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# Positive integer solutions of certain diophantine equations

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**Abstract.** In this study, the diophantine equations $x^2 - 32B_nxy - 32y^2 = \pm 32^r$, $x^4 - 32B_nxy - 32y^2 = \pm 32^r$ and $x^2 - 32B_nxy - 32y^4 = \pm 32^r$ are considered and determined when these equations have positive integer solutions. Moreover, all positive integer solutions of these diophantine equations in terms of balancing and Lucas-balancing numbers are also found out.

**Keywords.** Diophantine equations; balancing numbers; balancers; Lucas balancing numbers.

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## 1. Introduction

An important area of number theory is devoted to finding solutions of equations where the solutions are restricted to the set of integers. Diophantine equations get their name from Diophantus of Alexandria and they are algebraic equations for which rational or integer solutions are sought. Many researchers considered the diophantine equations $ax^2 + bxy + cy^2 = d$ for different fixed integer values of $a$, $b$, $c$ and $d$ [2–4]. In [6], Keskin *et al.* considered the values of $a = 1$, $b = -5F_n$, $c = -5(-1)^n$, $d = \pm 5^r$ which gives the diophantine equation

$$x^2 - F_nxy - 5(-1)^ny^2 = \pm 5^r,$$

and they determined the values of $n$ for which the equation has positive integer solutions $x$ and $y$. In a subsequent paper, Keskin *et al.* [7] considered $a = 1$, $b = -L_n$, $c = (-1)^n$, $d = \pm 5^r$ and determined when the equations $x^2 - L_nxy + (-1)^ny^2 = \pm 5^r$ have positive integer solutions. They also found all the positive integer solutions of the given equations in terms of Fibonacci and Lucas numbers.

Another interesting number sequence which is closely related to the sequence of Fibonacci numbers is the sequence of balancing numbers. In 1999, Behera *et al.* [1] introduced balancing numbers, which satisfy the diophantine equation $1 + 2 + \cdots + (x - 1) =$

 

$(x+1)+(x+2)+\cdots+(x+y)$, where $y$ is the balancer corresponding to the balancing number $x$. Balancing numbers are initiated from the recurrence relation $B_{n+1} = 6B_n - B_{n-1}$ with $B_0 = 0$ and $B_1 = 1$, where $B_n$ is the $n$-th balancing number. For each balancing number $B$, the square root of $8B^2+1$ also generates a sequence of numbers $\{C_n\}$ called as sequence of Lucas balancing numbers, where $C_n$ denotes the $n$-th Lucas balancing number. Lucas balancing numbers are recursively defined as $C_{n+1} = 6C_n - C_{n-1}$ with $C_0 = 1$ and $C_1 = 3$, where $C_n$ is the $n$-th Lucas balancing number. The Pell and associated Pell numbers are well known number sequences whose recurrence relations are given by

$$P_{n+1} = 2P_n + P_{n-1},\ P_0 = 0,\ P_1 = 1 \quad \text{and}$$
$$Q_{n+1} = 2Q_n + Q_{n-1},\ Q_0 = 1,\ Q_1 = 2.$$

Here, $P_n$ and $Q_n$ denote the $n$-th Pell and $n$-th Lucas Pell respectively. The balancing and Lucas balancing numbers are closely associated with Pell and their associated Pell numbers [11]. In particular, $P_{2n} = 2B_n$ and $Q_{2n} = C_n$. An interesting connection concerning balancing numbers, Pell numbers and Lucas Pell numbers is that, the $n$-th balancing number equals the product of $n$-th Pell number and $n$-th Lucas Pell number [11].

Consider the generalized Fibonacci sequence $\{U_n\}$ and generalized Lucas sequence $\{V_n\}$ as

$$U_{n+1} = PU_n - QU_{n-1} \ \text{ for } \ n \geq 1 \ \text{ with } \ U_0 = 0, U_1 = 1, \tag{1.1}$$
$$V_{n+1} = PV_n - QV_{n-1} \ \text{ for } \ n \geq 1 \ \text{ with } \ V_0 = 2, V_1 = P, \tag{1.2}$$

where $P$ and $Q$ are positive integers. Moreover, the generalized Fibonacci and Lucas numbers for negative subscripts are given as

$$U_{-n} = \frac{-U_n}{Q^n} \ \text{ and } \ V_{-n} = \frac{V_n}{Q^n}, \tag{1.3}$$

with $n \geq 1$. Noting that, for $P = 1$ and $Q = -1$ in (1.1) and (1.2), $U_n = F_n$ and $V_n = L_n$. For $P = 2$ and $Q = -1$, $\{U_n\}$ and $\{V_n\}$ are Pell sequence $\{P_n\}$ and Pell–Lucas sequence $\{Q_n\}$ respectively. Further, substituting $P = 6$ and $Q = 1$ into (1.1) and (1.2) gives $u_n = B_n$ and the sequence $v_n = 2C_n$. This means that both the sequence of balancing numbers and the sequence $\{v_n\}$ are special cases of the generalized Fibonacci and Lucas sequences for the case when $P = 6$ and $Q = 1$.

In the diophantine equation $ax^2 + bxy + cy^2 = d$, setting $a = 1$, $b = -8B_n$, $c = -2$ and $d = \pm 2^r$, Karaatli *et al.* [5] solved the diophantine equation $x^2 - 8B_nxy - 2y^2 = \pm 2^r$ and got the solution in terms of Pell and Pell–Lucas numbers. In this article, we consider the diophantine equation $x^2 - 32B_nxy - 32y^2 = \pm 32^r$ by setting $a = 1$, $b = -32B_n$, $c = -32$ and $d = \pm 32^r$. The aim of this article is to find some new positive integer solutions of the equation given in the title in terms of balancing and Lucas balancing numbers.

## 2. Preliminaries

The following are some known results regarding the sequences $\{P_n\}$, $\{Q_n\}$, $\{B_n\}$ and $\{v_n\}$ that will be needed later.

The following result is available in [5], that provides the information about the sum of the squares of balancing numbers.

*Lemma* 2.1. *Let $B_k$ denotes the $k$-th balancing number. Then*

$$\sum_{k=1}^{n} B_k^2 = \frac{1}{32}[B_{2n+1} - (2n+1)].$$

An immediate consequence of Lemma 2.1 arises in the following results.

COROLLARY 2.2

*If $n$ be an odd positive integer*, $B_n \equiv n (\mathrm{mod}\ 32)$.

COROLLARY 2.3

*Let $n$ be an even positive integer. Then* $B_n \equiv 3n (\mathrm{mod}\ 32)$.

COROLLARY 2.4

*Let $C_n$ be the $n$-th Lucas balancing number. Then*

$$C_n = \begin{cases} 1 \bmod 16, & \text{if } 2|n, \\ 3 \bmod 16, & \text{if } 2 \nmid n. \end{cases}$$

The following results are shown in [5].

*Lemma* 2.5. *Let $n$ be a positive integer. There is no balancing number except* 1 *satisfying the equation* $B_n = x^2$.

*Lemma* 2.6. *There is no positive integer $x$ such that* $v_n = x^2$.

*Lemma* 2.7. *If $n \geq 0$ and $x > 0$ are integers such that* $v_n = 2x^2$, *then* $(n, x) = (0, 1)$.

*Lemma* 2.8. *There is no positive integer $x$ such that* $B_n = v_m x^2$.

*Lemma* 2.9. *There is no positive integer $x$ such that* $B_n = 2v_m x^2$.

*Lemma* 2.10. *If $n \geq 1$, then the equation $P_n = x^2$ has positive solutions* $(n, x) = (1, 1)$ *or* $(7, 13)$, *where $P_n$ is the $n$-th Pell number.*

The following results are available in [12]. For $m, n \in Z$ and $n \geq 1$, the following holds.

*Lemma* 2.11. *$B_n$ divides $B_m$ if and only if $n \mid m$.*

*Lemma* 2.12. *$C_n$ divides $B_m$ if and only if $n \mid m$ and $\frac{m}{n}$ is an even integer.*

*Lemma* 2.13. *$C_n$ divides $C_m$ if and only if $n \mid m$ and $\frac{m}{n}$ is an odd integer.*

*Lemma* 2.14. *For any $n$, $4 \nmid v_n$.*

We state the following results from [10] and [8], respectively.

*Lemma* 2.15. *Let $P > 0$ and $Q = -1$. Then $U_n = wx^2$ with $w \in 1, 2, 3, 6$ and $n \leq 2$ except when $(P, n, w) = (2, 4, 3), (2, 7, 1), (4, 4, 2), (1, 12, 1), (1, 3, 2), (1, 4, 3), (1, 6, 2)$, or $(24, 4, 3)$.*

*Lemma* 2.16. *The generalized Lucas sequence has at most one non-trivial square-class. Furthermore, if $P \equiv 2 (\text{mod } 4)$, then we have no non-trivial square-class except $(v_1, v_2) = (338, 114242)$ when $P = 338$, $Q = 1$. If $P \equiv 0 (\text{mod } 4)$, then we have no non-trivial square-classes when $2 \nmid mn$ or $2|(m, n)$.*

## 3. Some new results of balancing and Lucas balancing numbers

In this section, we establish some new results which will be used subsequently. Following results are valid for balancing and Lucas balancing numbers.

*Lemma* 3.1. *$(B_n, v_n/v_m) = 1$ or $2$ and $(B_n, v_n/B_m) = 1$ or $2$.*

*Proof.* Suppose $(B_n, v_n/v_m) = k$ implies that $k|B_n$ and $k|v_n/v_m$. Then $k|B_n$ and $k|v_n$, but $(B_n, v_n) = 1$ or 2. Hence $(B_n, v_n/v_m) = 1$ or 2. Similarly, $(B_n, v_n/B_m) = 1$ or 2. □

**Theorem 3.2.** *$B_n = B_m x^2$ has no solution except $(n, x) = (1, 1)$.*

*Proof.* Let $B_n = B_m x^2$. Then $B_n/B_m = x^2$ which follows that $m|n$. Therefore $n = mk$ for some positive integer $k$. Then two cases will arise, i.e. $k$ is even or odd. Suppose $k$ is even, then $k = 2t$ for some integer $t$. It follows that $B_{2mt} = B_m x^2$. Since $B_{2n} = 2B_n C_n = B_n v_n$, we have $B_{2mt} = B_{mt} v_{mt}$. By virtue of Lemma 3.1,

$$d = \left(B_{mt}, \frac{v_{mt}}{B_m}\right) = 1 \text{ or } 2.$$

For $d = 1$, $B_{mt} = a^2$ and $v_{mt} = b^2 B_m$ for some integers $a$ and $b$. Then by Lemma 2.5, $x = 1$ is the only solution. Now for $d = 2$, $B_{mt} = 2a^2$ and $v_{mt} = 2b^2 B_m$. As $P_{2n} = 2B_n$, $P_{2mt} = (2a)^2$. Therefore by Lemma 2.10, $P_{2mt} = (2a)^2$ has positive solutions $(2mt, 2a) = (1, 1)$ or $(7, 13)$, which is not possible as $2mt$ is even. Suppose $k$ is odd. If $m$ is odd, then $B_{mk} = B_m x^2$ which implies that $B_{mk} (\text{mod } 32) = B_m x^2 (\text{mod } 32)$. By virtue of Corollary 2.2, $mk (\text{mod } 32) = mx^2 (\text{mod } 32)$, i.e. $x^2 = k$, where $k \equiv 1, 3, 5, 7 (\text{mod } 8)$, but $x^2 \equiv 0, 1, 4 (\text{mod } 8)$, which is not possible. In a similar manner, it can be shown when $m$ is even. Thus $B_n = B_m x^2$ has no solution except $x = 1$. □

*Lemma* 3.3. *$2v_m = v_n x^2$ has no solution for any $n$.*

*Proof.* Since $(v_n, v_m) = 2$, $x$ is even. This implies that $8|v_n x^2$. So $8|2v_m$ which implies that $4|v_m$, which is not possible as $2 \nmid C_m$. □

## 4. Positive integer solutions of some diophantine equations

In this section, we consider the diophantine equations $x^2 - 32B_nxy - 32y^2 = \pm 32^r$, $x^4 - 32B_nxy - 32y^2 = \pm 32^r$ and $x^2 - 32B_nxy - 32y^4 = \pm 32^r$ and find the positive integer solutions of these equations.

### 4.1 *Positive integer solutions of the diophantine equations* $x^2 - 32B_nxy - 32y^2 = \pm 32^r$

Consider the diophantine equation $x^2 - 32B_nxy - 32y^2 = \pm 32^r$, where $n, r \geq 0$. Setting $u = x - 16B_ny$ and $v = C_ny$, this diophantine equation reduces to $u^2 - 32v^2 = \pm 32^r$. Then the following result can be easily obtained by induction using the identities $Q_n^2 - 8P_n^2 = 4(-1)^n$ and $C_n^2 - 8B_n^2 = 1$.

**Theorem 4.1.** *Let $k \geq 0$ be an integer, then all positive integer solutions of the equation $u^2 - 32v^2 = 32^k$ are given by*

$$(u, v) = \begin{cases} \left(2^{\frac{5k}{2}} C_m,\ 2^{\frac{5k-2}{2}} B_m\right), & \text{if } k \text{ is even;} \\ \left(2^{\frac{5k+1}{2}} P_{2m+1},\ 2^{\frac{5k-7}{2}} Q_{2m+1}\right), & \text{if } k \text{ is odd,} \end{cases}$$

*and all positive integer solutions of the equation $u^2 - 32v^2 = -32^k$ are given by*

$$(u, v) = \begin{cases} \left(2^{\frac{5k-2}{2}} Q_{2m+1}, 2^{\frac{5k-4}{2}} P_{2m+1}\right), & \text{if } k \text{ is even;} \\ \left(2^{\frac{5k+3}{2}} B_m,\ 2^{\frac{5k-5}{2}} C_m\right), & \text{if } k \text{ is odd,} \end{cases}$$

*with $m \geq 0$.*

Using the above interesting facts, we have the following results:

**Theorem 4.2.** *All positive integer solutions of $x^2 - 32B_nxy - 32y^2 = 32^k$, where $k$ is even are given by $(x, y) = (2^{\frac{5k}{2}} \frac{C_{m+n}}{C_n}, 2^{\frac{5k-2}{2}} \frac{B_m}{C_n})$ with $m \geq 1$ and $n|m$ and $\frac{m}{n}$ is even. For $k$ is odd, all positive integer solutions of the equation $x^2 - 32B_nxy - 32y^2 = 32^k$ exist only if $n = 0$, and are given by $(x, y) = (2^{\frac{5k+1}{2}} P_{2m+1}, 2^{\frac{5k-7}{2}} Q_{2m+1})$.*

*Proof.* Since $k$ is even, by virtue of Theorem 4.1,

$$u = x - 16B_ny = 2^{5k/2}C_m, \quad v = C_ny = 2^{\frac{5k-2}{2}} B_m.$$

Putting $y = 2^{\frac{5k-2}{2}} \frac{B_m}{C_n}$, we get

$$x - 16B_n 2^{\frac{5k-2}{2}} \frac{B_m}{C_n} = 2^{5k/2} C_m.$$

Further simplification gives

$$x = 2^{\frac{5k}{2}} \frac{8B_nB_m + C_mC_n}{C_n}.$$

Since $C_mC_n + 8B_nB_m = C_{m+n}$ [11], $x = 2^{5k/2}\frac{C_{m+n}}{C_n}$. But these solutions will be positive integer solutions if and only if $C_n|B_m$ and $C_n|C_{m+n}$. This implies that $n|m$ and $\frac{m}{n}$ are even integers, which follows from Lemma 2.12. Further, for $k$ is odd,

$$x - 16B_ny = 2^{\frac{5k+1}{2}}P_{2m+1}, \quad C_ny = 2^{\frac{5k-7}{2}}Q_{2m+1}.$$

Since $C_n = Q_{2n}$, we have

$$Q_{2n}y = 2^{\frac{5k-7}{2}}Q_{2m+1} = 2^{\frac{5k-5}{2}}\frac{Q_{2m+1}}{2}.$$

In [9], McDaniel proved that for $m = 2^am'$, $n = 2^bn'$, where $m', n'$ are odd, $a, b \geq 0$ with $d = (m, n)$. Then

$$(V_m, V_n) = \begin{cases} v_d, & \text{if } a = b, \\ 1 \text{ or } 2, & \text{if } a \neq b. \end{cases}$$

Using this result one can see that $(Q_{2n}, \frac{Q_{2m+1}}{2}) = 1$. $Q_{2n}$ divides $2^{\frac{5k-5}{2}}$ and it is possible only when $n = 0$ and we get the desired result. □

It is noticed that, since $v_n = 2C_n$, all positive integer solutions of $x^2 - 32B_nxy - 32y^2 = 32^k$, where $k$ is even, are given by $(x, y) = (2^{\frac{5k}{2}}\frac{v_{m+n}}{v_n}, 2^{\frac{5k-4}{2}}\frac{B_m}{v_n})$ with $m \geq 1$ and, $n|m$ and $\frac{m}{n}$ are even.

**Theorem 4.3.** *For k is even, all positive integer solutions of the equation* $x^2 - 32B_nxy - 32y^2 = -32^k$ *exist only for* $n = 0$ *and are given by* $(x, y) = (2^{\frac{5k-2}{2}}Q_{2m+1}, 2^{\frac{5k-4}{2}}P_{2m+1})$. *For k is odd, all positive integer solutions of the equation* $x^2 - 32B_nxy - 32y^2 = -32^k$ *are given by* $(x, y) = (2^{\frac{5k+3}{2}}\frac{B_m+n}{C_n}, 2^{\frac{5k-5}{2}}\frac{C_m}{C_n})$ *with* $m \geq 1$ *and* $n|m$ *and,* $\frac{m}{n}$ *are odd integers.*

*Proof.* For $k$ is even, by virtue of Theorem 4.1,

$$u = x - 16B_ny = 2^{\frac{5k-2}{2}}Q_{2m+1}, \quad C_ny = 2^{\frac{5k-4}{2}}P_{2m+1}.$$

In [9], McDaniel proved that for $d = (m, n)$,

$$\begin{cases} (P_m, Q_n) = Q_d, & \text{if } m/d \text{ is even} \\ (P_m, Q_n) = 1, & \text{otherwise.} \end{cases}$$

Using the result one can see that $(Q_{2n}, P_{2m+1}) = 1$. $Q_{2n}$ divides $2^{\frac{5k-4}{2}}$ and it is possible when $n = 0$ and we get the desired result. Further for $k$ is odd, $x - 16B_ny = 2^{\frac{5k+3}{2}}B_m$, $C_ny = 2^{\frac{5k-5}{2}}C_m$. Putting $y = 2^{5k-5/2}\frac{C_m}{C_n}$, we get $x - 16B_n2^{5k-5/2}\frac{C_m}{C_n} = 2^{5k+3}B_m$. Further simplification gives

$$x = 2^{\frac{5k+3}{2}}\frac{B_mC_n + C_mB_n}{C_n}.$$

Since $B_m C_n + C_m B_n = B_{m+n}$, $x = 2^{\frac{5k+3}{2}} \frac{B_{m+n}}{C_n}$. But these solutions will be positive integer solutions if and only if $C_n$ divides $C_m$. This implies that $n|m$ and $\frac{m}{n}$ are odd integers and the result follows. □

### 4.2 *Nonnegative integer solutions of the diophantine equations* $x^4 - 32B_n xy - 32y^2 = \pm 32^r$ *and* $x^2 - 32B_n xy - 32y^4 = \pm 32^r$

The following two results identify the non-negative integer solutions of the diophantine equations $x^4 - 32B_n x^2 y - 32y^2 = \pm 32^r$.

**Theorem 4.4.** *For* $k \equiv 0, 1, 2 \pmod 4$, *the diophantine equation* $x^4 - 32B_n x^2 y - 32y^2 = 32^k$ *has no positive integer solutions* $x$ *and* $y$. *If* $k \equiv 3 \pmod 4$, *then all positive integer solutions of the equation* $x^4 - 32B_n x^2 y - 32y^2 = 32^k$ *are given by* $(x, y) = (2^{\frac{5k+1}{4}}, 2^{\frac{5k-5}{2}})$ *or* $(x, y) = (13.2^{\frac{5k+1}{4}}, 239.2^{\frac{5k-3}{2}})$.

*Proof.* Assume that $k$ is even. Then by virtue of Theorem 4.2, we have

$$(x^2, y) = \left(2^{\frac{5k}{2}} \frac{C_{m+n}}{C_n}, 2^{\frac{5k-2}{2}} \frac{B_m}{C_n}\right)$$

with $m \geq 1$ and, $n|m$ and $\frac{m}{n}$ are even. Hence, we get $x^2 = 2^{\frac{5k}{2}} \frac{C_{m+n}}{C_n}$.
*Case* 1: Let $k \equiv 0 \pmod 4$. We readily obtain from the equation $C_n u^2 = C_{n+m}$ for some $u > 0$. By Lemma 2.13, this is possible only when $n + m = n$, implying that $m = 0$, which contradicts the fact that $m \geq 1$.
*Case* 2: Let $k \equiv 2 \pmod 4$. So, we immediately have $C_n u^2 = 2C_{m+n}$, for some $u > 0$. Since $2 \nmid C_n$, this is impossible.

If $k$ is odd, then by virtue of Theorem 4.2, we have $(x^2, y) = (2^{\frac{5k+1}{2}} P_{2m+1}, 2^{\frac{5k-7}{2}} Q_{2m+1})$ with $m \geq 0$. This implies that $x^2 - 2^{\frac{5k+1}{2}} P_{2m+1}$.
*Case* 1. Let $k \equiv 1 \pmod 4$. Then from $x^2 = 2^{\frac{5k+1}{2}} P_{2m+1}$, we obtain $u^2 = 2P_{2m+1}$ for $u > 0$, which has no positive integer solution.
*Case* 2: Let $k \equiv 3 \pmod 4$. Then from the equation we obtain $u^2 = P_{2m+1}$ for $u > 0$. By Lemma 2.10, we get $2m + 1 = 1$ or $7$, implying that $m = 0$ or $3$. Substituting these values of $m$ into $(x^2, y) = (2^{\frac{5k+1}{2}} P_{2m+1}, 2^{\frac{5k-7}{2}} Q_{2m+1})$ with $m \geq 0$, we conclude that $(x, y) = (2^{\frac{5k+1}{4}}, 2^{\frac{5k-5}{2}})$ or $(13.2^{\frac{5k+1}{4}}, 239.2^{\frac{5k-7}{2}})$. □

**Theorem 4.5.** *The diophantine equation* $x^4 - 32B_n x^2 y - 32y^2 = -32^k$ *has only one positive integer solution* $(x, y) = (2^{\frac{5k}{4}}, 2^{\frac{5k-4}{2}})$ *for* $k \equiv 0 \pmod 4$ *and for* $k \equiv 1, 2, 3 \pmod 4$, *the diophantine equation* $x^4 - 32B_n x^2 y - 32y^2 = -32^k$ *has no solutions.*

*Proof.* Assume that $k$ is even. Then by virtue of Theorem 4.3, it follows that

$$(x^2, y) = \left(2^{\frac{5k-2}{2}} Q_{2m+1}, 2^{\frac{5k-4}{2}} P_{2m+1}\right),$$

with $m \geq 0$. Hence, we get $x^2 = 2^{\frac{5k-2}{2}} Q_{2m+1}$.

*Case* 1. Let $k \equiv 0 \pmod 4$. Hence, we immediately have from $x^2 = 2^{\frac{5k-2}{2}} Q_{2m+1}$ that $2u^2 = Q_{2m+1}$ for $u > 0$. By Theorem 2.15, we get $m = 0$. This yields $(x, y) = (2^{\frac{5k}{4}}, 2^{\frac{5k-4}{2}})$.
*Case* 2: Let $k \equiv 2 \pmod 4$. Then from the equation we obtain $u^2 = Q_{2m+1}$ for $u > 0$. Since $Q_{2m+1}$ is even as $2|Q_n$, it is clear that $u$ is even and therefore $4|Q_{2m+1}$, which is impossible because $Q_n^2 - 8Pn^2 = 4(-1)^n$.

Let $k$ is odd. Then by Theorem 4.3, it follows that

$$(x^2, y) = \left(2^{\frac{5k+3}{2}} \frac{B_{m+n}}{C_n}, 2^{\frac{5k-5}{2}} \frac{C_m}{C_n}\right),$$

with $m \geq 1$ and, $n|m$ and $\frac{m}{n}$ are odd integers.

*Case* 1: Let $k \equiv 1 \pmod 4$. Then from the equation, we obtain $C_n u^2 = B_{m+n}$ for $u > 0$. From Lemma 2.12, it has no solution.
*Case* 2: Let $k \equiv 3 \pmod 4$. Then from the equation, we obtain $C_n u^2 = 2B_{m+n}$ for $u > 0$, which has no solution. □

**Theorem 4.6.** *The diophantine equation $x^2 - 32B_n xy^2 - 32y^4 = 32^k$ has no positive integer solution for $k \equiv 0, 2, 3 \pmod 4$. If $k \equiv 1 \pmod 4$, then the equation has positive integer solutions only when $n = 0$ and the solution is $(x, y) = (2^{\frac{5k+1}{2}}, 2^{\frac{5k-5}{4}})$.*

*Proof.* Assume that $k$ is even. Then by Theorem 4.2, it follows that

$$(x, y^2) = \left(2^{\frac{5k}{2}} \frac{C_{m+n}}{C_n}, 2^{\frac{5k-2}{2}} \frac{B_m}{C_n}\right)$$

with $m \geq 1$ and, $n|m$ and $\frac{m}{n}$ are even. Hence we obtain $C_n y^2 = 2^{\frac{5k-2}{2}} B_m$.

*Case* 1: Let $k \equiv 0 \pmod 4$. Then from the equation, we obtain $B_m = 2C_n u^2 = v_n u^2$, which is impossible by Lemma 2.8.
*Case* 2: Let $k \equiv 2 \pmod 4$. Then we have $B_m = u^2 C_n = u^2 Q_{2n}$, which is impossible as $(B_m, Q_{2n}) = 1$.

Let $k$ be odd. Then the solution will be $(x, y^2) = (2^{\frac{5k+1}{2}} P_{2m+1}, 2^{\frac{5k-7}{2}} Q_{2m+1})$. Hence, we obtain $y^2 = 2^{\frac{5k-7}{2}} Q_{2m+1}$.

*Case* 1: Let $k \equiv 1 \pmod 4$. Then from the equation, we obtain $2u^2 = Q_{2m+1}$ for $u > 0$. By Theorem 2.15, we get $m = 0$. Thus, $x = 2^{\frac{5k+1}{2}}$ and $y = 2^{\frac{5k-5}{4}}$.
*Case* 2: Let $k \equiv 3 \pmod 4$. Then from the equation we obtain $u^2 = Q_{2m+1}$ for some $u > 0$. Since $Q_{2m+1}$ is even, it follows that $2|u$ and therefore $4|Q_{2m+1}$, which is impossible as $4 \nmid Q_n$. □

**Theorem 4.7.** *The diophantine equation $x^2 - 32B_n xy^2 - 32y^4 = -32^k$ has no positive integer solutions for $k \equiv 2, 3 \pmod 4$. For $k \equiv 0, 1 \pmod 4$ the diophantine equation $x^2 - 32B_n xy^2 - 32y^4 = -32^k$ has $(x, y) = (2^{\frac{5k}{2}}, 2^{\frac{5k-4}{4}})$ or $(239.2^{\frac{5k}{2}}, 13.2^{\frac{5k-4}{4}})$.*

*Proof.* Assume that $k$ is even. Then the solution will be $(x, y^2) = (2^{\frac{5k-2}{2}} Q_{2m+1}, 2^{\frac{5k-4}{2}} P_{2m+1})$ with $m \geq 0$. Hence we obtain $y^2 = 2^{\frac{5k-4}{2}} P_{2m+1}$.
*Case* 1: Let $k \equiv 0 \pmod 4$. Then from the equation, we obtain $u^2 = P_{2m+1}$ for some $u > 0$. By Lemma 2.10, we get $2m + 1 = 1$ or 7, i.e. $m = 0$ or 3. If $m = 0$, then we immediately have $x = 2^{\frac{5k}{2}}$ and $y = 2^{\frac{5k-4}{4}}$. If $m = 3$, we obtain $x = 239.2^{\frac{5k}{2}}$ and $13.2^{\frac{5k-4}{4}}$.
*Case* 2: Let $k \equiv 2 \pmod 4$. Then from the equation, we obtain $u^2 = 2.P_{2m+1}$ for some $u > 0$, which is not possible.

Now assume that $k$ is odd. Then the solution will be $(x, y^2) = (2^{\frac{5k+3}{2}} \frac{B_{m+n}}{C_n}, 2^{\frac{5k-5}{2}} \frac{C_m}{C_n})$ with $m \geq 1$ and, $n|m$ and $\frac{m}{n}$ are odd integers.
*Case* 1: Let $k \equiv 1 \pmod 4$. Then from the equation, we obtain $v_n u^2 = v_m$ for some $u > 0$. By Theorem 3.13, this is possible when $n = m$. Hence, we get $x = 2^{\frac{5k+3}{2}} \frac{B_{2n}}{C_n} = 2^{\frac{5k+5}{2}} B2_n$ and $y = 2^{\frac{5k-5}{4}}$.
*Case* 2: Let $k \equiv 3 \pmod 4$. Then from the equation, we obtain $C_{n+m} u^2 = 2C_m$ for $u > 0$, which is impossible by Lemma 3.3. □

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