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Allowable graphs of the nonlinear Schrödinger equation and their applications

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Abstract. We construct the definition of allowable graphs of the nonlinear Schrödinger equation of arbitrary degree and use it to verify the separation and irreducibility (over the ring of integers) of the characteristic polynomials of all the possible graphs giving 3-dimensional blocks of the normal form of the nonlinear Schrödinger equation. The method is purely algebraic and the obtained results will be useful in further studies of the nonlinear Schrödinger equation.

Keywords. Non-linear Schrödinger equation; graphs; characteristic polynomial; positive semi-definite; eigenvalue.

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1. Introduction

1.1 *The main purpose of this paper*

This paper is devoted to giving a general definition of allowable graphs of the nonlinear Schrödinger equation (NLS for brevity) and applying it to extend the main results of [4] and [6], namely proving the separation and irreducibility of the characteristic polynomials of the possible graphs giving 3-dimensional blocks of the normal form of the NLS.

The NLS of degree $2q + 1$ on an n-dimensional torus is a partial differential equation of the form

$$-\mathrm{i}\frac{\partial u}{\partial \tilde{t}} + \triangle u = \kappa |u|^{2q} u, \tag{1}$$

where $1 \le q \in \mathbb{N}, \kappa \in \mathbb{R}$ are constants, $u = u(\tilde{t}, \varphi) : \mathbb{R} \times \mathbb{T}^n \to \mathbb{C}, \varphi = (\varphi_1, \ldots, \varphi_n) \in \mathbb{T}^n = (\frac{\mathbb{R}}{2\pi\mathbb{Z}})^n$, $\tilde{t} \in \mathbb{R}$ is the time variable, $\triangle u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial \varphi_j^2}$, $|u|$ is the module of u.

1.2 *The motivation of this paper*

The following proposition is well-known.

PROPOSITION 1 [2,5]

After rescaling of u and of time, the NLS (1) can be written as an infinite dimensional Hamiltonian dynamical system

$$\frac{\partial u_k}{\partial t} = \{H, u_k\}, \frac{\partial \bar{u}_k}{\partial t} = \{H, \bar{u}_k\},$$

where by $\{-, -\}$ *we denote the Poisson bracket. The symplectic variables* u_k *are the Fourier coefficients of the functions*

$$u(t, \varphi) = \sum_{k \in \mathbb{Z}^n} u_k(t) e^{\mathrm{i}(k,\varphi)}. \tag{2}$$

The symplectic form is $\mathrm{i} \sum_{k\in\mathbb{Z}^n} du_k \wedge d\bar{u}_k$ *and the Hamiltonian is*

$$H := \sum_{k\in\mathbb{Z}^n} |k|^2 u_k \bar{u}_k \pm \sum_{k\in\mathbb{Z}^n : \sum_{i=1}^{2q+2} (-1)^i k_i = 0} u_{k_1} \bar{u}_{k_2} u_{k_3} \bar{u}_{k_4} \ldots u_{2q+1} \bar{u}_{2q+2}. \tag{3}$$

In order to study solutions of the NLS one needs to find suitable coordinates in which the Hamiltonian H decomposes into a dominant simple part called the *normal form* of the NLS and a *small perturbation*.

In [2], Procesi and Procesi have built a normal form for the NLS by the following steps:

- Choosing a finite number of frequencies called *tangential sites*, $S = \{v_1, \ldots, v_m\} \subset \mathbb{Z}^n$ in a *generic form* (i.e., S, thought as a point in $\mathbb{Z}^{mn}$, does not lie in any of the varieties, defined by a finite list of polynomial equations (cf. Constraints 1–6 in [2]).
- Looking at the Hamiltonian in polar coordinates for tangential sites:

$$u_k := z_k, k \in S^c, u_{v_i} := \sqrt{\xi_i + y_i} e^{\mathrm{i}x_i} = \sqrt{\xi_i} \left(1 + \frac{y_i}{2\xi_i} + \cdots \right) e^{\mathrm{i}x_i} \text{ for } i = 1, \ldots, m, \tag{4}$$

  considering ξ_i as parameters, $|y_i| < \xi_i$, while $x_i, y_i, w_k := (z_k, \bar{z}_k)$ are dynamical variables;
- Giving degree 0 to the angles x_i, 2 to y_i and 1 to w_k;
- Collecting the quadratic terms on the dynamical variables w_k of degree 1 and y_j of degree 2.

The full description of the properties of this normal form is given in Proposition 1 of [3].

In particular, this normal form is a quadratic Hamiltonian in infinitely many symplectic variables with coefficients depending on finitely many parameters ξ_i. The explicit formula for it will be given in Proposition 2. It defines a linear operator $ad(N) := \{N, -\}$ acting on a certain infinite dimensional vector space $F^{(0,1)}$ (see Definition 2.5 of functions). By Theorem 1 of [2], for the NLS on an n-dimensional torus the space $F^{0,1}$ decomposes as an infinite direct sum of finite dimensional invariant under the action of $ad(N)$ subspaces, each of dimension $\leq 2n$. In the frequency basis of $F^{0,1}$ (cf. (12)), the linear operator $ad(N)$ restricted to each invariant subspace is described by a matrix (cf. subsection 2.6) with coefficients polynomials in the variables $\sqrt{\xi_i}$ of degree q in ξ. Furthermore, these matrices are constructed from the combinatorial objects called *colored marked graphs* with vertices integral vectors (cf. Definition 2.8).

Remark 1.1. In order to study the solutions of the NLS by perturbing the simplified model after having the normal form of the NLS, one wishes to find a symplectic change of variables for which this normal form becomes diagonal with non-zero and distinct eigenvalues for generic values of the parameters. Before doing this one has to check that this linear operator is *regular semisimple*, i.e. the eigenvalues of its finite dimensional blocks are all non-zero and distinct. This condition is then needed to prove further non-degeneracy properties of this Hamiltonian and stability of solutions of the NLS. This is why we need to study the colored marked graphs giving finite-dimensional blocks of the normal form of the NLS and the characteristic polynomials of these blocks.

Remark 1.2. By Lemma 2.14 in [4] the characteristic polynomials $\det(tI - ad(N)_{\mathcal{G}})$ of the operator $ad(N)$ restricted to the infinitely many blocks $\mathcal{G}$ are all polynomials in the variables ξ_i and t with integer coefficients (i.e. the square roots of ξ_i disappear). The issue is thus to prove that a rather complicated infinite list of polynomials in a variable t, of degree increasing with the space dimension, and with coefficients polynomials in the parameters ξ_i have all non-zero distinct roots for generic values of the parameters.

Following the classical theory of Sylvester the condition for a polynomial to have distinct roots is the non-vanishing of the discriminant while the condition for two polynomials to have distinct roots in common is the non-vanishing of the resultant. In our case, we can consider all the characteristic polynomials as having coefficients in the field of rational functions in the parameters ξ_i, its algebraic closure is a *field of algebraic functions*. Thus if the discriminant of a given polynomial $D(\xi)$ and the resultant $R(\xi)$ of two distinct polynomials in $\mathbb{Q}(\xi_1, \dots, \xi_m)[t]$ are formally non-zero, we make sure that outside the real hypersurfaces $D(\xi) = 0$, $R(\xi) = 0$, the two polynomials have distinct roots.

Although both the discriminant and the resultant can be computed by explicit formulas, proof of their non-vanishing for the infinite list of complicated polynomials appearing seems to be a hopeless task.

In order to avoid this in [4] we have proposed the following conjecture which is stronger than the non-vanishing of the desired polynomials.

Conjecture 1.1 (*Separation and irreducibility conjecture*). The characteristic polynomials of the possible graphs giving blocks of the normal form of the NLS are all distinct and irreducible as polynomials with integer coefficients, i.e., in $\mathbb{Z}[\xi_1, \dots, \xi_m, t] \subset \mathbb{Q}(\xi_1, \dots, \xi_m)[t]$.

Let us explain why if this conjecture holds true, then the problem mentioned in 1.2 can be solved. Recall that, if we have a list of different polynomials in one variable t, with coefficients in a field F, a sufficient condition that all their roots (in the algebraic closure $\overline{F}$ of F) be non-zero and distinct is that they are all *irreducible* (over F) and distinct. This follows immediately from the fact that an irreducible polynomial $f(t)$ is uniquely determined as the minimal polynomial of each of its roots (cf. [1]) and, in characteristic 0, its derivative $f'(t)$ is non-zero. By the irreducibility of $f(t)$, the greatest common divisor between $f(t)$, $f'(t)$ is 1, so all the roots of $f(t)$ are distinct.

In [4], we have completely proved Conjecture 1.1 for the cubic NLS on a torus of arbitrary dimension (i.e. for $q = 1$ and for any n) using allowable graphs and induction on n. Later in [6], by arithmetic arguments the author has proved the conjecture for the NLS of arbitrary degree on 1- and 2-dimensional tori (i.e. for any q and for $n = 1, 2$), and verified the separation and irreducibility of the characteristic polynomials of the possible graphs containing only black edges and giving 3-dimensional blocks of the normal form of the NLS when $q + 1$ is a prime. In this paper, we shall try to use allowable graphs to prove the conjecture for all the possible graphs giving 3-dimensional blocks of the normal form of the NLS when $q + 1$ is a prime.

1.3 *The plan of this paper*

This paper consists of five sections. Section 1 gives the main purpose and the motivation of the paper. In § 2, we express the problem for the NLS in algebraic terms, namely in the language of Cayley graphs and their corresponding matrices. In § 3, we state a related geometric problem. The main results of the paper are summarized in §§ 4 and 5.

2. The algebraic problem

In this section, we recall the way proposed in [2] to convert the problem for the NLS mentioned in Remark 1.1 to an algebraic problem. For simplicity, the author will drop the concrete analysis details which interested readers can find in the same paper.

Starting from the lattice $\mathbb{Z}^m := \{\sum_{i=1}^m n_i e_i\}$ with the standard basis elements $e_i = (0, \dots, 0, \underbrace{1}_{i}, 0, \dots, 0)$, $i = 1, 2, \dots, m$, we define the linear maps: the *mass* η and the *momentum* π (the name comes from the mechanical considerations)

$$\eta : \mathbb{Z}^m \to \mathbb{Z}, \eta\left(\sum_i n_i e_i\right) = \sum_i n_i,$$
$$\pi : \mathbb{Z}^m \to \mathbb{Z}^n, \pi(e_i) = v_i.$$

We also define *edges* as follows:

DEFINITION 2.1

Elements of $\mathbb{Z}^m$ which are linear combinations of at most $2q$ basic vectors with coefficients ± 1 and which have mass 0 or -2, except elements of forms $0, -2e_i$, are called *edges*.

The set of all edges is denoted by X_q, hence

$$X_q := \left\{ \ell = \sum_{j=1}^{2q} \pm e_{i_j} = \sum_{i=1}^{m} \ell_i e_i, \ell \neq 0, -2e_i, \sum_i |\ell_i| \leq 2q, \eta(\ell) = \sum_i \ell_i \in \{0, -2\} \right\}. \tag{5}$$

DEFINITION 2.2

The support of an edge $\ell = \sum_i \ell_i e_i$ is $\mathrm{supp}(\ell) = \{i : \ell_i \neq 0\}$.

Notice that by our constraints the support of an edge contains at least 2 elements.

Example 2.1. For $q = 1$, we have edges of the forms $e_i - e_j, -e_i - e_j$ where $i \neq j$. For $q = 2$, we have all the terms for $q = 1$ and edges of the forms $e_i - e_j \pm e_a - e_b, 2e_i - 2e_j, -3e_i + e_j$, where i, j, a, b are pairwise different.

DEFINITION 2.3

We distinguish the edges by color, as $X_q^0 := \{\ell \in X_q | \eta(\ell) = 0\}$ to be black and $X_q^{-2} := \{\ell \in X_q | \eta(l) = -2\}$ red.

2.1 *The explicit formula for N*

PROPOSITION 2 [2]

For generic choices of the tangential sites S, we have

$$N = (\omega(\xi), y) + \sum_{k \in S^c} |k|^2 |z_k|^2 + \mathcal{Q}(x, w), \tag{6}$$

where

$$\omega = \omega_0 + \nabla_\xi A_{q+1}(\xi) - (q+1)^2 A_q(\xi)\underline{1}, \omega_0 = (|v_1|^2, \ldots, |v_m|^2) \tag{7}$$

does not depend on the dynamical variables. Here $\underline{1} \in \mathbb{N}^m$ *denotes the vector with all coordinates equal to 1,* $\mathcal{Q}$ *is given by formula* (11).

DEFINITION 2.4

- When $\ell \in X_q^0$, we define $\mathcal{P}_\ell$ as the set of pairs (h, k) satisfying

$$\sum_{j=1}^m \ell_j v_j + h - k = 0; \sum_{j=1}^m \ell_j |v_j|^2 + |h|^2 - |k|^2 = 0, \ell \in X_q^0. \tag{8}$$

- When $\ell \in X_q^{-2}$, we define $\mathcal{P}_\ell$ as the set of unordered pairs $\{h, k\}$ satisfying

$$\sum_{j=1}^m \ell_j v_j + k + h = 0; \sum_{j=1}^m \ell_j |v_j|^2 + |k|^2 + |h|^2 = 0, \ell \in X_q^{-2}. \tag{9}$$

For every edge $\ell = \sum_{j=1}^m l_j e_j$, set $\ell = \ell^+ - \ell^-$, where $\ell^+ = \sum_{l_j>0} l_j e_j$; $\ell^- = -\sum_{l_j<0} l_j e_j$ and define

$$c(\ell) = c_q(\ell)$$
$$:= \begin{cases} (q+1)^2 \xi^{\frac{\ell^+ + \ell^-}{2}} \sum_{\alpha\in\mathbb{N}^m; |\alpha+\ell^+|_1 = q} \binom{q}{\ell^+ + \alpha} \binom{q}{\ell^- + \alpha} \xi^\alpha, & \ell \in X_q^0, \\ (q+1)q \xi^{\frac{\ell^+ + \ell^-}{2}} \sum_{\alpha\in\mathbb{N}^m; |\alpha+\ell^+|_1 = q-1} \binom{q+1}{\ell^- + \alpha} \binom{q-1}{\ell^+ + \alpha} \xi^\alpha, & \ell \in X_q^{-2}, \end{cases} \tag{10}$$

where for every $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{Z}^m$, we define $|\beta|_1 := \sum_{i=1}^m \beta_1, \ldots, \beta_m$, $\xi^\beta := \xi_1^{\beta_1} \ldots \xi_m^{\beta_m}$,

$$\mathcal{Q}(x, w) = \sum_{\ell\in X_q^0} c(\ell) e^{\mathrm{i}(\ell,x)} \sum_{(h,k)\in\mathcal{P}_\ell} z_h \bar{z}_k + \sum_{\ell\in X_q^{-2}} c(\ell) \sum_{h,k\in\mathcal{P}_\ell} (e^{\mathrm{i}(\ell,x)} z_h z_k + e^{-\mathrm{i}(\ell,x)} \bar{z}_h \bar{z}_k). \tag{11}$$

2.2 *The vector space $F^{0,1}$*

DEFINITION 2.5

$V^{0,1}$ is the vector space of functions with basis of the elements $\{e^{\mathrm{i}\sum_j \nu_j x_j} z_k, \quad e^{-\mathrm{i}\sum_j \nu_j x_j} \bar{z}_k\}$ and $F^{0,1}$ is the subspace of $V^{0,1}$ commuting with momentum and mass.

The condition of commuting with momentum (respectively with mass) selects the elements, called *frequency basis* of $F^{0,1}$,

$$F_B = \{e^{\mathrm{i}\sum_j \nu_j x_j} z_k, \quad e^{-\mathrm{i}\sum_j \nu_j x_j} \bar{z}_k\}; \quad \sum_j \nu_j v_j + k = \pi(\nu) + k = 0$$
$$\left(\text{respectively}, \sum_j \nu_j + 1 = 0\right), \quad k \in S^c. \tag{12}$$

Remark 2.1. An element of F_B is completely determined by the value of ν and the fact that the z variable may or may not be conjugated, thus sometimes we refer to $e^{\mathrm{i}\sum_j \nu_j x_j} z_{-\pi(\nu)}$ as $(\sum_j \nu_j e_j, +1)$ and to $e^{-\mathrm{i}\sum_j \nu_j x_j} \bar{z}_{-\pi(\nu)}$ as $(\sum_j \nu_j e_j, -1)$.

By construction $\nu \in \mathbb{Z}_c^m$, where

$$\mathbb{Z}_c^m := \{\mu \in \mathbb{Z}^m \mid -\pi(\mu) \in S^c\}. \tag{13}$$

2.3 *The Cayley graphs*

We try to describe the operator $ad(N)$ into the language of group theory and, in particular, of the *Cayley graphs*. In fact, to a matrix $C = (c_{i,j})$ we can always associate a graph with vertices, the indices of the matrix and an edge between i, j if and only if $c_{i,j} \neq 0$. For the matrix of $ad(N)$ in the frequency basis, the relevant graph comes from a special Cayley graph.

Let G be a group and $X = X^{-1} \subset G$ a subset.

DEFINITION 2.6

An X-marked graph is an oriented graph Γ such that each oriented edge is marked with an element $x \in X$,

$$a \xrightarrow{x} b \qquad\qquad a \xleftarrow{x^{-1}} b.$$

We mark the same edge, with opposite orientation, with x^{-1}. Notice that if $x^2 = 1$, we may drop the orientation of the edge.

A typical way to construct an X-marked graph is the following. Consider an action $G \times A \to A$ of G on a set A. We then define, as follows.

DEFINITION 2.7 (Cayley graph)

The graph A_X has as vertices the elements of A and, given $a, b \in A$ we join them by an oriented edge $a \xrightarrow{x} b$, marked x, if $b = xa$, $x \in X$.

From the definition, one immediately deduces the following.

Remark 2.2. The connected components of the Cayley graph are the orbits in A under the subgroup generated by X.

Now we consider the group $G = \mathbb{Z}^m \rtimes_\phi \{\pm 1\}$ (the semidirect direct product of two groups: $\mathbb{Z}^m$ with the summation of vectors and $\{\pm 1\}$ with the multiplication), where

$$\phi : \{\pm 1\} \to \mathrm{Aut}(\mathbb{Z}^m);\ (\phi(1))(a) = a,\ (\phi(-1))(a) = -a \quad \forall a \in \mathbb{Z}^m.$$

Then it is easy to see that the multiplication in G is defined as follows:

$$(a, 1)(b, \pm 1) = (a + b, \pm 1);\ (a, -1)(b, \pm 1) = (a - b, \mp 1). \tag{14}$$

We denote by $\tau := (0, -1)$ so $G = \mathbb{Z}^m \cup \mathbb{Z}^m\tau$ (i.e. we identify $(a, +1)$ with $a \in \mathbb{Z}^m$, $(a, -1)$ with $a\tau$).

Remark 2.3. Following Remark 2.1, we think of an element $a = e^{\mathrm{i}\sum_j \nu_j x_j} z_k$ as being associated to the group element which, by abuse of notation, we still denote by $a = \sum_j \nu_j e_j \in \mathbb{Z}^m$. Then $\bar{a} = e^{-\mathrm{i}\sum_j \nu_j x_j} \bar{z}_k$ is associated to the group element $a\tau = (\sum_j \nu_j e_j)\tau \in \mathbb{Z}^m\tau$.

Thus by (12) the frequency basis is indexed by elements of $G^1 \setminus \bigcup_{i=1}^m \{-e_i, -e_i\tau\}$, where

$$G^1 := \{a, a\tau,\ a \in \mathbb{Z}^m \mid \eta(a) = -1\}.$$

2.4 *Colored marked graphs*

We now consider the action of G on itself by left multiplication. Let $X = X_q^0 \cup X_q^{-2}\tau$ (see Definitions 2.1 and 2.3). It is easy to see that $X = X^{-1} \subset G$. Then we obtain the Cayley graph G_X of G with respect to the elements X (see Definition 2.1). By Definition 2.3, we see that

Remark 2.4. The edges of the Cayley graph G_X is colored. The edges of mass 0 is black, the edges of mass -2 is red.

For convenience of the readers, we describe the graph G_X more clearly. The graph G_X is exactly the graph with the set of vertices G and with the edges as follows:

Two vertices (a, σ_1) and (b, σ_2) in G are connected by

- a black oriented edge marked by ℓ if $\sigma_1 = \sigma_2$ and $b - a = \ell \in X_q^0$ (or in other words, $(b, \sigma_2) = (\ell, 1)(a, \sigma_1)$ where $\ell \in X_q^0$):

$$a \xrightarrow[\text{black}]{\ell} b\ ;$$

- a red unoriented edge marked by ℓ if $\sigma_1 = -\sigma_2$ and $a + b = \ell \in X_q^{-2}$ (or in other words, $(b, \sigma_2) = (\ell, -1)(a, \sigma_1)$ where $\ell \in X_q^{-2}$):

$$a \overset{\ell}{\underset{\text{red}}{=\!=\!=}} b.$$

(Remark that we shall denote a red edge by a double unoriented line.)

Otherwise two vertices (a, σ_1) and (b, σ_2) are not connected by any edge.

DEFINITION 2.8

An induced subgraph of G_X is called a colored marked graph.

By the construction of the colored marked graphs it is easy to see as follows.

Remark 2.5. In a connected colored marked graph, two vertices of the same sign are connected by a path in which the number of the red edges is even. Hence two vertices of the same sign have the same mass.

Similarly, the sum of masses of two vertices of different signs is equal to -2.

If $p \in \mathbb{Z}$, it is easily seen as in the following remark.

Remark 2.6. The set $G_p := \{a,\ \eta(a) = 0,\ a\tau \mid \eta(a) = p\}$ form a subgroup of G.

In particular, by Remark 2.2 we have as follows.

Remark 2.7.

(1) G_{-2} is generated by the elements $X_q := X_q^0 \cup X_q^{-2}$ and it is a connected component of the Cayley graph.
(2) G^1 is also a coset of G_{-2} and it is also a connected component of the Cayley graph.

2.5 *The matrix description of $ad(N)$*

We introduce polynomials A_r with indeterminates $\xi_1, \ldots, \xi_m$,

$$A_r(\xi_1, \ldots, \xi_m) = \sum_{\sum_i k_i = r} \binom{r}{k_1, \ldots, k_m}^2 \prod_i \xi_i^{k_i}. \tag{15}$$

Define iM as the matrix of the linear operator $ad(N) := \{N, -\} : F^{0,1} \to F^{0,1}$ in the frequency basis F_B (12). After computations (cf. 4.4 of [5]) and use of notations in 2.3, we get for $a = \sum_{i=1}^m \mu_i e_i$, $b \in \mathbb{Z}^m$, $\tau = (0, -1) \in G$,

$$M_{a,a} = K(a) + \sum_i \mu_i \frac{\partial A_{q+1}(\xi)}{\partial \xi_i} - \sum_i \mu_i (q+1)^2 A_q(\xi), \tag{16}$$

$$M_{a\tau,a\tau} = K(a\tau) - \sum_i \mu_i \frac{\partial A_{q+1}(\xi)}{\partial \xi_i} + \sum_i \mu_i (q+1)^2 A_q(\xi), \tag{17}$$

$$M_{a\tau,b\tau} = -c(\ell),\ M_{a,b} = c(\ell), \quad \text{if } a,\ b \text{ are connected by a black edge } \ell, \tag{18}$$

$$M_{a,b\tau} = -c(\ell),\ M_{a\tau,b} = c(\ell), \quad \text{if } a,\ b\tau \text{ are connected by a red edge } \ell, \tag{19}$$

where $K((a, \sigma)) := \frac{\sigma}{2}(|\sum_i \mu_i v_i|^2 + \sum_i \mu_i |v_i|^2)$ is the *quadratic energy* of (a, σ) with $a = \sum_{i=1}^m \mu_i e_i$, $\sigma = \pm 1$.

Other entries of M are zero.

In 11.1.1 of [2], it is shown that the blocks M on $F^{0,1}$ come into pairs of conjugate Lagrangian blocks Γ, $\Gamma\tau$. With respect to the frequency basis the blocks are described as the connected components of a graph Λ_S which we now describe.

DEFINITION 2.9

Given an edge $u \xrightarrow{x} v$, $u = (a, \sigma)$, $v = (b, \rho) = xu$, $x \in X_q$, we say that the edge is *compatible* with S or π if $K(u) = K(v)$.

Remark now that $K(-e_i) = K(-e_i)\tau = 0$. We call the elements $\{-e_i, -e_i\tau\}$ the *special components*. Let $\Theta = \mathrm{Ker}(\pi)$.

DEFINITION 2.10

The graph Λ_S is the subgraph of G_{X_q} inside $G^1 \setminus \bigcup_i \{-e_i + \Theta, (-e_i + \Theta)\tau\}$ in which we only keep the compatible edges.

We then have

Theorem 2.1. *The indecomposable blocks of the matrix M in the frequency basis correspond to the connected components of the graph Λ_S. The entries of M are given by* (16), (17), (18), (19).

Remark 2.8. The fact that in the graph Λ_S we keep only compatible edges implies, in particular, that the *scalar part* $K((a,\sigma))$ (which is an integer) is constant on each block. On the other hand, in general, there are infinitely many blocks with the same scalar part.

Remark 2.9. One of the main ingredients of our work is to understand the possible connected components of the graph Λ_S. We do this by analyzing such a component as a translation $\Gamma = Au$ where A is a complete subgraph of the Cayley graph but contained in G_{-2} and containing the element 0. In particular, we have shown (cf. [2], §9) that A can be chosen among a finite number of graphs which we call *combinatorial*.

2.6 *The new matrices*

By Remark 2.8, if we ignore the constant scalar term $K((a,\sigma))$, i.e. if we consider the matrix $C_{\mathcal{G}} := M_{\mathcal{G}} - K((a,\sigma))I$, then

$$\det(tI - C_{\mathcal{G}}) = \det((t - K((a,\sigma)))I - M_{\mathcal{G}})$$

for each connected component $\mathcal{G}$ of Λ_S, hence the separation and irreducibility of the characteristic polynomials of blocks of $ad(N)$ do not change. Thus it will be convenient to work with $C_{\mathcal{G}}$.

Given (a,σ_1), $a = \sum_{i=1}^m n_i e_i$, set

$$(q+1)\tilde{a}(\xi) := \sum_{i=1}^m n_i \frac{\partial}{\partial \xi_i} A_{q+1}(\xi) - \sum_{i=1}^m n_i (q+1)^2 A_q(\xi). \tag{20}$$

For every colored marked graph of type (m,q) $\mathcal{G}$, we will consider the matrix $C_{\mathcal{G}}$ indexed by vertices of $\mathcal{G}$ as follows:

- In the diagonal at the position (a,σ_1), $a = \sum_{i=1}^m n_i e_i$, we put

$$\sigma_1 (q+1)\tilde{a}(\xi). \tag{21}$$

- At the position $((a,\sigma_1),(b,\sigma_2))$ we put 0 if they are not connected, otherwise we put $\sigma_2 c(\ell)$ (cf. (10), where ℓ is the the edge connecting a, b).

2.7 *Characteristic polynomials of colored marked graphs*

DEFINITION 2.11

$\chi_{\mathcal{G}} := \chi_{C_{\mathcal{G}}}(t) = \det(tI - C_{\mathcal{G}})$ is called the characteristic polynomial of graph $\mathcal{G}$.

Remark 2.10. By equation (12), Theorem 2.1 and Definition 2.8, the separation and irreducibility conjecture 1.1 is a corollary of the following conjecture.

Conjecture 2.1. The characteristic polynomials of the connected colored marked graphs with vertices of mass -1 are all distinct and irreducible.

From now we shall work with this conjecture. We recall the following:

Theorem 2.2 [6].

$$C_{\tau_c(\mathcal{G})} = (q+1)\tilde{c}(\xi)I + C_{\mathcal{G}}, \quad C_{\bar{\mathcal{G}}} = -C_{\mathcal{G}}. \tag{22}$$

where τ_c is the translation map by vector $c \in \mathbb{Z}^m$:

$$(d, \sigma_4) \mapsto (d, \sigma_4)c = (d + \sigma_4 c, \sigma_4), \quad \forall d \in \mathbb{Z}^m \tag{23}$$

and where $\bar{\mathcal{G}}$ is the image of $\mathcal{G}$ under the sign change

$$(d, \sigma_4) \mapsto (d, \sigma_4)(0, -1) = (d, -\sigma_4), \quad \forall d \in \mathbb{Z}^m. \tag{24}$$

COROLLARY 2.1 [6]

$$\chi_{\tau_c(\mathcal{G})}(t) = \chi_{\mathcal{G}}(t - (q+1)\tilde{c}(\xi)). \tag{25}$$

2.8 *The reduced matrices*

Lemma 2.1 [6]. *For any $a \in \mathbb{Z}^m$, $\tilde{a}(\xi)$ has integer coefficients.*

Hence all diagonal elements of $C_{\mathcal{G}}$ are divisible by $q+1$. Besides by formula (10) all off-diagonal elements of $C_{\mathcal{G}}$ are also divisible by $q+1$. Thus we can write

$$\begin{aligned} C_{\mathcal{G}} = (q+1)\tilde{C}_{\mathcal{G}} \Rightarrow \chi_{C_{\mathcal{G}}}(t) &= \det(tI - C_{\mathcal{G}}) = \det((q+1)t_1 I - (q+1)\tilde{C}_{\mathcal{G}}) \\ &= (q+1)^{|V(\mathcal{G})|}\chi_{\tilde{C}_{\mathcal{G}}}(t_1), \end{aligned}$$

where $V(\mathcal{G})$ is the set of vertices of $\mathcal{G}$. So in order to prove the irreducibility and the separation of the polynomials $\chi_{C_{\mathcal{G}}}$, it is enough to prove the irreducibility and the separation of the polynomials $\chi_{\tilde{C}_{\mathcal{G}}}$. For simplicity, we will denote $\tilde{C}_{\mathcal{G}}$ by $C_{\mathcal{G}}$, $\chi_{\tilde{C}_{\mathcal{G}}}$ by $\chi_{\mathcal{G}}$, and we will redefine $c(\ell)$ by dividing the right-hand sides of (10) by $q+1$,

$$c(\ell) = c_q(\ell) := \begin{cases} (q+1)\xi^{\frac{\ell^+ + \ell^-}{2}} \sum_{\alpha\in\mathbb{N}^m; |\alpha+\ell^+|_1 = q} \binom{q}{\ell^+ + \alpha}\binom{q}{\ell^- + \alpha}\xi^\alpha, & \ell \in X_q^0, \\ q\xi^{\frac{\ell^+ + \ell^-}{2}} \sum_{\alpha\in\mathbb{N}^m; |\alpha+\ell^+|_1 = q-1} \binom{q+1}{\ell^- + \alpha}\binom{q-1}{\ell^+ + \alpha}\xi^\alpha, & \ell \in X_q^{-2}. \end{cases} \tag{26}$$

The matrix $C_{\mathcal{G}}$ will be redefined as follows:

- In the diagonal at the position (a, σ_1), $a = \sum_{i=1}^m a_i e_i$, we put

$$\sigma_1 \tilde{a}(\xi). \tag{27}$$

- At the position $((a, \sigma_1), (b, \sigma_2))$, we put 0 if they are not connected, otherwise we put $\sigma_2 c(\ell)$ (c.f. (26), where ℓ is the edge connecting a, b).

2.9 *Factorization of a specialized characteristic polynomial*

For each $i \in \{1, \ldots, m\}$ consider the map $\pi_i, : \mathbb{Z}^m \rtimes_\phi \{\pm 1\} \to \mathbb{Z}^m \rtimes_\phi \{\pm 1\}$ by which we set the i-th coordinate to be zero. Then

$$\pi_i((a_1, a_2, \ldots, a_m), \delta) = ((a_1, \ldots, \underbrace{0}_{i}, \ldots, a_m), \delta).$$

Let $A \subset \mathbb{Z}^m \rtimes_\phi \{\pm 1\}$. Consider the colored marked graph with the set of vertices A, denoted by $\mathcal{G}_A$. Let $\mathcal{G}_A^i$ be the graph obtained from $\mathcal{G}_A$ by deleting all the edges which have i in the supports. We have the following proposition:

PROPOSITION 3 [6]

If $\mathcal{G}_A$ is connected, the map π_i restricted to A defines a graph isomorphism between $\mathcal{G}_A^i$ and $\mathcal{G}_{\pi_i(A)}$ (the colored marked graph with the set of vertices $\pi_i(A)$).

COROLLARY 2.2

When we set a variable $\xi_i = 0$ in $\chi_{\mathcal{G}_A}(t)$, we obtain the product of the polynomials $\chi_{\pi_i(\mathcal{B}_j)}(t)$ where B_j are the connected components of the graph $\mathcal{G}_A^i$.

3. The geometric problem

3.1 *The geometric graph*

In order to understand the possible components of the graph Λ_S we need to study a purely geometric graph defined on $\mathbb{R}^n$ using formulas (8) and (9) (Fig. 1).

DEFINITION 3.1

An edge $\ell \in X_q^{-2}$ defines a sphere S_ℓ through the relation

$$|x|^2 + \left(x, \sum_i \ell_i v_i\right) = \frac{-1}{2}\left(\left|\sum_i \ell_i v_i\right|^2 + \sum_i \ell_i |v_i|^2\right). \tag{28}$$

An edge $\ell \in X_q^0$ defines a plane H_ℓ through the relation

$$\left(x, \sum_i \ell_i v_i\right) = \frac{1}{2}\left(\left|\sum_i \ell_i v_i\right|^2 + \sum_i \ell_i |v_i|^2\right). \tag{29}$$

DEFINITION 3.2

Each $\ell \in S_\ell$ is joined by a *red* unoriented edge to $-x - \sum_i \ell_i v_i \in S_\ell$. Each $x \in H_\ell$ is joined by a *black* oriented edge to $x - \sum_i \ell_i v_i \in H_{-\ell}$. We construct the geometric graph Γ_S with vertices all the points of $\mathbb{R}^n$ and edges the black and edges described.

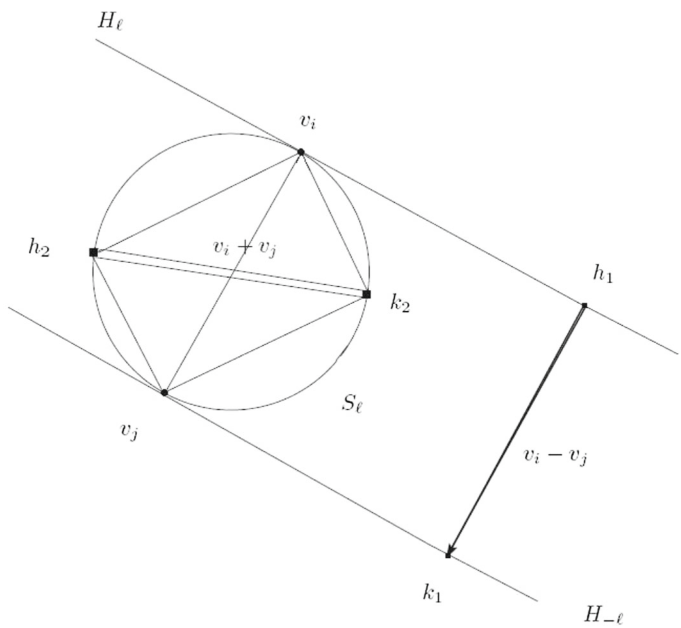


Figure 1. The plane H_ℓ with $\ell = e_j - e_i$ and the sphere S_ℓ with $\ell = -e_i - e_j$. The points h_1, k_1, v_j, v_i form the vertices of a rectangle. Same for the points h_2, v_i, k_2, v_j.

It is convenient to mark each edge of the graph with the element $-\pi(\ell)$ from which it comes from. Remark that for generic choices of S (cf. Constraint 1(iii) in [2]), the edge ℓ is uniquely determined by the vector $-\pi(\ell)$.

Remark 3.1. It is immediate by the definitions that the points in S are all pairwise connected by black and red edges and it is not hard to see that, for generic values of S, the set S is itself a connected component which we call the *special component*.

3.2 *The geometric graph as a subgraph of a Cayley graph*

We can think of $G = \mathbb{Z}^m \rtimes \pm 1$ also as linear operators on $\mathbb{R}^n$ by setting

$$ak := -\pi(a) + k,\ k \in \mathbb{R}^n,\ a \in \mathbb{Z}^m, \quad \tau k = -k. \tag{30}$$

We extend $\pi : \mathbb{Z}^m \to \mathbb{R}^n$ to $\mathbb{Z}^m \rtimes \pm 1$ by setting $\pi(a\tau) := \pi(a)$ so that $-\pi$ is just the orbit map of 0 associated to the action (30) (the sign convention is suggested by the conservation of momentum in the NLS).

We then have as follows.

Remark 3.2. X_q defines a Cayley graph on $\mathbb{R}^n$ and, in fact, the graph Γ_S is a subgraph of this graph.

3.3 *The connection of Γ_S with Λ_S*

This connection comes from the following remark:

Remark 3.3 [5]. Equations which define edges in the graph Γ_S are exactly the ones which define compatible edges in Λ_S. In other words, set $a, b \in \mathbb{Z}^m$ such that $-\pi(a) = x, -\pi(b) = y$, we have

(1) $x, y \in S_\ell$ are connected by a red edge marked by $-\pi(\ell)$ if and only if $a, b\tau$ are connected by a red edge marked by ℓ and $K(a) = K(b)$.
(2) $x \in H_\ell, y \in H_{-\ell}$ are connected by a black edge marked by $-\pi(\ell)$ if and only if a, b are connected by a black edge marked by ℓ and $K(a) = K(b)$

Therefore we have the following remark:

Remark 3.4. The map $-\pi$ gives an isomorphism between connected components of Λ_S and their images in Γ_S.

Thus in order to understand connected components of Λ_S one can study the connected components of the geometric graph Γ_S. Of course, the nature of this graph depends upon the choice of S but one expects a relatively simple behavior for generic choices of S.

3.4 *The equations defining a connected component of* Γ_S

Each connected component of the graph Γ_S has a combinatorial description based on (28) and (29) which encodes the information on the various types of edges which connect the vertices of the component.

In fact, it is clear that a connected component of Γ_S is obtained from a point $x \in \mathbb{R}^n$ and then applying the elements of an induced subgraph $A \subset G_{X_q}$ of the Cayley graph containing 0. Thus the problem consists in understanding when, given $x \in \mathbb{R}^n$, the elements $hx, h \in A$ describe the vertices of a connected component of the geometric graph Γ_S with *root* x in Γ_S.

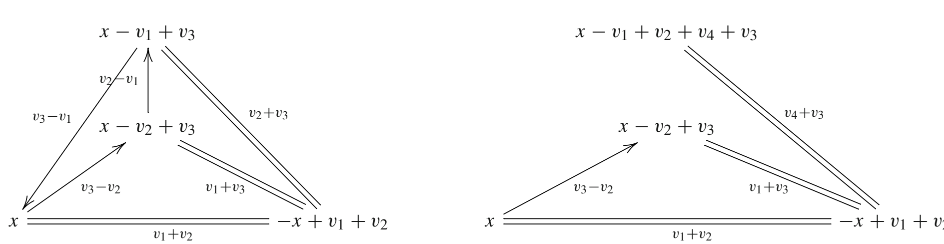
It is easy to prove the following proposition (cf. Lemma 5 in [2]).

PROPOSITION 4

The elements hx describe the vertices in a component C of the geometric graph Γ_S if and only if for each $h = (a, \sigma)$, we have

$$\begin{cases} (x, \pi(a)) = K(h), \text{ if } \sigma = 1 \Leftrightarrow \eta(a) = 0), \\ |x|^2 + (x, \pi(a)) = K(h), \text{ if } \sigma = -1 \Leftrightarrow \eta(a) = -2. \end{cases} \tag{31}$$

Example 3.1.



Following equations (31) we can write the equations that x has to satisfy:

$$\begin{array}{ll}(x, v_2 - v_3) = |v_2|^2 - (v_2, v_3), & (x, v_2 - v_3) = |v_2|^2 - (v_2, v_3),\\ |x|^2 - (x, v_1 + v_2) = -(v_1, v_2), & |x|^2 + (x, v_1 + v_2) = -(v_1, v_2),\\ (x, v_1 - v_3) = |v_1|^2 - (v_2, v_3), & (x, v_1 - v_2 - v_3 - v_4) = |v_1|^2 + (v_1, v_2) + (v_1, v_3)\\ & -(v_2, v_3) + (v_1, v_4) - (v_2, v_4) - (v_3, v_4).\end{array}$$

One may immediately ask the following question: For which graphs Γ we can say that equations (31) have at least one solution in $\mathbb{R}^n \backslash S$ (i.e. real solutions outside the special component) for generic values of the points v_i?

Such a graph is called *compatible*. We also give the following definition:

DEFINITION 3.3

We say that a connected colored marked graph $\mathcal{G}$ has a geometric realization outside the special component if its image $-\pi(\mathcal{G})$ is compatible.

4. Allowable graphs of the NLS

In [4], in order to study the connected components of Λ_S in the case of the cubic NLS we have given the definition of allowable graphs (cf. Definition 3.13 in [4]). Now we extend this definition for the case of the NLS of arbitrary degree.

DEFINITION 4.1

We say that a connected colored marked graph $\mathcal{G}$ is *allowable* if it does not have any pair of vertices $((a_1, \sigma_1), (a_2, \sigma_2))$ satifying

$$(a_1, \sigma_1)(a_2, \sigma_2)^{-1} = (\mu, -1), \tag{32}$$

where $\mu = (\mu_1, \ldots, \mu_m)$ is a vector of mass -2 in $\mathbb{Z}^m$ such that only one coordinate of μ is negative and $1 < \sum_{i=1}^m |\mu_i| \leq 2q + 2$. Otherwise we say that $\mathcal{G}$ is *not allowable*.

We will explain the name 'allowable graph' by showing the following lemma:

Lemma 4.1. *If a graph is not allowable then it has no geometric realization outside the special component.*

Proof. By the rules of multiplication in the group G (cf. (14)) from (32) we see that $\sigma_1 = -\sigma_2$.

By the assumptions of the lemma changing indices, if necessary, we may suppose that μ has the form

$$\mu = \sum_{i=1}^{r+1} \mu_i e_i, \tag{33}$$

where

$$\mu_1, \ldots, \mu_r, -\mu_{r+1} \in \mathbb{Z}^+ \backslash \{0\}, \sum_{i=1}^{r+1} \mu_i = -2. \tag{34}$$

Furthermore by a suitable translation, we may assume that $(a_1, \sigma_1) = (0, +1)$. Then by (32), we have $(a_2, \sigma_2) = (\mu, -1)$.

We write the quadratic equation (31) for a vertex x given by the vertex $(a_2, \sigma_2) = (\mu, -1)$,

$$|x|^2 + (x, \pi(\mu)) = K((\mu, -1)). \tag{35}$$

We have

$$K((\mu, -1)) = \frac{-1}{2}\left(\left|\sum_{i=1}^{r+1} \mu_i v_i\right|^2 + \sum_{i=1}^{r+1} \mu_i |v_i|^2\right). \tag{36}$$

Denote $(r+1)$-tuple $(v_1, \ldots, v_{r+1})$ by v. It is easy to see that equation (35) defines a sphere with the center $-\pi(\mu)/2$ and the radius

$$\begin{aligned} |\pi(\mu)/2|^2 + K((\mu, -1)) &= \frac{1}{4}\left|\sum_{i=1}^{r+1} \mu_i v_i\right|^2 + \frac{-1}{2}\left(\left|\sum_{i=1}^{r+1} \mu_i v_i\right|^2 + \sum_{i=1}^{r+1} \mu_i |v_i|^2\right) \\ &= -\frac{1}{4}\left|\sum_{i=1}^{r+1} \mu_i v_i\right|^2 - \frac{1}{2}\sum_{i=1}^{r+1} \mu_i |v_i|^2 = -\frac{1}{4} R_\mu(v), \end{aligned} \tag{37}$$

where

$$R_\mu(v) = \left(\sum_{i=1}^{r+1} (\mu_i^2 + 2\mu_i)|v_i|^2 + \sum_{i \neq j, i, j \in \{1, \ldots, r+1\}} \mu_i \mu_j (v_i, v_j)\right). \tag{38}$$

If $\mu = -2e_{i_0}$, then the center of the sphere is $\pi(-2e_{i_0})/2 = v_{i_0}$. The radius of the sphere is $-\frac{1}{4} R_{-2e_{i_0}}(v) = 0$ for all v. Hence equation (35) is equivalent to $|x - v_i|^2 = 0$ and has the unique real solution $x = v_i$. Then we apply Remark 15 of [2] where it is shown that the special component is an isolated component of the graph.

In other cases, at first, we will show that the quadratic form $R_\mu(v)$ is positive semi-definite. In fact, the matrix of this quadratic form is the following $(r+1) \times (r+1)$ matrix:

$$\mathcal{R} = \begin{pmatrix} \mu_1^2 + 2\mu_1 & \mu_1\mu_2 & \ldots & \mu_1\mu_r & \mu_1\mu_{r+1} \\ \mu_1\mu_2 & \mu_2^2 + 2\mu_2 & \ldots & \mu_2\mu_r & \mu_2\mu_{r+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \mu_1\mu_r & \ldots & \ldots & \mu_r^2 + 2\mu_r & \mu_r\mu_{r+1} \\ \mu_1\mu_{r+1} & \ldots & \ldots & \mu_r\mu_{r+1} & \mu_{r+1}^2 + 2\mu_{r+1} \end{pmatrix}. \tag{39}$$

Hence by Sylvester criterion for the positive semi-definiteness of quadratic forms, it is sufficient to show that all the principle minors of the matrix $\mathcal{R}$ are non-negative.

The r first principle minors of the matrix $\mathcal{R}$ are exactly all the principle minors of the quadratic form

$$\begin{aligned}\tilde{R}_{\mu}(v_1,\dots,v_r) &= \left(\sum_{i=1}^{r}(\mu_i^2+2\mu_i)|v_i|^2+\sum_{i\neq j,i,j\in\{1,\dots,r\}}\mu_i\mu_j(v_i,v_j)\right)\\ &= \left|\sum_{i=1}^{r}\mu_i v_i\right|^2+2\sum_{i=1}^{r}\mu_i|v_i|^2, \end{aligned}\tag{40}$$

which is obviously positive definite since $\mu_1,\dots,\mu_r>0$ by assumptions (34). Therefore all these minors are positive. Now we consider the last principal minor of the matrix $\mathcal{R}$. It is $\det(\mathcal{R})$. If we take the sum of all the row vectors, of $\mathcal{R}$, we get a vector the j-th coordinate of which is

$$\sum_{i=1;i\neq j}^{r+1}\mu_i\mu_j+(\mu_j^2+2\mu_j)=\mu_j\left(\sum_{i=1}^{r+1}\mu_i+2\right)=0, \tag{41}$$

since $\sum_{i=1}^{r+1}\mu_i=-2$. Therefore $\det(\mathcal{R})=0$.

Thus we have proved that the quadratic form $R_\mu(v)$ is positive semi-definite. If we write $R_\mu(v)$ as a sum of squares of linear polynomials in variables $v_1,\dots,v_{r+1}$, the first term will be

$$(\mu_1^2+2\mu_1)\left(v_1+\sum_{j=2}^{r+1}\frac{\mu_1\mu_j}{\mu_1^2+2\mu_1}v_j\right)^2=(\mu_1^2+2\mu_1)\left(v_1+\sum_{j=2}^{r+1}\frac{\mu_j}{\mu_1+2}v_j\right)^2. \tag{42}$$

This term is zero if and only if

$$v_1+\sum_{j=2}^{r+1}\frac{\mu_j}{\mu_1+2}v_j=\frac{(\mu_1+2)v_1+\sum_{j=2}^{r+1}\mu_j v_j}{\mu_1+2}=0. \tag{43}$$

We have $\mu_1+2+\sum_{j=2}^{r+1}\mu_j=\sum_{j=1}^{r+1}\mu_j+2=0$ by assumptions (34). Hence it is clear that for generic choices of $v_1,\dots,v_m$ (cf. Constraint 4.1), equality (43) is impossible.

Constraint 4.1. We assume that $\sum_{j=1}^{m}n_j v_j\neq 0$ for all $n_i\in\mathbb{Z}$ such that $\sum_{i=1}^{m}n_i=0$, $1<\sum_{i=1}^{m}|n_i|\leq 2q+2$.

Therefore for generic choices of $v_1,\dots,v_m$, $R_\mu(v)$ is positive. Hence the radius $-\frac{1}{4}R_\mu(v)$ of the sphere defined by (35) is negative. Thus the graph $\mathcal{G}$ has no geometric realization outside the special component. □

From this lemma one immediately deduces the following corollary:

COROLLARY 4.1

If a colored marked $\mathcal{G}$ has a red edge $\ell = \ell^+ - \ell^-$, where the vector ℓ^- has only one nonzero coordinate, then $\mathcal{G}$ has no geometric realization outside the special component.

Proof. In this case, by the construction of the colored marked graph, there is a pair of vertices $((a_1, \sigma_1), (a_2, \sigma_2))$ in $\mathcal{G}$ satisfying $(a_1, \sigma_1)(a_2, \sigma_2) = (\ell, -1)$, where by definition of edges 2.1, $\ell = (\ell_1, \ldots, \ell_m)$ is a vector of mass -2 in $\mathbb{Z}^m$ such that only one coordinate of ℓ is negative and $1 < \sum_{i=1}^m |\ell_i| \leq 2q + 2$. Therefore $\mathcal{G}$ is not allowable and by Lemma 4.1, it has no geometric realization outside the special component. □

5. On the separation and irreducibility of the characteristic polynomials of the possible graphs giving blocks of the NLS of arbitrary degree

We recall the following useful remark.

Remark 5.1 [6]. Let $\ell = \ell^+ - \ell^-$ be an edge. We have

(i) if ℓ is a black edge, then $|\ell^+|_1 = |\ell^-|_1 \leq q$,
(ii) if ℓ is a red edge, then $|\ell^+|_1 \leq q - 1$, $|\ell^-|_1 \leq q + 1$.

By Lemma 4.1, we need to consider only allowable graphs.

5.1 *Irreducibility*

Remark 5.2. When we look at subsubsections 4.4.1 and 4.4.2 of [6], it is easy to see that all the proofs are applicable also to the graphs containing red edges if we can prove that for every edge ℓ, the absolute value of each of its coordinates is strictly less than $q + 1$ and $c(\ell)$ is divisible by $q + 1$.

Now by using Corollary 4.1, we are able to do this.

Lemma 5.1. *If ℓ is an edge in an allowable graph and $q + 1$ is a prime, then the absolute value of each of its coordinates of ℓ is strictly less than $q + 1$ and $c(\ell)$ is divisible by $q + 1$.*

Proof.

- We prove the first part of the remark. By Remark 5.1, all the positive coordinates of ℓ have absolute values strictly less than $q + 1$. Suppose that there exists i_0 such that $\ell_{i_0}^- = q + 1$. Since by Remark 5.1, $|\ell^-|_1 \leq q + 1$, we deduce that $\ell_i^- = 0$ for all $i \neq i_0$, i.e. the graph is not allowable by Corollary 4.1.
- To prove the second part of the remark,
  - if $\ell \in X_q^0$ by formula (26), it is clear that $c(\ell)$ is divisible by $q + 1$.
  - If $\ell \in X_q^{-2}$, then

$$c(\ell) = q\xi^{\frac{\ell^+ + \ell^-}{2}} \sum_{\alpha \in \mathbb{N}^m;\, |\alpha + \ell^+|_1 = q-1} \binom{q+1}{\ell^- + \alpha} \binom{q-1}{\ell^+ + \alpha} \xi^\alpha. \tag{44}$$

We write

$$\binom{q+1}{\ell^- + \alpha} = \frac{(q+1)!}{(\ell_1^- + \alpha_1)! \dots (\ell_m^- + \alpha_m)!}. \tag{45}$$

Since $q+1$ is a prime, in order to prove that $c(\ell)$ is divisible by $q+1$ by formulas (44) and (45), it is sufficient to show that $\ell_i^- + \alpha_i < q+1$ for all $i \in \{1, \dots, m\}$. In fact, suppose that there exists i_0 such that $\ell_{i_0}^- + \alpha_{i_0} = q+1$. Since $\sum_{i=1}^m (\ell_i^- + \alpha_i) = q+1$, we deduce that $\sum_{j=1, j \neq i_0}^m (\ell_j^- + \alpha_j) = 0$. Moreover, $\ell^-, \alpha \in \mathbb{N}^m \implies \ell_j^- = 0, \alpha_j = 0$ for all $j \neq i_0$, i.e. ℓ^- has only nonzero coordinate $\ell_{i_0}^-$. This means the graph is not allowable by Corollary 4.1. □

Following Remark 5.2, by the same arguments in [6], we have the following.

Theorem 5.1. *If $q+1$ is a prime then the characteristic polynomials of the connected allowable graphs giving 3-dimensional blocks of the normal form of the NLS of degree $2q+1$ are irreducible.*

5.2 *Separation*

We give the following definition:

DEFINITION 5.1

For every colored marked graph $\mathcal{G}$ with the set of vertices $V_{\mathcal{G}} = \{(w_1, \delta_1), \dots, (w_k, \delta_k)\}$ we call

$$L_{\mathcal{G}} := \{\delta_1 w_1, \dots, \delta_k w_k\} \tag{46}$$

the information list of $\mathcal{G}$.

The role of the information list is as follows:

Remark 5.3 [6]. The vertices of every colored marked graph are determined from its information list uniquely up to the left multiplication by $\tau = (0, -1)$.

We also recall the following important facts from [6] which hold true for any q.

Remark 5.4. Irreducibility of the characteristic polynomials of the colored marked graphs reduced to one vertex is trivial, since they are of degree 1 in t.

Theorem 5.2. *Every colored marked graph reduced to one vertex is determined from its characteristic polynomial uniquely up to the left multiplication by $\tau = (0, -1)$.*

Theorem 5.3. *The characteristic polynomial of every connected colored marked graph which contains only two vertices is irreducible.*

Theorem 5.4. *Every connected colored marked graph containing only two vertices is determined from its characteristic polynomial uniquely up to the left multiplication by* $\tau = (0, -1)$.

Now we are ready to prove the separation property of the characteristic polynomials of the connected allowable graphs with 3 vertices.

Theorem 5.5. *For any* q*, the connected allowable graphs with* 3 *vertices are determined from their characteristic polynomials uniquely up to the left multiplication by* $\tau = (0, -1)$.

Proof. Given $\chi_{\mathcal{G}}$, in order to recover $\mathcal{G}$ by Remark 5.3, it is sufficient to recover $L_{\mathcal{G}} = \{w'_1, w'_2, w'_3\}$. The coefficient of t^2 in $\chi_{\mathcal{G}}$ is $Tr(C_{\mathcal{G}})$, i.e., it is equal to $\tilde{w}'_1(\xi) + \tilde{w}'_2(\xi) + \tilde{w}'_3(\xi)$. From this, by similar arguments as in the proof of Theorem 4.1 of [6], we can determine the coordinates of $w'_1 + w'_2 + w'_3$, in particular, we can determine $s = \eta(w'_1 + w'_2 + w'_3) = \eta(w'_1) + \eta(w'_2) + \eta(w'_3)$. Let k be the mass of a vertex with positive sign of $\mathcal{G}$. We see that

- $\mathcal{G}$ contains 3 vertices of the same sign if and only if s is divisible by 3,
- $\mathcal{G}$ contains 2 vertices of positive signs and 1 vertex of negative sign if and only if $s = 2k + k + 2 = 3k + 2 = 2(\text{mod } 3)$,
- $\mathcal{G}$ contains 1 vertex of positive sign and 2 vertices of negative signs if and only if $s = k + 2(k + 2) = 3k + 4 = 1(\text{mod } 3)$.

Hence by computing s, we can define the number of vertices of positive signs and their masses. Set one variable $\xi_i = 0$, where i appears in the support of at least one edge of $\mathcal{G}$, say $\xi_1 = 0$, we have by Theorem 2.2,

$$\chi_{\mathcal{G}}|_{\xi_1=0} = \prod_j \chi_{\pi_1(\mathcal{B}_j)},$$

where $\mathcal{B}_j$ are the connected components of the graph obtained from $\mathcal{G}$ by removing all the edges which have 1 in the supports, π_1 is the projection $\mathbb{Z}^m \rtimes \{\pm 1\} \to \mathbb{Z}^m \rtimes \{\pm 1\}$ by which we set the first coordinate to be zero. Moreover, each $\pi_1(\mathcal{B}_j)$ contains at most 2 vertices. Therefore by Remark 5.4 and Theorem 5.3, their characteristic polynomials are irreducible and thus are determined by the unique factorization of $\chi_{\mathcal{G}}$. So by Theorems 5.2 and 5.4, from $\chi_{\pi_1(\mathcal{B}_j)}$, we can recover all the coordinates except the first coordinate of the vectors of $L_{\mathcal{B}_j}$. Thus all the coordinates, except the first coordinate of all the vectors of $L_{\mathcal{G}}$ can be recovered:

$$\begin{cases} w'_1 = (*, w'_{1,2}, \dots, w'_{1,m}), \\ w'_2 = (*, w'_{2,2}, \dots, w'_{2,m}), \\ w'_3 = (*, w'_{3,2}, \dots, w'_{3,m}). \end{cases} \tag{47}$$

Similarly, by setting another variable, say $\xi_2 = 0$, we can recover all the coordinates, except the second coordinate of the vectors of $L_{\mathcal{G}}$,

$$\begin{cases} w'_1 = (w'_{1,1}, *, \dots, w'_{1,m}), \\ w'_2 = (w'_{2,1}, *, \dots, w'_{2,m}), \\ w'_3 = (w'_{3,1}, *, \dots, w'_{3,m}). \end{cases} \tag{48}$$

If $(w'_{i,3}, \dots, w'_{i,m})$, $i = 1, 2, 3$ are pairwise different, then from the systems of equations (47),(48) we can recover all coordinates of w'_1, w'_2, w'_3:

$$w'_i = (w'_{i,1}, w'_{i,2}, \dots, w'_{i,m}), \quad i = 1, 2, 3. \tag{49}$$

Define $c = \sum_{j=3}^{m} w'_{i,j}$.

If there are only two vectors in $L_{\mathcal{G}}$ with the same $m-2$ last coordinates, we consider only the case when they have different signs. The rest of the case is treated similarly. Since the number of vertices of positive signs is already known and can not change, we may suppose that

$$\begin{cases} w'_{1,1} + w'_{1,2} + c = k, \\ w'_{2,1} + w'_{2,2} + c = k + 2, \end{cases} \tag{50}$$

$$\begin{cases} w'_{1,1} + w'_{2,2} + c = k + 2, \\ w'_{2,1} + w'_{1,2} + c = k. \end{cases} \tag{51}$$

From (50) and (51), we deduce that $w'_{2,2} - w'_{1,2} = 2, w'_{2,1} = w'_{1,1} \implies (w'_1, +1)$ $(-w'_2, -1)^{-1} = (-2e_2, -1)$. One deduces that the graph is not allowable by Definition 4.1.

If three vectors in $L_{\mathcal{G}}$ with the same $m-2$ last coordinates, we consider only the case when in $\mathcal{G}$ there are two vertices of positive signs. The other cases are treated similarly. Since the number of vertices of positive signs is already known and can not change, we may suppose that

$$\begin{cases} w'_{1,1} + w'_{1,2} + c = k, \\ w'_{2,1} + w'_{2,2} + c = k + 2, \\ w'_{3,1} + w'_{3,2} + c = k, \end{cases} \tag{52}$$

$$\begin{cases} w'_{1,1} + w'_{2,2} + c = k, \\ w'_{2,1} + w'_{3,2} + c = k + 2, \\ w'_{3,1} + w'_{3,2} + c = k. \end{cases} \tag{53}$$

From (52) and (53) we deduce that $w'_{1,1} = w'_{2,1}, w'_{1,2} = w'_{2,2}$, i.e. $L(\mathcal{G})$ contains 2 equal vectors. This is not possible. Hence

$$\begin{cases} w'_{1,1} + w'_{1,2} + c = k, \\ w'_{2,1} + w'_{2,2} + c = k + 2, \\ w'_{3,1} + w'_{3,2} + c = k, \end{cases} \tag{54}$$

$$\begin{cases} w'_{1,1} + w'_{2,2} + c = k + 2, \\ w'_{2,1} + w'_{3,2} + c = k, \\ w'_{3,1} + w'_{3,2} + c = k. \end{cases} \tag{55}$$

From (54) and (55) we deduce that $w'_{3,2} = w'_{2,2}, w'_{2,1} - w'_{3,1} = -2 \implies (w'_2, +1)$ $(-w'_3, -1)^{-1} = (-2e_1, -1)$. One deduces that $\mathcal{G}$ is not allowable by Definition 4.1. $\square$

From Remark 2.10, Theorem 5.1 and Theorem 5.5, we deduce the following corollary:

COROLLARY 5.1

For any q, the characteristic polynomials of the possible graphs giving 3*-dimensional blocks of the normal form, the NLS of degree* $2q + 1$ *are distinct. Moreover, if* $q + 1$ *is a prime, all of them are irreducible.*

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