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On 3-way combinatorial identities

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Abstract. In this paper, we provide combinatorial meanings to two generalized basic series with the aid of associated lattice paths. These results produce two new classes of infinite 3-way combinatorial identities. Five particular cases are also discussed. These particular cases provide new combinatorial versions of Göllnitz–Gordon identities and Göllnitz identity. Seven q-identities of Slater and five q-identities of Rogers are further explored using the same combinatorial object. These results are an extension of the work of Goyal and Agarwal (*Utilitas Math*. **95** (2014) 141–148), Agarwal and Rana (*Utilitas Math*. **79** (2009) 145–155), and Agarwal (*J. Number Theory* **28** (1988) 299–305).

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1. Introduction

An effective elementary tool for studying partitions and compositions is the graphical representation. Such a representation is very useful when applications of partitions and compositions are considered. Many combinatorial objects, viz., Ferrer's graph, lattice paths, etc. are used to graphically represent partitions and compositions. Recently, in [18], an attempt was made to give an explicit formula to get an exact number of such representations for restricted n-color compositions and in [12], a relation between the number of integer partitions and compositions of an integer under certain conditions was studied. Several basic series identities had been interpreted combinatorially using ordinary partitions, colored partitions, Frobenius partitions, lattice paths etc. in [1, 2, 4, 6, 8, 11, 20]. Anand and Agarwal [9] introduced a new combinatorial object which they called associated lattice paths. This tool is very useful in graphical representation of an n-color partition as well as an n-color composition. In this paper, we are using this combinatorial object as an aid to interpret some Rogers and Slater's q-identities combinatorially. Before we state our main results we, first recall some definitions from [3].

DEFINITION 1.1

A *partition with* ‘$(n+t)$ *copies of* n’, $t \geq 0$, is a partition in which a part of size n, $n \geq 0$, can come in $(n+t)$ different colors denoted by subscripts: $n_1, n_2, \ldots, n_{n+t}$. Note that zeros are permitted if and only if t is greater than or equal to one. Also, in no partition zeros are permitted to repeat.

Remark 1.1. We note that if we take $t = 0$, then these are nothing but the n-color partitions. If order of the parts is considered then these are nothing but the n-color compositions.

DEFINITION 1.2

The *weighted difference* of two parts g_k, h_l $(g \geq h)$ is defined by $g - h - k - l$ and is denoted by $((g_k - h_l))$.

In [4], the *lattice paths* are described as follows.

DEFINITION 1.3

All paths will be of finite length lying in the first quadrant. They will begin on the y-axis and terminate on the x-axis. Only three moves are allowed at each step:

northeast: from (i, j) to $(i+1, j+1)$.
southeast: from (i, j) to $(i+1, j-1)$, only allowed if $j > 0$.
horizontal: from $(i, 0)$ to $(i+1, 0)$, only allowed along x-axis.

All our lattice paths are either empty or terminate with a southeast step: from $(i, 1)$ to $(i+1, 0)$.

In describing lattice paths, the following terminologies are used:

Peak: Either a vertex on the y-axis which is followed by a southeast step or a vertex preceded by a northeast step and followed by a southeast step.
Valley: A vertex preceded by a southeast step and followed by a northeast step. Note that a southeast step followed by a horizontal step followed by a northeast step does not constitute a valley.
Mountain: A section of the path which starts on either the x- or y-axis, which ends on the x-axis and which does not touch the x-axis anywhere in between the end points. Every mountain has at least one peak and may have more than one.
Plain: A section of the path consisting of only horizontal steps which starts either on the y-axis or at a vertex preceded by a southeast step and ends at a vertex followed by a northeast step.
Height of a vertex is its y-coordinate, *weight* of a vertex is its x-coordinate and *weight of a lattice path* is the sum of the weights of its peaks.

Anand and Agarwal [9] gave the following description of *associated lattice paths*.

DEFINITION 1.4

All paths will be of finite length lying in the first quadrant. They will begin on the y-axis and terminate on the x-axis. Only three moves are allowed at each step:

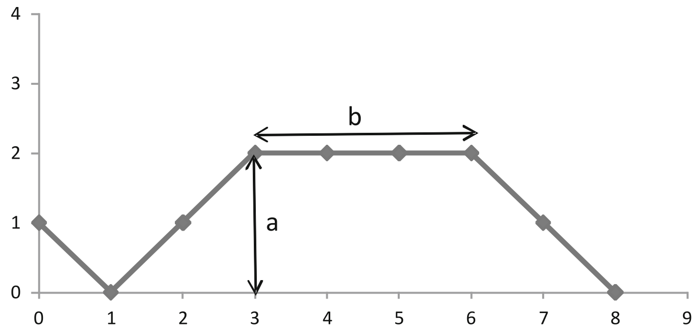


Figure 1. One SS of height 1 and one TITS with ordered pair {2,3}.

northeast: from (i, j) to $(i+1, j+1)$.
southeast: from (i, j) to $(i+1, j-1)$, only allowed if $j > 0$.
horizontal: from $(i, 0)$ to $(i+1, 0)$, only allowed when the first step is preceded by a northeast step and the last is followed by a southeast step.

The following terminology is used in describing associated lattice paths:

Truncated isosceles trapezoidal section (*TITS*): A section of the path which starts on the x-axis with northeast steps followed by horizontal steps and then followed by southeast steps ending on the x-axis forms a truncated isosceles trapezoidal section. Since the lower base lies on the x-axis and is not a part of the path, hence the term truncated.
Slant section (*SS*): A section of the path consisting of only southeast steps which starts on the y-axis (origin not included) and ends on the x-axis.
Height of a slant section: It is 't' if it starts from $(0, t)$. Clearly, a path can have an SS only in the beginning of the path. An associated lattice path can have at most one SS.
Weight of a TITS: This is defined by representing every TITS by an ordered pair $\{a, b\}$ where a denotes its altitude and b the length of the upper base. Weight of a TITS with ordered pair $\{a, b\}$ is a units.
Weight of an associated lattice path is the sum of weights of its TITSs. Note that slant section is assigned weight zero.

Example 1.1. In figure 1, the associated lattice path has one SS of height 1 and one TITS with ordered pair {2, 3} and its weight is 2 units (figure 1).

2. Two generalized basic series

The following two generalized basic series

$$\sum_{n=0}^{\infty} \frac{q^{n(n+k-1)}(-q; q^2)_n}{(q^4; q^4)_n}, \tag{2.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+k-1)}(-q; q^2)_n}{(q^2; q^2)_n}, \tag{2.2}$$

where

$$(a;q)_n = \prod_{i=0}^{\infty} \frac{(1-aq^i)}{(1-aq^{n+i})}$$

for k to be a positive integer, had been interpreted as the generating function of some restricted n-color partitions and certain weighted lattice path functions by Goyal and Agarwal [17] and, Agarwal and Rana [7] respectively. They proved the following theorems, respectively:

Theorem 2.1. *For a positive integer k, let $A_k(\mu)$ denote the number of n-color partitions of μ such that* (*i*) *the parts are of the form $(2j-1)_1$ or $(2j)_2$, if k is odd and of the form $(2j-1)_2$ or $(2j)_1$, if k is even,* (*ii*) *if m_i is the smallest or the only part in the partition, then $m \equiv i+k-1 \pmod 4$ and* (*iii*) *the weighted difference between any two consecutive parts is nonnegative and is $\equiv 0 \pmod 4$. Let $B_k(\mu)$ denote the number of lattice paths of weight μ which start at $(0,0)$, such that* (*iv*) *they have no valley above height 0,* (*v*) *there is a plain of length $\equiv k-1 \pmod 4$ in the beginning of the path, other plains, if any, are of lengths which are multiples of 4 and* (*vi*) *the height of each peak of odd (resp., even) weight is 1(resp., 2) if k is odd and 2 (resp., 1) if k is even. Then $A_k(\mu) = B_k(\mu)$, for all μ and*

$$\sum_{\mu=0}^{\infty} A_k(\mu) q^{\mu} = \sum_{\mu=0}^{\infty} B_k(\mu) q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{n(n+k-1)}(-q;q^2)_n}{(q^4;q^4)_n} . \tag{2.3}$$

Theorem 2.2. *Given a positive integer k, let $D_k(\mu)$ denote the number of n-color partitions of μ such that* (*i*) *the parts are of the form $(2j-1)_1$ or $(2j)_2$, if k is odd and of the form $(2j-1)_2$ or $(2j)_1$, if k is even,* (*ii*) *the parts are greater than or equal to k and* (*iii*) *the weighted difference between any two consecutive parts is nonnegative and even. Let $E_k(\mu)$ denote the number of lattice paths of weight μ which start at $(0,0)$, such that* (*iv*) *they have no valley above height 0,* (*v*) *there is a plain of length $(k-1)+2s$, $s \geq 0$, in the beginning of the path, other plains, if any, are of even lengths and* (*vi*) *the height of each peak of odd* (*resp.*, *even*) *weight is 1 (resp., 2) if k is odd and 2 (resp., 1) if k is even. Then $D_k(\mu) = E_k(\mu)$, for all μ and*

$$\sum_{\mu=0}^{\infty} D_k(\mu) q^{\mu} = \sum_{\mu=0}^{\infty} E_k(\mu) q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{n(n+k-1)}(-q;q^2)_n}{(q^2;q^2)_n} . \tag{2.4}$$

Our objective in this section is to further extend these results using associated lattice paths. We will show that certain restricted associated lattice paths are also generated by the extreme r.h.s. of (2.3) and (2.4). This extends Theorems 2.1 and 2.2 to two new infinite class of 3-way combinatorial identities.

2.1 *Main results*

Theorem 2.3. *For a positive integer k, let $C_k(\mu)$ denote the number of associated lattice paths of weight μ such that* (*i*) *for any TITS with ordered pair $\{a,b\}$, b does not exceed a,* (*ii*) *the TITSs are arranged in order of nondecreasing altitudes and the TITSs with same altitude are ordered by the length of their upper base,* (*iii*) *the length of the upper* *base of a TITS with odd (resp., even) weight is* 1 (*resp.*, 2) *if* k *is odd and* 2 (*resp.*, 1) *if* k *is even and* (*iv*) *for any two TITSs with respective ordered pairs* $\{a_1, b_1\}$ *and* $\{a_2, b_2\}, (a_1 \le a_2)$, $a_2 - b_2 = a_1 + b_1 + h$, *where* h *is a nonnegative multiple of* 4. *Then* $A_k(\mu) = B_k(\mu) = C_k(\mu)$, *for all* μ *and*

$$\sum_{\mu=0}^{\infty} A_k(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} B_k(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} C_k(\mu)q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{n(n+k-1)}(-q;q^2)_n}{(q^4;q^4)_n}. \tag{2.5}$$

Theorem 2.4. *For a positive integer* k, *let* $F_k(\mu)$ *denote the number of associated lattice paths of weight* μ *such that* (*i*) *for any TITS with ordered pair* $\{a,b\}$, b *does not exceed* a, (*ii*) *the TITSs are arranged in order of nondecreasing altitudes and the TITSs with same altitude are ordered by the length of their upper base*, (*iii*) *the length of the upper base of a TITS with odd* (*resp.*, *even*) *weight is* 1 (*resp.*, 2) *if* k *is odd and* 2 (*resp.*, 1) *if* k *is even and*, (*iv*) *for any two TITSs with respective ordered pairs* $\{a_1, b_1\}$ *and* $\{a_2, b_2\}$, $(a_1 \le a_2)$, $a_2 - b_2 = a_1 + b_1 + l$, *where* l *is a nonnegative even number. Then* $D_k(\mu) = E_k(\mu) = F_k(\mu)$, *for all* μ *and*

$$\sum_{\mu=0}^{\infty} D_k(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} E_k(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} F_k(\mu)q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{n(n+k-1)}(-q;q^2)_n}{(q^2;q^2)_n}. \tag{2.6}$$

As both theorems have similar proofs, we will discuss the detailed proof of Theorem 2.3 and provide an outline of the proof of Theorem 2.4.

2.2 *Proof of the Theorem* 2.3

We will prove this theorem in three steps. First we will show that the extreme r.h.s. of (2.5) generates the associated lattice paths enumerated by $C_k(\mu)$. Then we will show a bijection between n-color partitions enumerated by $A_k(\mu)$ and the the associated lattice paths enumerated by $C_k(\mu)$. Finally we will establish a bijection between weighted lattice paths enumerated by $B_k(\mu)$ and the associated lattice paths enumerated by $C_k(\mu)$.

Step I. We shall prove that

$$\sum_{\mu=0}^{\infty} C_k(\mu)q^{\mu} = \sum_{m=0}^{\infty} \frac{q^{m(m+k-1)}(-q;q^2)_m}{(q^4;q^4)_m}. \tag{2.7}$$

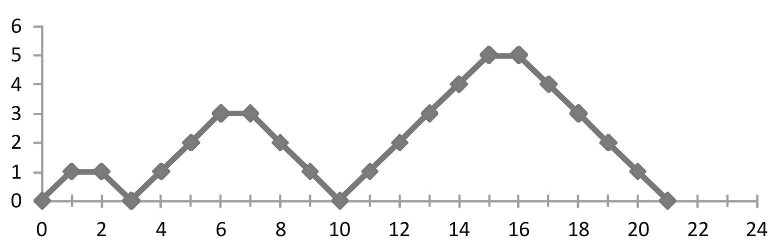


Figure 2. Associated lattice path for $k = 1$ and $m = 3$.

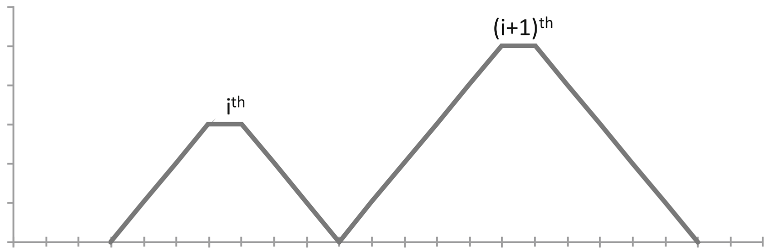


Figure 3. i-th and $(i+1)$-th TITSs.

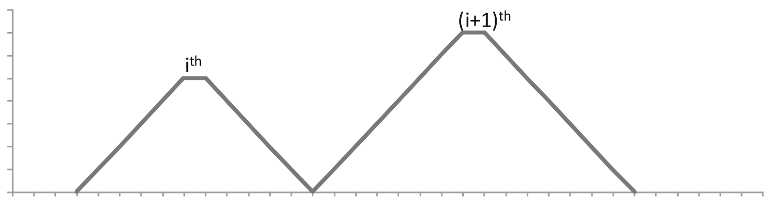


Figure 4. i-th and $(i+1)$-th TITSs.

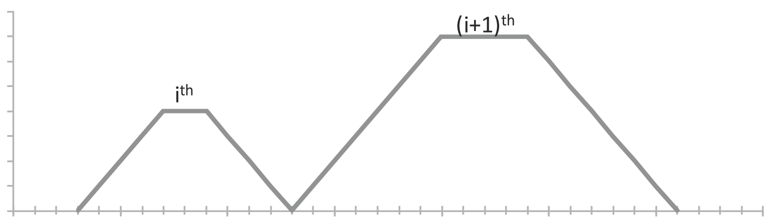


Figure 5. i-th and $(i+1)$-th TITSs.

In $\frac{q^{m(m+k-1)}(-q;q^2)_m}{(q^4;q^4)_m}$, the factor $q^{m(m+k-1)}$ generates an associated lattice path having m TITSs such that i-th TITS has the ordered pair $\{2i-1+(k-1), 1\}$.

For $k=1$ and $m=3$, the path begins as in figure 2.

In figure 2, we consider two successive TITSs, say, i-th and $(i+1)$-th. Their corresponding ordered pairs are $\{2i-1+(k-1), 1\}$ and $\{2i+1+(k-1), 1\}$ respectively as shown in figure 3.

The factor $\frac{1}{(q^4;q^4)_m}$ generates m nonnegative multiples of 4, say $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m \geq 0$, which are encoded by increasing the altitude of the i-th TITS by α_{m-i+1}, $1 \leq i \leq m$. Thus the ordered pair associated with the i-th TITS becomes $\{(2i-1)+k-1+\alpha_{m-i+1}, 1\}$.

Figure 3 now becomes figure 4.

The factor $(-q;q^2)_m$ generates m nonnegative multiples of $(2i-1)$, $1 \leq i \leq m$, say, $\beta_1 \times 1$, $\beta_2 \times 3, \ldots, \beta_m \times (2m-1)$, where each β_i $(1 \leq i \leq m)$ is 0 or 1. This is encoded by increasing the altitude of i-th TITS by $2(\beta_m + \beta_{m-1} + \cdots + \beta_{m-i+2}) + \beta_{m-i+1}$ and the length of the upper base by β_{m-i+1}. So the associated ordered pair becomes $\{2i-1+(k-1)+\alpha_{m-i+1}+2(\beta_m+\beta_{m-1}+\cdots+\beta_{m-i+2})+\beta_{m-i+1}, 1+\beta_{m-i+1}\}$.

Figure 4 now changes to figure 5.

Every associated lattice path enumerated by $C_k(\mu)$ is uniquely generated in this manner. This proves (2.7).

Step II. We now establish a 1–1 correspondence between the associated lattice paths enumerated by $C_k(\mu)$ and the n-color partitions enumerated by $A_k(\mu)$. We do this by encoding each associated lattice path as the sequence of weights of TITSs with each altitude of the TITS subscripted by the length of the respective upper base.

Thus, if we denote the two TITS in figure 5 by P_r and Q_s respectively, then

$$\begin{aligned} P &= (2i-1)+(k-1)+\alpha_{m-i+1}+2(\beta_m+\beta_{m-1}+\cdots+\beta_{m-i+2})+\beta_{m-i+1}\\ r &= \beta_{m-i+1}+1\\ Q &= (2i+1)+(k-1)+\alpha_{m-i}+2(\beta_m+\beta_{m-1}+\cdots+\beta_{m-i+1})+\beta_{m-i}\\ s &= \beta_{m-i}+1. \end{aligned}$$

Clearly, when k is odd and P (or Q) is odd (resp. even), r (or s) is 1 (resp. 2). Similarly, when k is even and P (or Q) is odd (resp. even), r (or s) is 2 (resp. 1).

The weighted difference of these two parts is $((Q_s - P_r)) = Q - P - r - s = \alpha_{m-i} - \alpha_{m-i+1}$ which is nonnegative and is a multiple of 4.

Obviously, if $\{P, r\}$ is the ordered pair of first TITS in the associated lattice path then it will correspond to the smallest part in the corresponding n-color partition or to the singleton part if the n-color partition has only one part and in both cases $P - r = k - 1 + \alpha_m \equiv k - 1 (\text{mod } 4)$.

To see the reverse implication, we consider two n-color parts of a partition enumerated by $A_k(\mu)$, say, P_r and Q_s with $Q \geq P$; $1 \leq r, s \leq 2$. Clearly $r \leq P$ and $s \leq Q$.

Since P_r and Q_s are the parts of n-color partition enumerated by $A_k(\mu)$, weighted difference $= ((Q_s - P_r)) \equiv 0 (\text{mod } 4)$

$\Rightarrow Q - P - r - s = h$, where h is nonnegative multiple of 4.
$\Rightarrow Q - s = P + r + h$, where h is nonnegative multiple of 4.

Again, if k is odd and P is odd (or even), then r is 1 (or 2). So this implies that the corresponding TITS has the length of the upper base 1 (or 2) according as the altitude is odd (or even).

Similarly when k is even and P is odd (or even), then r is 2 (or 1). So this implies that the corresponding TITS has the length of the upper base 2 (or 1) according as the altitude is odd (or even).

Step III. Finally, we establish a bijection between the weighted lattice paths enumerated by $B_k(\mu)$ and the associated lattice paths enumerated by $C_k(\mu)$. We do this by mapping each peak of weight a and height b of a weighted lattice path enumerated by $B_k(\mu)$ to a TITS with ordered pair $\{a, b\}$ of an associated lattice path enumerated by $C_k(\mu)$ and conversely. Under this mapping, all the conditions on the weighted lattice paths enumerated by $B_k(\mu)$ are translated to the conditions on the associated lattice paths enumerated by $C_k(\mu)$ and vice-versa. Hence this completes the bijection between the weighted lattice paths enumerated by $B_k(\mu)$ and the associated lattice paths enumerated by $C_k(\mu)$.

2.3 *Outline of the proof of the Theorem 2.4*

This is treated in the same manner as Theorem 2.3. The only difference is the change in condition which states $a_2 - b_2 = a_1 + b_1 + l$, where l is a nonnegative even number.

3. Five particular cases of Theorems 2.3–2.4

For some particular values of k, Theorems 2.3 and 2.4 enable us to provide new combinatorial meanings to the following identities.

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q^4;q^4)_n} = \frac{(-q;q^2)_\infty (q^3;q^3)_\infty (q^3;q^6)_\infty}{(q^2;q^2)_\infty}, \tag{3.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}(-q;q^2)_n}{(q^4;q^4)_n} = \frac{(-q;q^2)_\infty (q^6;q^6)_\infty (q;q^6)_\infty (q^5;q^6)_\infty}{(q^2;q^2)_\infty}, \tag{3.2}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q;q^8)_\infty (q^4;q^8)_\infty (q^7;q^8)_\infty}, \tag{3.3}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}(-q;q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q^3;q^8)_\infty (q^4;q^8)_\infty (q^5;q^8)_\infty}, \tag{3.4}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}(-q;q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q^2;q^8)_\infty (q^3;q^8)_\infty (q^7;q^8)_\infty}. \tag{3.5}$$

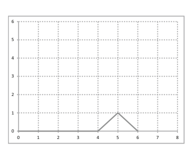
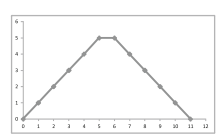
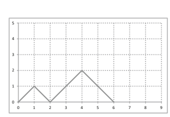
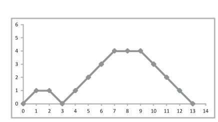
Identity (3.1) was due to Slater ([19], p. 154, eq. (25)), identity (3.2) was given by Andrews ([10], p. 105), the combinatorial interpretations (using ordinary partitions) of identities (3.3) and (3.4) are known as Göllnitz–Gordon identities [13, 15] and that of the identity (3.5) was discovered by Göllnitz independently [14]. Now an appeal to Theorems 2.3–2.4 give the following 4-way combinatorial interpretations of identities (3.1)–(3.5), respectively:

Theorem 3.1. *Let $X_1(\mu)$ denote the number of partitions of μ into parts* $\equiv \pm 2, \pm 3, 6 \pmod{12}$ *and let $Y_1(\mu)$ denote the number of partitions of μ into parts* $\equiv \pm 1, \pm 2 \pmod 6$. *Then*

$$Y_1(\mu) = \sum_{i=0}^{\mu} A_1(i) X_1(\mu - i) = \sum_{i=0}^{\mu} B_1(i) X_1(\mu - i) = \sum_{i=0}^{\mu} C_1(i) X_1(\mu - i),$$

where $A_1(\mu)$, $B_1(\mu)$ and $C_1(\mu)$ are as defined in Theorem 2.3 for $k = 1$.

The following table illustrates this theorem more precisely:

Partitions enum. by $X_1(5)$	Partitions enum. by $Y_1(5)$	Partitions enum. enum. by $A_1(5)$	Lattice paths enum. by $B_1(5)$	Asso. lattice paths by $C_1(5)$
3+2	5, 4+1, 2+2+1, 2+1+1+1, 1+1+1+1+1	$5_1,\ 4_2 + 1_1$		
				

Also, $Y_1(5) = \sum_{i=0}^{5} A_1(i)X_1(5-i) = \sum_{i=0}^{5} B_1(i)X_1(5-i) = \sum_{i=0}^{5} C_1(i)X_1(5-i) = 5$.

Theorem 3.2. *Let* $X_3(\mu)$ *denote the number of partitions of* μ *into parts* $\equiv 2 (\mathrm{mod}\ 4)$ *and let* $Y_3(\mu)$ *denote the number of partitions of* μ *into parts* $\equiv \pm 2, 3 (\mathrm{mod}\ 6)$. *Then*

$$Y_3(\mu) = \sum_{i=0}^{\mu} A_3(i)X_3(\mu-i) = \sum_{i=0}^{\mu} B_3(i)X_3(\mu-i) = \sum_{i=0}^{\mu} C_3(i)X_3(\mu-i),$$

where $A_3(\mu)$, $B_3(\mu)$ *and* $C_3(\mu)$ *are as defined in Theorem* 2.3 *for* $k = 3$.

Theorem 3.3. *Let* $Z_1(\mu)$ *denote the number of partitions of* μ *into parts* $\equiv 1, 4$ *or* 7 $(\mathrm{mod}\ 8)$. *Then*

$$Z_1(\mu) = D_1(\mu) = E_1(\mu) = F_1(\mu) \quad \textit{for all } \mu,$$

where $D_1(\mu)$, $E_1(\mu)$ *and* $F_1(\mu)$ *are as defined in Theorem* 2.4 *for* $k = 1$.

Theorem 3.4. *Let* $Z_3(\mu)$ *denote the number of partitions of* μ *into parts* $\equiv 3, 4$ *or* 5 $(\mathrm{mod}\ 8)$. *Then*

$$Z_3(\mu) = D_3(\mu) = E_3(\mu) = F_3(\mu) \quad \textit{for all } \mu,$$

where $D_3(\mu)$, $E_3(\mu)$ *and* $F_3(\mu)$ *are as defined in Theorem* 2.4 *for* $k = 3$.

Theorem 3.5. *Let* $Z_2(\mu)$ *denote the number of partitions of* μ *into parts* $\equiv 2, 3$ *or* 7 $(\mathrm{mod}\ 8)$. *Then*

$$Z_2(\mu) = D_2(\mu) = E_2(\mu) = F_2(\mu) \quad \textit{for all } \mu,$$

where $D_2(\mu)$, $E_2(\mu)$ *and* $F_2(\mu)$ *are as defined in Theorem* 2.4 *for* $k = 2$.

4. Seven q-identities of Slater

The following identities are listed in Slater's compendium [19]:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q^2)_n(q^2;q^2)_n} = \frac{(q^8;q^{10})_\infty(q^2;q^{10})_\infty(q^6;q^{20})_\infty(q^{14};q^{20})_\infty(q^{10};q^{10})_\infty}{(q;q)_\infty}, \tag{4.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q^2)_{n+1}(q^2;q^2)_n} = \frac{(q^7;q^{10})_\infty(q^3;q^{10})_\infty(q^4;q^{20})_\infty(q^{16};q^{20})_\infty(q^{10};q^{10})_\infty}{(q;q)_\infty}, \tag{4.2}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q;q^2)_{n+1}(q^2;q^2)_n} = \frac{(q^6;q^{10})_\infty (q^4;q^{10})_\infty (q^2;q^{20})_\infty (q^{18};q^{20})_\infty (q^{10};q^{10})_\infty}{(q;q)_\infty}, \tag{4.3}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q^2)_{n}(q^2;q^2)_n} = \frac{(q^9;q^{10})_\infty (q^1;q^{10})_\infty (q^{12};q^{20})_\infty (q^{8};q^{20})_\infty (q^{10};q^{10})_\infty}{(q;q)_\infty}, \tag{4.4}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_{n}(q^2;q^2)_n} = \frac{(q^7;q^{8})_\infty (q^1;q^{8})_\infty (q^{6};q^{16})_\infty (q^{10};q^{16})_\infty (q^{8};q^{8})_\infty}{(q;q)_\infty}, \tag{4.5}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2-n}}{(q;q^2)_{n}(q^2;q^2)_n} = \frac{1}{(q;q^2)_\infty}, \tag{4.6}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}(q^2;q^2)_n} = \frac{(q^5;q^{8})_\infty (q^3;q^{8})_\infty (q^{2};q^{16})_\infty (q^{14};q^{16})_\infty (q^{8};q^{8})_\infty}{(q;q)_\infty}. \tag{4.7}$$

Identities (4.1)–(4.7) were combinatorially interpreted by Agarwal [1, 2] in terms of $(n+t)$-color partitions and weighted lattice path functions in the form of the following theorems.

Theorem 4.1. *Let $G_1(\mu)$ denote the number of n-color partitions of μ such that even parts appear with even subscripts and odd with odd. The weighted difference of any two consecutive parts is nonnegative. Let $H_1(\mu)$ denote the number of lattice paths of weight μ which start at $(0, 0)$, have no valley above height 0 and no plain with odd length. Then $G_1(\mu) = H_1(\mu)$ for all μ and*

$$\sum_{\mu=0}^{\infty} G_1(\mu) q^\mu = \sum_{\mu=0}^{\infty} H_1(\mu) q^\mu = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q^2)_n(q^2;q^2)_n}.$$

Theorem 4.2. *Let $G_2(\mu)$ denote the number of $(n+1)$-color partitions of μ such that even parts appear with odd subscripts and odd with even, for some i, i_{i+1} is a part and the weighted difference of any two consecutive parts is nonnegative. Let $H_2(\mu)$ denote the number of lattice paths of weight μ which start at $(0, 1)$, have no valley above height 0 and no plain with odd length. Then $G_2(\mu) = H_2(\mu)$ for all μ and*

$$\sum_{\mu=0}^{\infty} G_2(\mu) q^\mu = \sum_{\mu=0}^{\infty} H_2(\mu) q^\mu = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q^2)_{n+1}(q^2;q^2)_n}.$$

Theorem 4.3. *Let $G_3(\mu)$ denote the number of $(n+2)$-color partitions of μ such that even parts appear with even subscripts and odd with odd, for some i, i_{i+2} is a part and the weighted difference of any two consecutive parts is nonnegative. Let $H_3(\mu)$ denote the number of lattice paths of weight μ which start at $(0, 2)$, have no valley above height 0 and no plain with odd length. Then $G_3(\mu) = H_3(\mu)$ for all μ and*

$$\sum_{\mu=0}^{\infty} G_3(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} H_3(\mu)q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q;q^2)_{n+1}(q^2;q^2)_n} .$$

Theorem 4.4. *Let $G_4(\mu)$ denote the number of n-color partitions of μ such that the even parts appear with even subscripts and odd with odd, all subscripts are >1 and the weighted difference of any two consecutive parts is either nonnegative or equal to -2. Let $H_4(\mu)$ denote the number of lattice paths of weight μ which start at $(0, 0)$, have no valley above height 0 and no plain with odd length except in the beginning of the path where there is always a plain with odd length. Then $G_4(\mu) = H_4(\mu)$, for all μ and*

$$\sum_{\mu=0}^{\infty} G_4(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} H_4(\mu)q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q^2)_n(q^2;q^2)_n} .$$

Theorem 4.5. *Let $G_5(\mu)$ denote the number of n-color partitions of μ such that even parts appear with even subscripts and odd with odd, all subscripts are >1 and the weighted difference of any two consecutive parts is either ≥ 2 or equal to 0. Let $H_5(\mu)$ denote the number of lattice paths of weight μ which start at $(0, 0)$, have no valley above height 0, no plain with odd length and for which the minimal height of each peak is 2. Then $G_5(\mu) = H_5(\mu)$, for all μ and*

$$\sum_{\mu=0}^{\infty} G_5(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} H_5(\mu)q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n(q^2;q^2)_n} .$$

Theorem 4.6. *Let $G_6(\mu)$ denote the number of n-color partitions of μ such that even parts appear with even subscripts and odd with odd. The weighted difference of any two consecutive parts is >1. Let $H_6(\mu)$ denote the number of lattice paths of weight μ which start at $(0, 0)$, have no valley at all, no plain with odd length and for which there is always a plain of minimal length 2 between any two consecutive peaks. Then $G_6(\mu) = H_6(\mu)$ for all μ and*

$$\sum_{\mu=0}^{\infty} G_6(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} H_6(\mu)q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{2n^2-n}}{(q;q^2)_n(q^2;q^2)_n} .$$

Theorem 4.7. *Let $G_7(\mu)$ denote the number of $(n+2)$-color partitions of μ such that even parts appear with even subscripts and odd with odd, all subscripts are >1, for some i, i_{i+2} is a part and the weighted difference of any two consecutive parts is either ≥ 2 or equal to 0. Let $H_7(\mu)$ denote the number of lattice paths of weight μ which start at $(0, 2)$, have no valley above height 0, no plain with odd length and for which the minimal height of each peak is 2. Then $G_7(\mu) = H_7(\mu)$, for all μ and*

$$\sum_{\mu=0}^{\infty} G_7(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} H_7(\mu)q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}(q^2;q^2)_n} .$$

This section is fully devoted to further extend these results using associated lattice paths.

4.1 *Main results*

Theorem 4.8. *Let $I_1(\mu)$ denote the number of associated lattice paths of weight μ such that (i) for any TITS with ordered pair $\{a, b\}$, b does not exceed a, (ii) the TITSs are arranged in order of nondecreasing altitudes and the TITSs with same altitude are ordered by the length of their upper base and (iii) for any two TITSs with respective ordered pairs $\{a_1, b_1\}$ and $\{a_2, b_2\}$, $(a_1 \leq a_2)$, $a_2 - b_2 \geq a_1 + b_1$. Then $G_1(\mu) = H_1(\mu) = I_1(\mu)$, for all μ and*

$$\sum_{\mu=0}^{\infty} G_1(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} H_1(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} I_1(\mu)q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_n(q;q^2)_n} .$$

Theorem 4.9. *Let $I_2(\mu)$ denote the number of associated lattice paths of weight μ such that (i) for any TITS with ordered pair $\{a, b\}$, b does not exceed $(a+1)$, (ii) the TITSs are arranged in order of nondecreasing altitudes and the TITSs with same altitude are ordered by the length of their upper base, (iii) there is one TITS with ordered pair $\{a, a+1\}$ or an SS of height 1 and (iv) for any two TITSs with respective ordered pairs $\{a_1, b_1\}$ and $\{a_2, b_2\}$, $(a_1 \leq a_2)$, $a_2 - b_2 \geq a_1 + b_1$. Then $G_2(\mu) = H_2(\mu) = I_2(\mu)$ for all μ and*

$$\sum_{\mu=0}^{\infty} G_2(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} H_2(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} I_2(\mu)q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2;q^2)_n(q;q^2)_{n+1}} .$$

Theorem 4.10. *Let $I_3(\mu)$ denote the number of associated lattice paths of weight μ such that (i) for any TITS with ordered pair $\{a, b\}$, b does not exceed $(a+2)$, (ii) the TITSs are arranged in order of nondecreasing altitudes and the TITSs with same altitude are ordered by the length of their upper base, (iii) there is one TITS with ordered pair $\{a, a+2\}$ or an SS of height 2 and (iv) for any two TITSs with respective ordered pairs $\{a_1, b_1\}$ and $\{a_2, b_2\}$, $(a_1 \leq a_2)$, $a_2 - b_2 \geq a_1 + b_1$. Then $G_3(\mu) = H_3(\mu) = I_3(\mu)$ for all μ and*

$$\sum_{\mu=0}^{\infty} G_3(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} H_3(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} I_3(\mu)q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q^2;q^2)_n(q;q^2)_{n+1}} .$$

Theorem 4.11. *Let $I_4(\mu)$ denote the number of associated lattice paths of weight μ such that (i) for any TITS with ordered pair $\{a, b\}$, b does not exceed a, (ii) the TITSs are arranged in order of nondecreasing altitudes and the TITSs with same altitude are ordered by the length of their upper base, (iii) the length of each upper base is greater than 1 and (iv) for any two TITSs with respective ordered pairs $\{a_1, b_1\}$ and $\{a_2, b_2\}$, $(a_1 \leq a_2)$, $a_2 - b_2 \geq a_1 + b_1 - 2$. Then $G_4(\mu) = H_4(\mu) = I_4(\mu)$ for all μ and*

$$\sum_{\mu=0}^{\infty} G_4(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} H_4(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} I_4(\mu)q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^2;q^2)_n(q;q^2)_n} .$$

Theorem 4.12. *Let $I_5(\mu)$ denote the number of associated lattice paths of weight μ such that (i) for any TITS with ordered pair $\{a, b\}$, b does not exceed a, (ii) the TITSs are arranged in order of nondecreasing altitudes and the TITSs with same altitude are ordered by the length of their upper base, (iii) the length of each upper base is greater than 1 and (iv) for any two TITSs with respective ordered pairs $\{a_1, b_1\}$ and $\{a_2, b_2\}$, $(a_1 \leq a_2)$, $a_2 - b_2 \geq a_1 + b_1$. Then $G_5(\mu) = H_5(\mu) = I_5(\mu)$ for all μ and*

$$\sum_{\mu=0}^{\infty} G_5(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} H_5(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} I_5(\mu)q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2;q^2)_n(q;q^2)_n}.$$

Theorem 4.13. *Let $I_6(\mu)$ denote the number of associated lattice paths of weight μ such that (i) for any TITS with ordered pair $\{a, b\}$, b does not exceed a, (ii) the TITSs are arranged in order of nondecreasing altitudes and the TITSs with same altitude are ordered by the length of their upper base and (iii) for any two TITSs with respective ordered pairs $\{a_1, b_1\}$ and $\{a_2, b_2\}$, $(a_1 \leq a_2)$, $a_2 - b_2 \geq a_1 + b_1 + 2$. Then $G_6(\mu) = H_6(\mu) = I_6(\mu)$ for all μ and*

$$\sum_{\mu=0}^{\infty} G_6(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} H_6(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} I_6(\mu)q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{n(2n-1)}}{(q^2;q^2)_n(q;q^2)_n}.$$

Theorem 4.14. *Let $I_7(\mu)$ denote the number of associated lattice paths of weight μ such that (i) for any TITS with ordered pair $\{a, b\}$, b does not exceed $(a + 2)$, (ii) the TITSs are arranged in order of nondecreasing altitudes and the TITSs with the same altitude are ordered by the length of their upper base, (iii) there is one TITS with ordered pair $\{a, a+2\}$ or an SS of height 2, (iv) the length of each upper base is greater than 1 and (v) for any two TITSs with respective ordered pairs $\{a_1, b_1\}$ and $\{a_2, b_2\}$, $(a_1 \leq a_2)$, $a_2 - b_2 \geq a_1 + b_1$. Then $G_7(\mu) = H_7(\mu) = I_7(\mu)$, for all μ and*

$$\sum_{\mu=0}^{\infty} G_7(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} H_7(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} I_7(\mu)q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q^2;q^2)_n(q;q^2)_{n+1}}.$$

4.2 *Proof of Theorem 4.8*

Again the proof of this theorem comprises of three steps which are discussed below.

Step I. We shall prove that

$$\sum_{\mu=0}^{\infty} I_1(\mu)q^{\mu} = \sum_{m=0}^{\infty} \frac{q^{m^2}}{(q^2;q^2)_m(q;q^2)_m}. \tag{4.8}$$

In $\frac{q^{m^2}}{(q^2;q^2)_n(q;q^2)_n}$, the factor q^{m^2} generates an associated lattice path having m TITSs such that i-th TITS have the ordered pair $\{2i - 1, 1\}$.

Now we consider two successive TITSs, say, i-th and $(i + 1)$-th. Their corresponding ordered pairs are $\{2i - 1, 1\}$ and $\{2i + 1, 1\}$, respectively. The factor $\frac{1}{(q^2;q^2)_m}$ generates m nonnegative multiples of 2, say $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m \geq 0$, which are encoded by increasing the altitude of the i-th TITS by α_{m-i+1}, $1 \leq i \leq m$. Thus the ordered pair associated with the i-th TITS becomes $\{(2i-1)+\alpha_{m-i+1}, 1\}$. The factor $\frac{1}{(q;q^2)_m}$ generates m nonnegative multiples of $(2i-1)$, $1 \le i \le m$, say, $\beta_1 \times 1$, $\beta_2 \times 3, \ldots, \beta_m \times (2m-1)$. This is encoded by increasing the altitude of i-th TITS by $2(\beta_m + \beta_{m-1} + \cdots + \beta_{m-i+2}) + \beta_{m-i+1}$ and the length of the upper base by β_{m-i+1}. So the associated ordered pair becomes $\{2i-1+\alpha_{m-i+1}+2(\beta_m+\beta_{m-1}+\cdots+\beta_{m-i+2})+\beta_{m-i+1}, 1+\beta_{m-i+1}\}$.

Every associated lattice path enumerated by $I_1(\mu)$ is uniquely generated in this manner. This proves (4.8).

Step II. We now establish a 1–1 correspondence between the associated lattice paths enumerated by $I_1(\mu)$ and the n-color partitions enumerated by $G_1(\mu)$. We do this by encoding each associated lattice path as the sequence of weights of TITSs with each altitude of the TITS subscripted by the length of the respective upper base.

Thus, if we consider two TITS, say, P_r and Q_s of the path enumerated by $I_1(\mu)$, then

$$\begin{aligned} P &= (2i-1)+\alpha_{m-i+1}+2(\beta_m+\beta_{m-1}+\cdots+\beta_{m-i+2})+\beta_{m-i+1}, \\ r &= \beta_{m-i+1}+1, \\ Q &= (2i+1)+\alpha_{m-i}+2(\beta_m+\beta_{m-1}+\cdots+\beta_{m-i+1})+\beta_{m-i}, \\ s &= \beta_{m-i}+1. \end{aligned}$$

Clearly, the parity of P and r depends upon β_{m-i+1}. If β_{m-i+1} is odd, then both P and r are even and when β_{m-i+1} is even then both P and r are odd. This proves that even parts are appearing with even subscripts and odd with odd subscripts.

The weighted difference of these two parts is $((Q_s - P_r)) = Q - P - r - s = \alpha_{m-i} - \alpha_{m-i+1}$, which is nonnegative.

To see the reverse implication, we consider two n-color parts of a partition enumerated by $G_1(\mu)$, say, P_r and Q_s with $Q \ge P$. Clearly $r \le P$ and $s \le Q$. Since P_r and Q_s are the parts of n-color partition enumerated by $G_1(\mu)$, weighted difference $= ((Q_s - P_r)) \ge 0$ $\Rightarrow Q - P - r - s \ge 0 \Rightarrow Q - s \ge P + r$.

Step III. Finally we establish a bijection between the weighted lattice paths enumerated by $H_1(\mu)$ and the associated lattice paths enumerated by $I_1(\mu)$. We do this by mapping each peak of weight a and height b of a weighted lattice path enumerated by $H_1(\mu)$ to a TITS with ordered pair $\{a, b\}$ of an associated lattice path enumerated by $I_1(\mu)$ and conversely. Under this mapping, all the conditions on the weighted lattice paths enumerated by $H_1(\mu)$ are translated to the conditions on the associated lattice paths enumerated by $I_1(\mu)$ and *vice-versa*. Hence this completes the bijection between the weighted lattice paths enumerated by $H_1(\mu)$ and the associated lattice paths enumerated by $I_1(\mu)$.

4.3 *Outline of the proofs of Theorems* 4.9–4.14

Now we will discuss the changes required to prove the remaining theorems:

Proof of Theorem 4.9. An appeal to Theorem 4.8, the extra factor $\frac{q^m}{(1-q^{2m+1})}$ puts an SS of height 1 in the beginning of the path or a TITS with ordered pair $\{a, a+1\}$. Clearly, it will correspond to a_{a+1} or we can say i_{i+1} part of the corresponding colored partition.

Proof of Theorem 4.10. An appeal to Theorem 4.8, the extra factor $\frac{q^{2m}}{(1-q^{2m+1})}$ puts an SS of height 2 in the beginning of the path or a TITS with ordered pair $\{a, a+2\}$. Clearly, it will correspond to a_{a+2} or we can say i_{i+2} part of the corresponding colored partition.

Proof of Theorem 4.11. An appeal to Theorem 4.8, the extra factor q^m causes an increase by 1 in the altitude of each of the TITSs and in the length of their upper bases.

Proof of Theorem 4.12. This is treated in the same manner as Theorem 4.8. The only difference is now the path begins with m TITSs with i-th TITS having the ordered pair $\{4i-2, 2\}$.

Proof of Theorem 4.13. An appeal to Theorem 4.12, the extra factor q^{-m} causes a decrease by 1 in the altitude of each of the TITSs and in the length of their upper bases.

Proof of Theorem 4.14. An appeal to Theorem 4.12, the extra factor $\frac{q^{2m}}{(1-q^{2m+1})}$ puts an SS of height 2 in the beginning of the path or a TITS with ordered pair $\{a, a+2\}$. Clearly, it will correspond to a_{a+2} or we can say i_{i+2} part of the corresponding colored partition.

5. Five q-identities of Rogers

In [5, 16], the following five identities of Rogers were interpreted combinatorially using $(n+t)$-color partitions and weighted lattice path functions

$$\sum_{n=0}^{\infty} \frac{q^{3n^2}}{(q;q^2)_n(q^4;q^4)_n} = \frac{(-q^3,-q^5,-q^7;q^{10})_\infty}{(q^4,q^6;q^{10})_\infty}, \tag{5.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{3n^2-2n}}{(q;q^2)_n(q^4;q^4)_n} = \frac{(-q,-q^5,-q^9;q^{10})_\infty}{(q^2,q^8;q^{10})_\infty}, \tag{5.2}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n(q^4;q^4)_n} = \frac{(-q^3,-q^7,-q^{11};q^{14})_\infty}{(q^2,q^6,q^8,q^{12};q^{14})_\infty}, \tag{5.3}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_n(q^4;q^4)_n} = \frac{(-q^5,-q^7,-q^9;q^{14})_\infty}{(q^4,q^6,q^8,q^{10};q^{14})_\infty}, \tag{5.4}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}(q^4;q^4)_n} = \frac{(-q,-q^7,-q^{13};q^{14})_\infty}{(q^2,q^4,q^{10},q^{12};q^{14})_\infty}. \tag{5.5}$$

These identities have their combinatorial meanings in the form of the following five theorems.

Theorem 5.1. *Let $J_1(\mu)$ denote the number of n-color partitions of μ such that even parts appear with even subscripts and odd with odd, and all subscripts are greater than* 2. *If m_i is the smallest or the only part in the partition, then $m \equiv i \pmod 4$ and the weighted difference of any two consecutive parts is nonnegative and is $\equiv 0 \pmod 4$. Let $K_1(\mu)$ denote the number of lattice paths of weight μ which start from* $(0, 0)$, *have no valley above height* 0, *the lengths of the plains, if any, are $\equiv 0 \pmod 4$ and the height of each peak is greater than* 2. *Then $J_1(\mu) = K_1(\mu)$, for all μ and*

$$\sum_{\mu=0}^{\infty} J_1(\mu)q^\mu = \sum_{\mu=0}^{\infty} K_1(\mu)q^\mu = \sum_{n=0}^{\infty} \frac{q^{3n^2}}{(q;q^2)_n(q^4;q^4)_n}.$$

Theorem 5.2. *Let $J_2(\mu)$ denote the number of n-color partitions of μ such that even parts appear with even subscripts and odd with odd. If m_i is the smallest or the only part in the partition, then $m \equiv i \pmod 4$ and the weighted difference of any two consecutive parts is ≥ 4 and is $\equiv 0 \pmod 4$. Let $K_2(\mu)$ denote the number of lattice paths of weight μ which start from $(0, 0)$, have no valley above height 0, the lengths of the plains, if any, are $\equiv 0 \pmod 4$ and there is a plain of length ≥ 4 between any two peaks. Then $J_2(\mu) = K_2(\mu)$, for all μ and*

$$\sum_{\mu=0}^{\infty} J_2(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} K_2(\mu)q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{3n^2-2n}}{(q;q^2)_n(q^4;q^4)_n}.$$

Theorem 5.3. *Let $J_3(\mu)$ denote the number of n-color partitions of μ such that even parts appear with even subscripts and odd with odd, and all subscripts are > 1. If m_i is the smallest or the only part in the partition, then $m \equiv i \pmod 4$ and the weighted difference of any two consecutive parts is nonnegative and is $\equiv 0 \pmod 4$. Let $K_3(\mu)$ denote the number of lattice paths of weight μ which start from $(0, 0)$, have no valley above height 0, the lengths of the plains, if any, are $\equiv 0 \pmod 4$ and the height of each peak is greater than 1. Then $J_3(\mu) = K_3(\mu)$, for all μ and*

$$\sum_{\mu=0}^{\infty} J_3(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} K_3(\mu)q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n(q^4;q^4)_n}.$$

Theorem 5.4. *Let $J_4(\mu)$ denote the number of n-color partitions of μ such that even parts appear with even subscripts and odd with odd, and all subscripts are > 3. If m_i is the smallest or the only part in the partition, then $m \equiv i \pmod 4$ and the weighted difference of any two consecutive parts is ≥ -4 and is $\equiv 0 \pmod 4$. Let $K_4(\mu)$ denote the number of lattice paths of weight μ which start from $(0, 0)$, have no valley above height 0, the height of each peak is >3, and there is a plain of length $\equiv 2 \pmod 4$ in the beginning of the path and the lengths of the other plains, if any, are $\equiv 0 \pmod 4$. Then $J_4(\mu) = K_4(\mu)$, for all μ and*

$$\sum_{\mu=0}^{\infty} J_4(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} K_4(\mu)q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_n(q^4;q^4)_n}.$$

Theorem 5.5. *Let $J_5(\mu)$ denote the number of $(n+2)$-color partitions of μ such that the even parts appear with even subscripts and odd with odd, and all subscripts are >1. If m_i is the smallest or the only part in the partition, then $m \equiv i \pmod 4$, for some i, i_{i+2} is a part and the weighted difference of any two consecutive parts is nonnegative and is $\equiv 0 \pmod 4$. Let $K_5(\mu)$ denote the number of lattice paths of weight μ which start from $(0, 2)$, have no valley above height 0, the height of each peak is >1, the lengths of the plains, if any, are $\equiv 0 \pmod 4$. Then $J_5(\mu) = K_5(\mu)$, for all μ and*

$$\sum_{\mu=0}^{\infty} J_5(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} K_5(\mu)q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}(q^4;q^4)_n}.$$

Again with the aid of associated lattice paths we extend these results to 3-way combinatorial identities.

5.1 *Main results*

Theorem 5.6. *Let $L_1(\mu)$ denote the number of associated lattice paths of weight μ such that*

(*i*) *for any TITS with ordered pair $\{a, b\}$, b does not exceed a,*
(*ii*) *the TITSs are arranged in order of nondecreasing altitudes and the TITSs with same altitude are ordered by the length of their upper base,*
(*iii*) *the length of each upper base is greater than 2, and*
(*iv*) *for any two TITSs with respective ordered pairs $\{a_1, b_1\}$ and $\{a_2, b_2\}$, $(a_1 \leq a_2)$, $a_2 - b_2 = a_1 + b_1 + h$, where h is a nonnegative multiple of 4. Then $J_1(\mu) = K_1(\mu) = L_1(\mu)$ for all μ and*

$$\sum_{\mu=0}^{\infty} J_1(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} K_1(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} L_1(\mu)q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{3n^2}}{(q;q^2)_n(q^4;q^4)_n}.$$

Theorem 5.7. *Let $L_2(\mu)$ denote the number of associated lattice paths of weight μ such that*

(*i*) *for any TITS with ordered pair $\{a, b\}$, b does not exceed a,*
(*ii*) *the TITSs are arranged in order of nondecreasing altitudes and the TITSs with same altitude are ordered by the length of their upper base, and*
(*iii*) *for any two TITSs with respective ordered pairs $\{a_1, b_1\}$ and $\{a_2, b_2\}$, $(a_1 \leq a_2)$, $a_2 - b_2 = a_1 + b_1 + h$, where h is a positive multiple of 4. Then $J_2(\mu) = K_2(\mu) = L_2(\mu)$, for all μ and*

$$\sum_{\mu=0}^{\infty} J_2(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} K_2(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} L_2(\mu)q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{3n^2-2n}}{(q;q^2)_n(q^4;q^4)_n}.$$

Theorem 5.8. *Let $L_3(\mu)$ denote the number of associated lattice paths of weight μ such that*

(*i*) *for any TITS with ordered pair $\{a, b\}$, b does not exceed a,*
(*ii*) *the TITSs are arranged in order of nondecreasing altitudes and the TITSs with same altitude are ordered by the length of their upper base,*
(*iii*) *the length of each upper base is greater than 1, and*
(*iv*) *for any two TITSs with respective ordered pairs $\{a_1, b_1\}$ and $\{a_2, b_2\}$, $(a_1 \leq a_2)$, $a_2 - b_2 = a_1 + b_1 + h$, where h is a nonnegative multiple of 4. Then $J_3(\mu) = K_3(\mu) = L_3(\mu)$, for all μ and*

$$\sum_{\mu=0}^{\infty} J_3(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} K_3(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} L_3(\mu)q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n(q^4;q^4)_n}.$$

Theorem 5.9. *Let $L_4(\mu)$ denote the number of associated lattice paths of weight μ such that*

(*i*) *for any TITS with ordered pair* $\{a, b\}$, *b does not exceed a*,
(*ii*) *the TITSs are arranged in order of nondecreasing altitudes and the TITSs with same altitude are ordered by the length of their upper base*,
(*iii*) *the length of each upper base is greater than* 3, *and*
(*iv*) *for any two TITSs with respective ordered pairs* $\{a_1, b_1\}$ *and* $\{a_2, b_2\}$, $(a_1 \le a_2)$, $a_2 - b_2 = a_1 + b_1 + h$, *where* $h \ge -1$ *and is a multiple of* 4. *Then* $J_4(\mu) = K_4(\mu) = L_4(\mu)$, *for all* μ *and*

$$\sum_{\mu=0}^{\infty} J_4(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} K_4(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} L_4(\mu)q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q; q^2)_n (q^4; q^4)_n} .$$

Theorem 5.10. *Let* $L_5(\mu)$ *denote the number of associated lattice paths of weight* μ *such that*

(*i*) *for any TITS with ordered pair* $\{a, b\}$, *b does not exceed* $(a + 2)$,
(*ii*) *the TITSs are arranged in order of nondecreasing altitudes and the TITSs with same altitude are ordered by the length of their upper base*,
(*iii*) *there is an SS of height* 2 *or a TITS with ordered pair* $\{a, a + 2\}$,
(*iv*) *the length of each upper base is greater than* 1, *and*
(*v*) *for any two TITSs with respective ordered pairs* $\{a_1, b_1\}$ *and* $\{a_2, b_2\}$, $(a_1 \le a_2)$, $a_2 - b_2 = a_1 + b_1 + h$, *where h is a nonnegative multiple of* 4. *Then* $J_5(\mu) = K_5(\mu) = L_5(\mu)$, *for all* μ *and*

$$\sum_{\mu=0}^{\infty} J_5(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} K_5(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} L_5(\mu)q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q; q^2)_{n+1} (q^4; q^4)_n} .$$

5.2 *Proof of the Theorem 5.6*

The method of proving this theorem consists of three steps as discussed below:

Step I. We shall prove that

$$\sum_{\mu=0}^{\infty} L_1(\mu)q^{\mu} = \sum_{m=0}^{\infty} \frac{q^{3m^2}}{(q; q^2)_m (q^4; q^4)_m} . \tag{5.6}$$

In $\frac{q^{3m^2}}{(q;q^2)_m(q^4;q^4)_m}$ the factor q^{3m^2} generates an associated lattice path having m TITSs such that i-th TITS have the ordered pair $\{6i - 3, 3\}$.

For $m = 2$, the path begins as in Fig. 6. We consider two successive TITSs, say, i-th and $(i + 1)$-th. Their corresponding ordered pairs are $\{6i - 3, 3\}$ and $\{6i + 3, 3\}$, respectively.

The factor $\frac{1}{(q^4;q^4)_m}$ generates m nonnegative multiples of 4, say $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_m \ge 0$, which are encoded by increasing the altitude of the i-th TITS by α_{m-i+1}, $1 \le i \le m$. Thus the ordered pair associated with the i-th TITS becomes $\{6i - 3 + \alpha_{m-i+1}, 3\}$.

The factor $\frac{1}{(q;q^2)_m}$ generates m nonnegative multiples of $(2i - 1)$, $1 \le i \le m$, say, $\beta_1 \times 1$, $\beta_2 \times 3, \ldots, \beta_m \times (2m - 1)$. This is encoded by increasing the altitude of i-th TITS by $2(\beta_m + \beta_{m-1} + \cdots + \beta_{m-i+2}) + \beta_{m-i+1}$ and the length of the upper base by

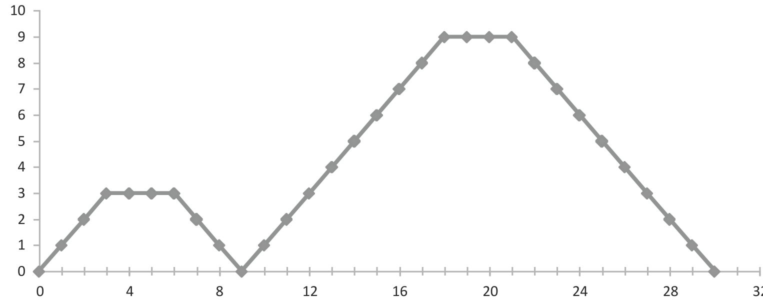


Figure 6. Associated lattice path for $m = 2$.

β_{m-i+1}. So the associated ordered pair becomes $\{6i - 3 + \alpha_{m-i+1} + 2(\beta_m + \beta_{m-1} + \cdots + \beta_{m-i+2}) + \beta_{m-i+1}, 3 + \beta_{m-i+1}\}$.

Every associated lattice path enumerated by $L_1(\mu)$ is uniquely generated in this manner. This proves (5.6).

Step II. We now establish a 1–1 correspondence between the associated lattice paths enumerated by $L_1(\mu)$ and the n-color partitions enumerated by $J_1(\mu)$.

We do this by encoding each associated lattice path as the sequence of weights of TITSs with each altitude of the TITS subscripted by the length of the respective upper base.

Thus, if we consider two TITS, say, P_r and Q_s of the path enumerated by $L_1(\mu)$, then

$$\begin{aligned} P &= 6i - 3 + \alpha_{m-i+1} + 2(\beta_m + \beta_{m-1} + \cdots + \beta_{m-i+2}) + \beta_{m-i+1},\\ r &= \beta_{m-i+1} + 3,\\ Q &= 6i + 3 + \alpha_{m-i} + 2(\beta_m + \beta_{m-1} + \cdots + \beta_{m-i+1}) + \beta_{m-i},\\ s &= \beta_{m-i} + 3. \end{aligned}$$

Clearly, all subscripts are greater 2. The parity of P and r depends upon β_{m-i+1}. If β_{m-i+1} is odd, then both P and r are even and when β_{m-i+1} is even then both P and r are odd. This proves that even parts are appearing with even subscripts and odd with odd subscripts.

The weighted difference of these two parts is $((Q_s - P_r)) = Q - P - r - s = \alpha_{m-i} - \alpha_{m-i+1}$ which is nonnegative and is a multiple of 4.

Obviously, if $\{P, r\}$ is the ordered pair of first TITS in the associated lattice path then it will correspond to the smallest part in the corresponding n-color partition or to the singleton part if the n-color partition has only one part and in both cases $P - r = \alpha_m \equiv 0 \pmod 4$.

To see the reverse implication, we consider two n-color parts of a partition enumerated by $J_1(\mu)$, say, P_r and Q_s with $Q \geq P$. Clearly $r \leq P$ and $s \leq Q$. Since P_r and Q_s are the parts of n-color partition enumerated by $J_1(\mu)$, weighted difference $= ((Q_s - P_r)) \equiv 0 \pmod 4$ $\Rightarrow Q - P - r - s = h$, where h is nonnegative multiple of 4 and $\Rightarrow Q - s = P + r + h$, where h is nonnegative multiple of 4.

As P_r is the part of n-color partition enumerated by $J_1(\mu)$, so $r > 2$. This implies the length of each upper base of the path enumerated by $L_1(\mu)$ is > 2.

Step III. Finally we establish a bijection between the weighted lattice paths enumerated by $K_1(\mu)$ and the associated lattice paths enumerated by $L_1(\mu)$. We do this by mapping each peak of weight a and height b of a weighted lattice path enumerated by $K_1(\mu)$ to a TITS with ordered pair $\{a, b\}$ of an associated lattice path enumerated by $L_1(\mu)$ and conversely. Under this mapping, all the conditions on the weighted lattice paths enumerated by $K_1(\mu)$ are translated to the conditions on the associated lattice paths enumerated by $L_1(\mu)$ and vice versa. Hence this completes the bijection between the weighted lattice paths enumerated by $K_1(\mu)$ and the associated lattice paths enumerated by $L_1(\mu)$.

5.3 *Outline of the proofs of Theorems* 5.7–5.10

Here, the changes required to prove the remaining theorems are discussed briefly.

Proof of Theorem 5.7. An appeal to Theorem 5.6, the extra factor q^{-2m} causes a decrease by 2 in the altitude of each of the TITSs and in the length of their upper bases. □

Proof of Theorem 5.8. This is treated in the same manner as Theorem 5.6. The only difference is now the path begins with m TITSs with i-th TITS having the ordered pair $\{4i-2, 2\}$. □

Proof of Theorem 5.9. An appeal to Theorem 5.8, the extra factor q^{2m} causes an increase by 2 in the altitude of each of the TITSs and in the length of their upper bases. Thus the length of each upper base is > 3. □

Proof of Theorem 5.10. An appeal to Theorem 5.8, the extra factor $\frac{q^{2m}}{(1-q^{2m+1})}$ puts an SS of height 2 in the beginning of the path or a TITS with ordered pair $\{a, a+2\}$. Clearly, it will correspond to a_{a+2} or we can say i_{i+2} part of the corresponding colored partition. □

6. Conclusion

A fine connection between the theory of basic series and combinatorics is observed in this paper. Theorems 2.3 and 2.4 yield two new infinite class of 3-way combinatorial identities. For $k = 1$ and $k = 3$, Theorem 2.3 gives us 4-way combinatorial interpretations of two q-identities of Slater and Andrews respectively. Theorem 2.4 for $k = 1, 2$ and 3, provide new combinatorial versions of the famous Göllnitz–Gordon and Göllnitz identities. Theorems 4.8–4.14 and Theorems 5.6–5.10 yield 3-way combinatorial interpretations of seven Slater's and five Rogers q-identities respectively. It would be of interest if more q-identities can be interpreted combinatorially using associated lattice paths.

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