with (2) and then know that $w(\widetilde{Q_k} \cap B) > \frac{1}{2}w\left(\widetilde{Q_k}\right)$. Finally, we can use Lemma 2.2 again to get that

$$w(A) \leq 2C_2\mu^\sigma w(B),$$

which implies that

$$\begin{aligned} &w(\{z \in D_i : \mathcal{M}(|D^2u|^q)(z) \geq N^q\}) \\ &\quad \leq 2C_2\mu^\sigma[w(\{z \in D_i : \mathcal{M}(|D^2u|^q)(z) > 1\}) \\ &\qquad + w(\{z \in D_i : \mathcal{M}(|f|^q)(z) > \delta^q\})]. \end{aligned}$$

Thus, we obtain the desired estimate (2.19). This completes our proof. □

Now we are ready to prove Theorem 2.1.

Proof. From Lemma 1.4(3), Lemmas 1.6 and 2.4, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}} |\mathcal{M}(|D^2u|^q)|^{\frac{p}{q}} w(z)\mathrm{d}z \\
&= \frac{p}{q}\int_0^\infty (N^q\lambda)^{\frac{p}{q}-1} w(\{z\in\mathbb{R}^{n+1} : \mathcal{M}(|D^2u|^q)(z) > N^q\lambda\})d[N^q\lambda] \\
&\le \frac{2C_2p\mu^\sigma}{q}\int_0^\infty (N^q\lambda)^{\frac{p}{q}-1} w(\{z\in\mathbb{R}^{n+1} : \mathcal{M}(|D^2u|^q)(z) > \lambda\})d[N^q\lambda] \\
&\quad + \frac{2C_2p\mu^\sigma}{q}\int_0^\infty (N^q\lambda)^{\frac{p}{q}-1} w(\{z\in\mathbb{R}^{n+1} : \mathcal{M}(|f|^q)(z) > \lambda\delta^q\})d[N^q\lambda] \\
&\le C_4\mu^\sigma \int_{\mathbb{R}^{n+1}} |\mathcal{M}(|D^2u|^q)|^{\frac{p}{q}} w(z)\mathrm{d}z + C_5\int_{\mathbb{R}^{n+1}} |\mathcal{M}(|f|^q)|^{\frac{p}{q}} w(z)\mathrm{d}z
\end{aligned}$$

for any $\mu\in(0,1)$ with $C_2\mu^\sigma < 1$, where $C_4 = C_4(p,n)$ and $C_5 = C_5(p,n,\mu,\sigma)$. Then choosing a suitable μ such that $C_4\mu^\sigma < 1$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}} |\mathcal{M}(|D^2u|^q)|^{\frac{p}{q}} w(z)\mathrm{d}z \le C\int_{\mathbb{R}^{n+1}} |\mathcal{M}(|f|^q)|^{\frac{p}{q}} w(z)\mathrm{d}z \\
&\le C\int_{\mathbb{R}^{n+1}} |f|^p\, w(z)\mathrm{d}z
\end{aligned}$$

in view of Lemma 1.4(1). From the fact that $|D^2u|^q(z) \le \mathcal{M}(|D^2u|^q)(z)$, we can obtain

$$\int_{\mathbb{R}^{n+1}} |D^2u|^p w(z)\mathrm{d}z \le C\int_{\mathbb{R}^{n+1}} |f|^p w(z)\mathrm{d}z.$$

Thus from (2.1) we observe that

$$\int_{\mathbb{R}^{n+1}} |u_t|^p w(z)\mathrm{d}z \le C\int_{\mathbb{R}^{n+1}} |f|^p w(z)\mathrm{d}z,$$

which completes the proof of Theorem 2.1, since $u \in C_0^\infty(Q_{R/2})$ with $Q_{R/2} =: Q_{R/2}(x_0,t_0)\subset\mathbb{R}^n_T$. □

2.2 *Final proof*

Observe that we have the artificial assumption $u\in C_0^\infty(Q_{R/2})$ in the statement of Theorem 2.1. Next, motivated by [15,16,20,21], we shall show that this assumption is redundant and can be removed.

Lemma 2.5. *Under the same assumptions on w, A, f as those in Theorem 1.8, let $u\in C_0^\infty(Q_{R/2})$ be the solution of*

$$\mathcal{L}u + \lambda u = f,\quad (x,t)\in\mathbb{R}^n_T, \tag{2.20}$$

$$u(x,0) = 0. \tag{2.21}$$

Then there exist positive constants δ_0, λ_0, depending only on n, w, p, Λ, R, such that if $\delta < \delta_0$, we have

$$\lambda^p \int_{\mathbb{R}^n_T} |u|^p w(z)\mathrm{d}z + \lambda^{\frac{p}{2}} \int_{\mathbb{R}^n_T} |Du|^p w(z)\mathrm{d}z + \int_{\mathbb{R}^n_T} |D^2u|^p w(z)\mathrm{d}z \\ \leq C \int_{\mathbb{R}^n_T} |f|^p w(z)\mathrm{d}z$$

for any $\lambda \geq \lambda_0$.

Proof. We divide into two cases.

Case 1: $\mathcal{L} = \mathcal{L}_1$. Assume that $\zeta(y) \in C_0^\infty(-R/2, R/2)$ is a cut-off function. We define

$$\tilde{u}(x, y, t) = \tilde{u}(x, y, t) = u(x, t)\zeta(y)\cos(\sqrt{\lambda}y)$$

and

$$\tilde{\mathcal{L}}_1\tilde{u}(x, y, t) = \mathcal{L}_1\tilde{u}(x, t) - \tilde{u}_{yy}, \quad (z, t) = (x, y, t) \in \mathbb{R}^{n+1}_T.$$

It is easy to check that the coefficients matrix $\tilde{A}_{(n+1)\times(n+1)} = \begin{bmatrix} A_{n\times n} & 0 \\ 0 & 1 \end{bmatrix}$ of the operator $\tilde{\mathcal{L}}_1$ still satisfies (1.3) and the small BMO condition. Moreover, from (2.20) to (2.21), we can find that

$$\begin{cases} \tilde{\mathcal{L}}_1\tilde{u}(x, y, t) = \tilde{f}, & \text{in } \mathbb{R}^{n+1}_T, \\ \tilde{u}(x, y, 0) \quad = 0, \end{cases} \tag{2.22}$$

where

$$\tilde{f} = \zeta f \cos(\sqrt{\lambda}y) + u\zeta'' \cos(\sqrt{\lambda}y) - 2\sqrt{\lambda}u\zeta' \sin(\sqrt{\lambda}y).$$

For the sake of convenience, we write

$$D^2\tilde{u}(x, y, t) = \{D^2_{xx}\tilde{u}, D_x\tilde{u}_y, \tilde{u}_{yy}\}.$$

Without loss of generality, we may assume that $w(x, y, t) \equiv w(x, t) = w(z)$. Using Theorem 2.1, we see that

$$\int_{\mathbb{R}^{n+1}_T} |D^2\tilde{u}(x, y, t)|^p w(z)\mathrm{d}x\mathrm{d}y\mathrm{d}t \leq C \int_{\mathbb{R}^{n+1}_T} |\tilde{f}(x, y, t)|^p w(z)\mathrm{d}x\mathrm{d}y\mathrm{d}t. \tag{2.23}$$

Since $D^2_{xx}\tilde{u}(x, y, t) = \zeta(y)\cos(\sqrt{\lambda}y)D^2_{xx}u(x, t)$ and $\zeta(y) \in C_0^\infty(-R/2, R/2)$, we find that

$$\int_{\mathbb{R}^n_T} |D^2_{xx}u|^p w(z)\mathrm{d}z$$

$$
\begin{aligned}
&= \left(\int_{\mathbb{R}} (|\zeta(y) \cos(\sqrt{\lambda} y)|)^p \mathrm{d}y \right)^{-1} \int_{\mathbb{R}^{n+1}_T} |D^2_{xx} \tilde{u}(x, y, t)|^p w(z) \mathrm{d}x \mathrm{d}y \mathrm{d}t \\
&\le C \int_{\mathbb{R}^{n+1}_T} |D^2_{xx} \tilde{u}(x, y, t)|^p w(z) \mathrm{d}x \mathrm{d}y \mathrm{d}t \\
&\le C \int_{\mathbb{R}^{n+1}_T} |D^2 \tilde{u}(x, y, t)|^p w(z) \mathrm{d}x \mathrm{d}y \mathrm{d}t. \qquad (2.24)
\end{aligned}
$$

Similarly, we compute that

$$
\begin{aligned}
&\int_{\mathbb{R}^n_T} |D^2_x u|^p w(z) \mathrm{d}z \\
&\le C \int_{\mathbb{R}^{n+1}_T} |\zeta(y) \sin(\sqrt{\lambda} y) D_x u|^p w(z) \mathrm{d}x \mathrm{d}y \mathrm{d}t \\
&\le C \sum_{i=1}^{n} \int_{\mathbb{R}^{n+1}_T} \frac{1}{\lambda^{p/2}} |\tilde{u}_{x_i y}(x, y, t) - u_{x_i} \zeta'(y) \cos(\sqrt{\lambda} y)|^p w(z) \mathrm{d}x \mathrm{d}y \mathrm{d}t \\
&\le \frac{C}{\lambda^{p/2}} \left(\int_{\mathbb{R}^{n+1}_T} |D_x \tilde{u}_y(x, y, t)|^p w(z) \mathrm{d}x \mathrm{d}y \mathrm{d}t + \int_{\mathbb{R}^n_T} |D_x u|^p w(z) \mathrm{d}z \right),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\lambda^{\frac{p}{2}} \int_{\mathbb{R}^n_T} |D_x u|^p w(z) \mathrm{d}z &\le C \int_{\mathbb{R}^{n+1}_T} |D_x \tilde{u}_y(x, y, t)|^p w(z) \mathrm{d}x \mathrm{d}y \mathrm{d}t \\
&\le C \int_{\mathbb{R}^{n+1}_T} |D^2 \tilde{u}(x, y, t)|^p w(z) \mathrm{d}x \mathrm{d}y \mathrm{d}t \qquad (2.25)
\end{aligned}
$$

by choosing $\lambda \ge \lambda_0$ large enough. Similarly, by choosing $\lambda \ge \lambda_0$ large enough, we find that

$$
\begin{aligned}
\lambda^p \int_{\mathbb{R}^n_T} |u|^p w(z) \mathrm{d}z &\le C \int_{\mathbb{R}^{n+1}_T} |\tilde{u}_{yy}(x, y, t)|^p w(z) \mathrm{d}x \mathrm{d}y \mathrm{d}t \\
&\le C \int_{\mathbb{R}^{n+1}_T} |D^2 \tilde{u}(x, y, t)|^p w(z) \mathrm{d}x \mathrm{d}y \mathrm{d}t, \qquad (2.26)
\end{aligned}
$$

since

$$
|u| \le \frac{C}{\lambda} |\tilde{u}_{yy}(x, t) - u(x)(\zeta''(y) \cos(\sqrt{\lambda} y) - 2\sqrt{\lambda} \zeta'(y) \sin(\sqrt{\lambda} y))|.
$$

Therefore, combining (2.23)–(2.26) and taking $\lambda \ge \lambda_0 > 0$ large enough, we conclude that

$$
\begin{aligned}
&\int_{\mathbb{R}^n_T} |D^2_{xx} u|^p w(z) \mathrm{d}z + \lambda^{\frac{p}{2}} \int_{\mathbb{R}^n_T} |D_x u|^p w(z) \mathrm{d}z + \lambda^p \int_{\mathbb{R}^n_T} |u|^p w(z) \mathrm{d}z \\
&\quad \le C \int_{\mathbb{R}^{n+1}_T} |D^2 \tilde{u}(x, y, t)|^p w(z) \mathrm{d}x \mathrm{d}y \mathrm{d}t \le C \int_{\mathbb{R}^{n+1}_T} |\tilde{f}(x, y, t)|^p w(z) \mathrm{d}x \mathrm{d}y \mathrm{d}t. \qquad (2.27)
\end{aligned}
$$

Furthermore, since $\zeta(y) \in C_0^\infty(-R/2, R/2)$ and

$$-\sqrt{\lambda}u(z)\zeta'(y)\sin(\sqrt{\lambda}y) = u(z)((\zeta'(y)\cos(\sqrt{\lambda}y))_y - \zeta''(y)\cos(\sqrt{\lambda}y)),$$

we can find that

$$\int_{\mathbb{R}_T^{n+1}} |\tilde{f}(x, y, t)|^p w(z)\mathrm{d}x\mathrm{d}y\mathrm{d}t \leq C\left(\int_{\mathbb{R}_T^n} |f|^p w(z)\mathrm{d}z + \int_{\mathbb{R}_T^n} |u|^p w(z)\mathrm{d}z\right). \tag{2.28}$$

So, from (2.27) to (2.28), we can complete the proof by taking $\lambda \geq \lambda_0 > 0$ large enough.
Case 2: $\mathcal{L} \neq \mathcal{L}_1$. Since $u \in C_0^\infty(Q_{R/2})$ satisfies (2.20)–(2.21), we find that

$$\mathcal{L}_1 u + \lambda u = f - b_i(x, t)u_{x_i} - c(x, t)u, \quad (x, t) \in \mathbb{R}_T^n, \tag{2.29}$$

$$u(x, 0) = 0. \tag{2.30}$$

Therefore, from the result of Case 1, we obtain

$$\lambda^p \int_{\mathbb{R}_T^n} |u|^p w(z)\mathrm{d}z + \lambda^{\frac{p}{2}} \int_{\mathbb{R}_T^n} |Du|^p w(z)\mathrm{d}z + \int_{\mathbb{R}_T^n} |D^2u|^p w(z)\mathrm{d}z$$
$$\leq C\left(\int_{\mathbb{R}_T^n} |f|^p w(z)\mathrm{d}z + \int_{\mathbb{R}_T^n} |Du|^p w(z)\mathrm{d}z + \int_{\mathbb{R}_T^n} |u|^p w(z)\mathrm{d}z\right).$$

Finally, we can complete the proof by taking $\lambda \geq \lambda_0 > 0$ large enough. □

Furthermore, we can remove the additional condition $u \in C_0^\infty(Q_{R/2})$ and then obtain the following result.

Lemma 2.6. *Under the same assumptions on w, A, f as those in Theorem 1.8, let $u \in W_p^{2,1}(\mathbb{R}_T^n)$ be the solution of (2.20)–(2.21). Then there exist positive constants δ_0, λ_0, depending only on n, w, p, Λ, R, such that if $\delta < \delta_0$, we have*

$$\lambda^p \int_{\mathbb{R}_T^n} |u|^p w(z)\mathrm{d}z + \lambda^{\frac{p}{2}} \int_{\mathbb{R}_T^n} |Du|^p w(z)\mathrm{d}z + \int_{\mathbb{R}_T^n} |D^2u|^p w(z)\mathrm{d}z$$
$$\leq C \int_{\mathbb{R}_T^n} |f|^p w(z)\mathrm{d}z$$

for any $\lambda \geq \lambda_0$.

Proof. By an elementary approximation argument, we may assume that

$$u^0(x, t) = u(x, t)\rho(x - x_0, t - t_0) =: u(x, t)\rho^0(x, t),$$

where $(x_0, t_0) \in \mathbb{R}_T^n$ and $\rho^0(x, t) \in C_0^\infty(Q_{R/2})$. Then we compute

$$\mathcal{L}u^0(x, t) + \lambda u^0(x, t) = f^0(x, t),$$

where

$$f^0(x,t) =: f\rho^0 + u(\rho^0)_t - a_{ij}(\rho^0)_{x_i}u_{x_j} - a_{ij}(\rho^0)_{x_j}u_{x_i} - u(\rho^0)_{x_ix_j} + b_i u(\rho^0)_{x_i}.$$

Assume that $\lambda \geq \lambda_0 > 1$. From Lemma 2.5, we find that

$$\begin{aligned}
&\lambda^p \int_{\mathbb{R}^n_T} |u^0|^p w(z)\mathrm{d}z + \lambda^{\frac{p}{2}} \int_{\mathbb{R}^n_T} |Du^0|^p w(z)\mathrm{d}z + \int_{\mathbb{R}^n_T} |D^2u^0|^p w(z)\mathrm{d}z \\
&\quad \leq C \int_{\mathbb{R}^n_T} |f^0|^p w(z)\mathrm{d}z \\
&\quad \leq C \int_{\mathbb{R}^n_T} (|f\cdot\chi_{Q_{R/2}(x_0,t_0)}|^p + |u\cdot\chi_{Q_{R/2}(x_0,t_0)}|^p + |Du\cdot\chi_{Q_{R/2}(x_0,t_0)}|^p) w(z)\mathrm{d}z.
\end{aligned}$$

Since

$$D_{x_i}u^0 = \rho^0 D_{x_i}u + uD_{x_i}\rho^0$$

and

$$D_{x_ix_j}u^0 = \rho^0 D_{x_ix_j}u + uD_{x_ix_j}\rho^0 + D_{x_j}uD_{x_i}\rho^0 + D_{x_i}uD_{x_j}\rho^0,$$

we deduce that

$$\begin{aligned}
&\lambda^p \int_{\mathbb{R}^n_T} |\rho^0|^p|u|^p w(z)\mathrm{d}z + \lambda^{\frac{p}{2}} \int_{\mathbb{R}^n_T} |\rho^0|^p|Du|^p w(z)\mathrm{d}z + \int_{\mathbb{R}^n_T} |\rho^0|^p|D^2u|^p w(z)\mathrm{d}z \\
&\quad \leq C \int_{\mathbb{R}^n_T} (|f\cdot\chi_{Q_{R/2}(x_0,t_0)}|^p + \lambda^{\frac{p}{2}}|u\cdot\chi_{Q_{R/2}(x_0,t_0)}|^p + |Du\cdot\chi_{Q_{R/2}(x_0,t_0)}|^p) w(z)\mathrm{d}z.
\end{aligned}$$

Finally, by integrating with (x_0, t_0) over $\mathbb{R}^n_T$ and choosing proper $\lambda \geq \lambda_0 > 1$, we can obtain the desired result. □

Now we are ready to complete the proofs of the main results, Theorems 1.8 and 1.7.

Proof of Theorem 1.8. Fix $\lambda = \lambda_0$ and let

$$w(x,t) = u(x,t)\mathrm{e}^{-\lambda_0 t}.$$

From (1.1) to (1.2), we find that w is the solution of

$$\mathcal{L}w + \lambda_0 w = \mathrm{e}^{-\lambda_0 t} f$$

with $w(x, 0) = 0$. Therefore, it follows from Lemma 2.6 that

$$\int_{\mathbb{R}^n_T} |u|^p w(z)\mathrm{d}z + \int_{\mathbb{R}^n_T} |Du|^p w(z)\mathrm{d}z + \int_{\mathbb{R}^n_T} |D^2u|^p w(z)\mathrm{d}z \leq C \int_{\mathbb{R}^n_T} |f|^p w(z)\mathrm{d}z.$$

This completes our proof. □

Proof of Theorem 1.7. Let

$$v(x,t)=\zeta\left(\frac{t}{k}\right)u(x) \quad \text{and} \quad w(z)=w(x),$$

where $k\in\mathbb{N}$ and $\zeta(t)\in C_0^\infty(\mathbb{R})$ is a cut-off function. From (1.10), $v(x,t)$ satisfies

$$\mathcal{L}v+\lambda v=\zeta\left(\frac{t}{k}\right)f+\frac{1}{k}\zeta'\left(\frac{t}{k}\right)u.$$

Therefore, from Lemma 2.6, we have

$$\begin{aligned}
&\lambda^p\int_{\mathbb{R}^n_T}\left|\zeta\left(\frac{t}{k}\right)u\right|^p w(x)\mathrm{d}z+\lambda^{\frac{p}{2}}\int_{\mathbb{R}^n_T}\left|\zeta\left(\frac{t}{k}\right)Du\right|^p w(x)\mathrm{d}z\\
&\quad+\int_{\mathbb{R}^n_T}\left|\zeta\left(\frac{t}{k}\right)D^2u\right|^p w(x)\mathrm{d}z\\
&\quad\le C\int_{\mathbb{R}^n_T}\left|\zeta\left(\frac{t}{k}\right)f\right|^p w(x)\mathrm{d}z+\frac{C}{k}\int_{\mathbb{R}^n_T}\left|\zeta'\left(\frac{t}{k}\right)u\right|^p w(x)\mathrm{d}z
\end{aligned}$$

for any $\lambda\ge\lambda_0$. Selecting $\lambda=\lambda_0$ and $k>0$ large enough, we can obtain (1.12). This completes our proof. □

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