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# No hexavalent half-arc-transitive graphs of order twice a prime square exist

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**Abstract.** A graph is *half-arc-transitive* if its automorphism group acts transitively on its vertex set and edge set, but not arc set. Let $p$ be a prime. Wang and Feng (*Discrete Math.* **310** (2010) 1721–1724) proved that there exists no tetravalent half-arc-transitive graphs of order $2p^2$. In this paper, we extend this result to prove that no hexavalent half-arc-transitive graphs of order $2p^2$ exist.

**Keywords.** Half-arc-transitive; bi-Cayley graph; vertex transitive; edge transitive.

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## 1. Introduction

All groups considered in this paper are finite, and all graphs are finite, connected, simple and undirected. For group-theoretic and graph-theoretic terminologies not defined here, we refer the reader to [4,24].

For a graph $\Gamma$, let $V(\Gamma)$, $E(\Gamma)$, $A(\Gamma)$ and Aut $(\Gamma)$ be the vertex set, the edge set, the arc set and the full automorphism group of $\Gamma$, respectively. A graph $\Gamma$ is said to be *vertex-transitive*, *edge-transitive* or *arc-transitive* if Aut $(\Gamma)$ acts transitively on $V(\Gamma)$, $E(\Gamma)$ or $A(\Gamma)$, respectively. A graph is said to be *half-arc-transitive* provided that it is vertex- and edge-transitive, but not arc-transitive.

The investigation of half-arc-transitive graphs was initiated by Tutte [20] who proved that a vertex- and edge-transitive graph with odd valency must be arc-transitive. In 1970, Bouwer [2] constructed an infinite family of half-arc-transitive graphs, and proved the existence of half-arc-transitive graphs of every even valency at least 4. After Bouwer's work, many infinite families of half-arc-transitive graphs have been constructed (see, for example, [11,16–18,21]). In 1981, Holt [15] found a half-arc-transitive graph with 27 vertices and valency 4. Alspach *et al.* [1] proved that Holt's graph is the smallest half-arc-transitive graph. In the literature, tetravalent half-arc-transitive graphs have also received a lot of attention (see [7,19]), and much work focuses on the classification problem. Let $p$ be a prime. Xu [25] classified the tetravalent half-arc-transitive graphs of order $p^3$ and Feng *et al.* [13] classified the tetravalent half-arc-transitive graphs of order $p^4$. It is known that no half-arc-transitive graphs of order $p$ or $p^2$ exist (see [1]). In [8], Cheng and Oxley proved that there exist no half-arc-transitive graphs of order $2p$. In [23], Wang and Feng proved that no tetravalent half-arc-transitive graphs of order $2p^2$ exist. In this paper, we shall extend this result to the hexavalent half-arc-transitive graphs of order $2p^2$.

The following is the main result of this paper.

**Theorem 1.1.** *No hexavalent half-arc-transitive graphs of order twice a prime square exist.*

## 2. Preliminaries

For a finite, simple and undirected graph $\Gamma$ and a subset $B$ of $V(\Gamma)$, the subgraph of $\Gamma$ induced by $B$ will be denoted by $\Gamma[B]$.

Let $\Gamma$ be a connected vertex-transitive graph, and let $G \leq \mathrm{Aut}(\Gamma)$ be vertex-transitive on $\Gamma$. For a $G$-invariant partition $\mathcal{B}$ of $V(\Gamma)$, the *quotient graph* $\Gamma_{\mathcal{B}}$ is defined as the graph with vertex set $\mathcal{B}$ such that for any two vertices $B, C \in \mathcal{B}$, $B$ is adjacent to $C$ if and only if there exist $u \in B$ and $v \in C$ which are adjacent in $\Gamma$. Let $N$ be a normal subgroup of $G$. Then the set $\mathcal{B}$ of orbits of $N$ in $V(\Gamma)$ is a $G$-invariant partition of $V(\Gamma)$. In this case, the symbol $\Gamma_{\mathcal{B}}$ will be replaced by $\Gamma_N$.

For a positive integer $n$, let $\mathbb{Z}_n$ denote the cyclic group of order $n$ and $\mathbb{Z}_n^*$ be the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to $n$. For two groups $M$ and $N$, $N \rtimes M$ denotes a semidirect product of $N$ by $M$. For a subgroup $H$ of a group $G$, denote by $C_G(H)$ the centralizer of $H$ in $G$ and by $N_G(H)$ the normalizer of $H$ of $G$.

Let $G$ be a permutation group on a set $\Omega$ and $\alpha \in \Omega$. Denoted by $G_\alpha$ the stabilizer of $\alpha$ in $G$, that is, the subgroup of $G$ fixing the point $\alpha$. We say that $G$ is *semiregular* on $\Omega$ if $G_\alpha = 1$ for every $\alpha \in \Omega$ and *regular* if $G$ is transitive and semiregular. It is well-known that a graph $\Gamma$ is a *Cayley graph* if it has a group of automorphisms acting regularly on $V(\Gamma)$. If we, instead, require the graph $\Gamma$ to have a group of automorphisms acting semiregularly on $V(\Gamma)$ with two orbits, then we obtain a so-called *bi-Cayley graph*.

The following proposition is easy to obtain (see, for example, [23, Proposition 1.2]).

PROPOSITION 2.1

*Every edge-transitive Cayley graph over an abelian group is also arc-transitive.*

Note that every bi-Cayley graph admits the following concrete realization. Given a group $H$, let $R$, $L$ and $S$ be subsets of $H$ such that $R^{-1} = R$, $L^{-1} = L$ and $R \cup L$ does not contain the identity element of $H$. The *bi-Cayley graph* over $H$ relative to the triple $(R, L, S)$, denoted by $\mathrm{BiCay}(H, R, L, S)$ is the graph having vertex set the union of the right part $H_0 = \{h_0 \mid h \in H\}$ and the left part $H_1 = \{h_1 \mid h \in H\}$, and edge set the union of the right edges $\{\{h_0, g_0\} \mid gh^{-1} \in R\}$, the left edges $\{\{h_1, g_1\} \mid gh^{-1} \in L\}$ and the spokes $\{\{h_0, g_1\} \mid gh^{-1} \in S\}$. If $|R| = |L| = s$, then $\mathrm{BiCay}(H, R, L, S)$ is said to be an *$s$-type bi-Cayley graph* (see [27]). Let $\Gamma = \mathrm{BiCay}(H, R, L, S)$. For $g \in H$, define a permutation $R(g)$ on the vertices of $\Gamma$ by the rule

$$h_i^{R(g)} = (hg)_i, \quad \forall\, i \in \mathbb{Z}_2,\ h \in H.$$

Then $R(H) = \{R(h) \mid h \in H\}$ is a semiregular subgroup of $\mathrm{Aut}(\Gamma)$ which is isomorphic to $H$ and has $H_0$ and $H_1$ as its orbits. When $R(H)$ is normal in $\mathrm{Aut}(\Gamma)$, the bi-Cayley graph

$\Gamma = \mathrm{BiCay}(H, R, L, S)$ will be called a *normal bi-Cayley graph* over $H$ (see [28]). A bi-Cayley graph over an abelian group is simply called a *bi-abelian*.

The following proposition is about half-arc-transitive bi-abelians.

PROPOSITION 2.2 [9, Proposition 1.3]

*Every connected regular edge transitive bi-Cayley graph* $\Gamma = \mathrm{BiCay}(H, R, L, S)$ *over an abelian group is vertex-transitive. Moreover, if* $\Gamma$ *is half-arc-transitive, then* $R \cup L$ *is non-empty and does not contain involutions* $|R| = |L|$ *is even,* $|S| > 2$, *and the valency of* $\Gamma$ *is at least* 6.

In view of [22], we have the following proposition.

PROPOSITION 2.3 [22, Proposition 2.6]

*Let* $X$ *be a connected half-arc-transitive graphs of valency* $2n$. *Let* $A = \mathrm{Aut}\,(X)$ *and let* $A_u$ *be the stabilizer of* $u \in V(X)$ *in* $A$. *Then each prime divisor of* $|A_u|$ *is a divisor of* $n!$. *In particular, if* $X$ *has valency 6 then* $A_u$ *is a* $\{2, 3\}$*-group.*

By Theorems 1 and 3 of [5], we have the following proposition.

PROPOSITION 2.4 [5]

*Let* $\Gamma$ *be a arc-transitive graph of order* $p$ *and valency* $n$ *with* $p$ *a prime and* $0 < n < p$. *Then* $n$ *is even and divides* $p - 1$ *and* $|\mathrm{Aut}\,(\Gamma)| = np$.

Let $p$ be a prime. The following two propositions are all cubic arc-transitive graphs of order $2p$ and $4p$, respectively.

PROPOSITION 2.5 [12, Proposition 2.8]

*Let* $p$ *be a prime and let* $\Gamma$ *be a connected cubic arc-transitive graph of order* $2p$. *Then* $\Gamma$ *is* 1-, 2-, 3- *or* 4-*regular. Furthermore,*

(1) $\Gamma$ *is* 1-*regular if and only if* $\Gamma$ *is isomorphic to the graph* $\mathrm{Cay}\,(D_{2p}, \{a, ab, ab^{-1}\})$, *where* $D_{2p} = \langle a, b \mid a^p = b^2 = 1, bab = 1\rangle$, $p \geq 13$ *and* $p - 1$ *is a multiple of* 3. *In this case, the Cayley graph* $\mathrm{Cay}\,(D_{2p}, \{a, ab, ab^{-\lambda}\})$ *is normal and it is independent of the choice of an element* $\lambda$ *of order* 3 *in* $\mathbb{Z}_p^*$.
(2) $\Gamma$ *is* 2-*regular if and only if* $\Gamma$ *is isomorphic to the complete graph* $K_4$ *of order* 4.
(3) $\Gamma$ *is* 3-*regular if and only if* $\Gamma$ *is isomorphic to the complete bipartite graph* $K_{3,3}$ *of order* 6 *or the Petersen graph* $O_3$ *of order* 10.
(4) $\Gamma$ *is* 4-*regular if and only if* $\Gamma$ *is isomorphic to the Heawood graph of order* 14.

PROPOSITION 2.6 [26, Theorem 1.3]

*Let* $\Gamma$ *be a connected cubic arc-transitive graph of order* $4p$, $p$ *a prime. Then* $\Gamma$ *is one of the following:* $Q_3$, *the* 3*-dimensional cube;* $D_{20}$, *the dodecahedron;* $C_{28}$, *the Coxeter graph; and* $GP(10, 3)$, *the generalized Peterson graph, which is also the standard double cover of Petersen graph.*

## 3. Proof of Theorem 1.1

Suppose to the contrary, $\Gamma$ is a hexavalent half-arc-transitive graph of order $2p^2$ for a prime $p$. As the smallest half-arc-transitive graph has order 27, we can assume that $p \geq 5$. Let $A = \mathrm{Aut}\,(\Gamma)$ and let $P$ be a Sylow $p$-subgroup of $A$. By Proposition 2.3, the vertex-stabilizer $A_v$ of $v \in V(\Gamma)$ in $A$ is a $\{2, 3\}$-group and $P_v = P \cap A_v = 1$. Since $|A| = 2p^2|A_v|$ is a $\{2, 3, p\}$-group, we have $|P| = p^2$. It follows that every orbit of $P$ on $V(\Gamma)$ has length $|P : P_v| = |P| = p^2$, and so $\Gamma$ is a bi-Cayley graph on $P$.

Suppose that $\Gamma = \mathrm{BiCay}(P, R, L, S)$ is a hexavalent half-arc-transitive bi-Cayley graph over $P$. Since $|P| = p^2$, $P$ is abelian. By Proposition 2.2, we have $R \cup L \neq \emptyset$, $|R| = |L|$ is even and $|S| > 2$. As $|R| + |S| = 6$, it follows that $|R| = 2$ and $|S| = 4$, namely, $\Gamma$ is a 2-type bi-Cayley graph over $P$. If $p = 5$ or 7, then by using Magma [3], $\Gamma$ is not a half-arc-transitive. In the rest of the proof, we shall assume that $p \geq 11$. For convenience of the statement, we shall identify $R(P)$ with $P$. Let $G \leq A$ be such that $P \leq G$ and $G$ is vertex-, edge- but not arc-transitive on $\Gamma$. We shall first prove three claims.

*Claim* 1. $P$ is non-normal in $G$.

Suppose the contrary, $P \trianglelefteq G$. Then the orbits of $P$ are two blocks of imprimitivity of $G$ on $V(\Gamma)$. Since $G$ is edge-transitive on $\Gamma$, each induced subgraph $\Gamma[P_i]$ is null. This implies that $\Gamma$ is a 0-type bi-Cayley graph over $P$, which is contrary to the fact that $\Gamma$ is a 2-type bi-Cayley graph over $P$.

*Claim* 2. $G$ has no non-trivial normal 2-subgroups.

Let $M = O_2(G)$ be the largest normal 2-subgroup of $G$. Consider the quotient graph $\Gamma_M$, and let $Y$ be the kernel of $G$ that acts on $V(\Gamma_M)$. If $M \neq 1$, then $|V(\Gamma_M)| = p^2$. The valency of $\Gamma_M$ must be even. Since $\Gamma$ is edge-transitive, we have the valency of $\Gamma_M$ which divides the valency of $\Gamma$. So $\Gamma_M$ has valency 2 or 6. Let $\Lambda$ and $\Lambda'$ be two adjacent orbits of $M$. If the valency of $\Gamma_M$ is 2, then $\Gamma[\Lambda \cup \Lambda']$ has valency 3. This is impossible because $|\Lambda| = 2$. Then $\Gamma_M$ has valency 6, $Y = M$ is semiregular and $|M| = 2$. Then $PM/M$ is regular on $V(\Gamma_M)$, and hence $PM$ is regular on $V(\Gamma)$. It follows that $\Gamma$ is a Cayley graph on $PM$. As $M \cong \mathbb{Z}_2$ and $M \trianglelefteq G$, $M$ is contained in the center of $G$, and so $PM$ is abelian. This is contrary to Proposition 2.1.

*Claim* 3. $G$ has no normal subgroups with orbits of length $p$ or $2p$.

Let $N$ be a normal subgroup of $G$ such that $N$ has an orbit of length $\ell$ on $V(\Gamma)$. Consider the quotient graph $\Gamma_N$, and let $K$ be the kernel of $G$ which acts on $V(\Gamma_N)$. Then $|V(\Gamma_N)| = \frac{2p^2}{\ell}$ and $G/K \leq \mathrm{Aut}\,(\Gamma_N)$ is vertex- and edge-transitive on $\Gamma$. Furthermore, the valency of $\Gamma_N$ divides the valency of $\Gamma$ because $\Gamma$ is edge-transitive, and so the valency of $\Gamma_N$ is 2, 3 or 6. Let $\Delta$ and $\Delta'$ be two adjacent orbits of $N$. Then the induced subgraph $\Gamma[\Delta \cup \Delta']$ is also edge-transitive.

Suppose first that $\ell = p$. If $\Gamma_N$ has valency 2, then $\Gamma[\Delta \cup \Delta']$ is a cubic edge-transitive graph of order $2p$. Since $p \geq 11$, by Proposition 2.5, we have $p \equiv 1 \pmod 3$ and $\Gamma[\Delta \cup \Delta']$ is 1-arc-regular. Let $K^*$ be the subgroup of $K$ fixing $\Delta$ point-wise. If $K^*$ does not fix $\Delta'$ point-wise, then for any $u, v \in \Delta'$, if $u, v$ are contained in the same orbit of $K^*$, then $u, v$ would have the same neighborhood in $\Gamma[\Delta \cup \Delta']$. This implies that $\Gamma[\Delta \cup \Delta']$ is of girth 4. However, this is impossible because $\Gamma[\Delta \cup \Delta']$ has girth 6 by Proposition 2.5. Thus, $K^*$ fixes $\Delta'$ point-wise. By the connectivity of $\Gamma$, we have that $K^*$ fixes every orbit of $N$ point-wise and so $K^* = 1$. This implies that $K \leq \mathrm{Aut}\,(\Gamma[\Delta \cup \Delta']) \cong D_{2p} \rtimes \mathbb{Z}_3$. Since

$K$ fixes each orbit of $N$, we have $K \leq \mathbb{Z}_p \rtimes \mathbb{Z}_3$. As $\Gamma_N$ is a cycle of length $2p$, one has $\mathrm{Aut}(\Gamma_N) \cong D_{4p}$, and so $G/K \leq D_{4p}$. Consequently, $|G| \mid 12p^2$. Since $p \geq 11$ and $p \equiv 1 \pmod 3$, one has $p > 11$, and so the Sylow $p$-subgroup $P$ of $G$ is normal. This is contrary to Claim 1.

If $\Gamma_N$ has valency 3, then since $G/K$ is vertex- and edge-transitive on $\Gamma_N$, again by Proposition 2.5, we have $p \equiv 1 \pmod 3$ and $\Gamma_N$ is 1-arc-regular. It follows that $|G/K| = 6p$. On the other hand, $\Gamma[\Delta \cup \Delta']$ is a cycle of length $2p$. By the connectedness of $\Gamma$, it is easy to see that $K$ acts faithfully on each orbit of $N$. This means that $K \leq \mathrm{Aut}(\Gamma[\Delta \cup \Delta']) \cong D_{4p}$. Since $K$ fixes each orbit of $N$, one has $K \leq D_{2p}$. As $|G/K| = 6p$, one has $|G| \leq 12p^2$. Since $p \geq 11$ and $p \equiv 1 \pmod 3$, one has $p > 11$, and so the Sylow $p$-subgroup $P$ of $G$ is normal, contrary to Claim 1.

If $\Gamma_N$ has valency 6, then $K = N$ is semiregular on $V(\Gamma)$. Since $p \geq 11$, by [8, Theorem 2.4 and Table 1], we have $p \geq 13$ and $\Gamma_N$ is a 1-arc-regular graph of order $2p$. Then $|G/N| \mid 12p$ and $|G| \mid 12p^2$. This forces that $P \trianglelefteq G$, a contradiction.

Suppose now that $\ell = 2p$. Then $|V(\Gamma_N)| = p$ and so $\Gamma_N$ is a Cayley graph on a cyclic group of order $p$. Then the valency of $\Gamma_N$ is 2 or 6. If $\Gamma_N$ has valency 2, then let $V(\Gamma_N) = \{\Delta_0, \Delta_1, \ldots, \Delta_{p-1}\}$ be the set of $N$-orbit such that $\Delta_i$ is adjacent to $\Delta_{i+1}$ for each $i \in \mathbb{Z}_p$. It follows that $G/K \lesssim \mathrm{Aut}(\Gamma_N) \cong D_{2p}$ and $G/K$ is vertex- and edge-transitive on $\Gamma_N$. Therefore, for any $0 \leq i,\ j \leq p-1$, $\Gamma[\Delta_i \cup \Delta_{i+1}] \cong \Gamma[\Delta_j \cup \Delta_{j+1}]$. If $\Gamma[\Delta_i \cup \Delta_{i+1}]$ is connected, then $\Gamma[\Delta_i \cup \Delta_{i+1}]$ is a cubic vertex- and edge-transitive graph of order $4p$, and so it is also arc-transitive. However, by Proposition 2.6, no cubic arc-transitive graphs of order $4p$ with $p \geq 11$ exist, a contradiction. Thus, $\Gamma[\Delta_i \cup \Delta_{i+1}]$ is disconnected. Clearly, each component of $\Gamma[\Delta_i \cup \Delta_{i+1}]$ is a cubic arc-transitive graph with order dividing $4p$, so each component of $\Gamma[\Delta_i \cup \Delta_{i+1}]$ has order $2p$. Let $C_i^0, C_i^1$ be the two components of $\Gamma[\Delta_i \cup \Delta_{i+1}]$. Since $p \geq 11$, again by Proposition 2.5, we have $p \equiv 1 \pmod 3$, and $C_i^0$ and $C_i^1$ are 1-arc-regular and of girth 6. This implies that $K$ acts faithfully on $\Delta_i$, and so $K \leq \mathrm{Aut}(\Gamma[\Delta_i \cup \Delta_{i+1}]) \cong (\mathrm{Aut}(C_i^0) \times \mathrm{Aut}(C_i^1)) \rtimes \mathbb{Z}_2$. Since $C_i^0$ and $C_i^1$ are cubic 1-arc-regular graphs of order $2p$, one has $\mathrm{Aut}(C_i^0) \cong \mathrm{Aut}(C_i^1)) \cong \mathbb{Z}_p \rtimes \mathbb{Z}_6$. This implies that the Sylow $p$-subgroup of $\mathrm{Aut}(\Gamma[\Delta_i \cup \Delta_{i+1}])$ is normal, and so $K$ would have a normal Sylow $p$-subgroup, say $M$. Then $M$ is characteristic in $K$ and hence is normal in $G$. Since $p > 11$ and each orbit of $K$ has length $2p$, each orbit of $M$ would have length $p$. However, by the above arguments, this is impossible.

If $\Gamma_N$ has valency 6, then $K = N$ is semiregular on $V(\Gamma_N)$ and so $|K| = |N| = 2p$. By Proposition 2.1, $\Gamma_N$ is arc-transitive, and by Prosition 2.4, $\Gamma_N$ is 1-arc-regular and $6 \mid (p-1)$. So, $|G/N| \mid 6p$. This implies that $|G| \mid 12p^2$, which means $P \trianglelefteq G$, contrary to Claim 1. This completes the proof of Claim 3.

Now we are ready to complete the proof of the theorem. Let $N$ be a minimal normal subgroup of $A$. Then $N > 1$. Assume that $N$ is solvable. Since $|V(\Gamma)| = 2p^2$, $N$ is an elementary abelian 2- or $p$-group. If $N \cong \mathbb{Z}_p^2$, then $N = P$ is a normal Sylow $p$-subgroup of $A$, contrary to Claim 1. If $N \cong \mathbb{Z}_p$, then the length of each $N$-orbit is $p$, contrary to Claim 3. If $N$ is a 2-group, then by Claim 2, $N = 1$, a contradiction.

Assume now that $N$ is non-solvable. Then $N \cong T^m$, where $T$ is a non-abelian simple $\{2, 3, p\}$-group. As $p \geq 11$, by [6,14], a non-abelian simple $\{2, 3, p\}$-group is either $\mathrm{PSL}(2, 17)$ or $\mathrm{PSL}(3, 3)$. Since $p^3 \nmid |A|$, we have $m = 1$ or 2. If $m = 1$, then $N \cong T$, and the length of each orbit of $N$ would be $p$ or $2p$, contrary to Claim 3. If $m = 2$, then $N \cong T \times T$, and $P$ is a Sylow $p$-subgroup of $N$. Thus, $N$ has at most two orbits on $V(\Gamma)$. If $N$ has two orbits, then $N$ has the same orbits with $P$, and since $\Gamma$ is edge-transitive, each orbit of $N$ does not contain edges. This is impossible because $\Gamma$ is a 2-type bi-Cayley graph over $P$. Therefore, $N$ is transitive on $V(\Gamma)$. Since $\Gamma$ is half-arc-transitive, one has $A(\Gamma) = (u, v)^A \cup (v, u)^A$ for each $\{u, v\} \in E(\Gamma)$. Let $\Sigma = (V(\Gamma), (u, v)^A)$. Let $\Sigma^+(u) = \{v \mid (u, v) \in A(\Sigma)\}$ and $\Sigma^-(u) = \{v \mid (v, u) \in A(\Sigma)\}$. Then $|\Sigma^+(u)| = |\Sigma^-(u)| = 3$ and $\Gamma(u) = \Sigma^+(u) \cup \Sigma^-(u)$. Since $|\Sigma^+(u)| = 3$, $A_u$ acts primitively on $\Sigma^+(u)$. Since $1 \neq N_u \trianglelefteq A_u$, $N_u$ is transitive on $\Sigma^+(u)$. It follows that $N$ is transitive on $A(\Sigma)$, and so $N$ is transitive on $E(\Gamma)$. Therefore, $N$ is transitive on $V(\Gamma)$ and $E(\Gamma)$ but not on $A(\Gamma)$. Let $R$ be a normal subgroup of $N$ that is isomorphic to $T$. Then the length of each $R$-orbit is $p$ or $2p$. This is contrary to Claim 3. □

## Appendix

In this section, we give the Magma codes to check whether there hexavalent half-arc-transitive 2-type bi-Cayley graphs over $P$ exist or not, where $|P| = p^2$ and $p = 5$ or 7.

*Case* 1. $P \cong \mathbb{Z}_{p^2}$.

```
for p in {5,7} do
G⟨a, c⟩:=Group⟨a, c | a^(p^2), c^2, a*c=c*a⟩;
I:=sub⟨G|⟩;
phi, H, K:=CosetAction(G,I);
x:=phi(a); z:=phi(c);
H1:=sub⟨H | x⟩;
V:={};
for t in H1 do
Include(~V,t*z);
Include(~V, t);
end for;
for i1, i2, i3, i4, i5 in {1..p^2-1} do
E:={};
for h in H1 do
Include(~E, {h, x^i1*h});
Include(~E, {h, x^-i1*h});
Include(~E, {h*z, x^i2*h*z});
Include(~E, {h*z, x^-i2*h*z});
Include(~E, {h, h*z});
Include(~E, {h, x^i3*h*z});
Include(~E, {h, x^i4*h*z});
Include(~E, {h, x^i5*h*z});
end for;
X:=Graph⟨V|E⟩;
if IsConnected(X) and Valence(X) eq 6 and IsEdgeTransitive(X) and IsVertexTransitive(X)
and IsSymmetric(X) eq false then
print p, i1, i2, i3, i4, i5;
end if;
end for;
end for;
```

Case 2. $P \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

```
for p in {5,7} do
G⟨a,b,c⟩:=Group⟨a,b,c | a^p, b^p, c^2, a*b=b*a, a*c=c*a, b*c=c*b⟩;
I:=sub⟨G |⟩;
phi, H, K:=CosetAction(G,I);
x:=phi(a); y:=phi(b); z:=phi(c);
H1:=sub⟨H|x, y⟩;
V:={};
for t in H1 do
Include(~V,t*z);
Include(~V, t);
end for;
for i1, i2, i3, i4, i5, j1, j2, j3, j4, j5 in {0..p-1} do
if x^i1*y^j1 ne x^0*y^0 and x^i2*y^j2 ne x^0*y^0 and x^i3*y^j3 ne x^0*y^0 and x^i4*y^j4
ne x^0*y^0 and x^i5*y^j5 ne x^0*y^0 then
E:={};
for h in H1 do
Include(~E, {h, x^i1*y^j1*h});
Include(~E, {h, x^-i1*y^-j1*h});
Include(~E, {h*z, x^i2*y^j2*h*z});
Include(~E, {h*z, x^-i2*y^-j2*h*z});
Include(~E, {h, h*z});
Include(~E, {h, x^i3*y^j3*h*z});
Include(~E, {h, x^i4*y^j4*h*z});
Include(~E, {h, x^i5*y^j5*h*z});
end for;
X:=Graph⟨V|E⟩;
if IsConnected(X) and Valence(X) eq 6 and IsEdgeTransitive(X) and IsVertexTransitive(X)
and IsSymmetric(X) eq false then
print p, i1, i2, i3, i4, i5, j1, j2, j3, j4, j5;
end if;
end if;
end for;
end for;
```

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