Proc. Indian Acad. Sci. (Math. Sci.) Vol. 127, No. 5, November 2017, pp. 779–786.
https://doi.org/10.1007/s12044-017-0361-4

# Some infinite families of Ramsey $(P_3, P_n)$-minimal trees

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MS received 1 September 2016; revised 6 July 2017;
published online 20 November 2017

**Abstract.** For any given two graphs $G$ and $H$, the notation $F \rightarrow (G, H)$ means that for any red–blue coloring of all the edges of $F$ will create either a red subgraph isomorphic to $G$ or a blue subgraph isomorphic to $H$. A graph $F$ is a Ramsey $(G, H)$-minimal graph if $F \rightarrow (G, H)$ but $F - e \nrightarrow (G, H)$, for every $e \in E(F)$. The class of all Ramsey $(G, H)$-minimal graphs is denoted by $\mathcal{R}(G, H)$. In this paper, we construct some infinite families of trees belonging to $\mathcal{R}(P_3, P_n)$, for $n = 8$ and 9. In particular, we give an algorithm to obtain an infinite family of trees belonging to $\mathcal{R}(P_3, P_n)$, for $n \geq 10$.

**Keywords.** Ramsey minimal graph; coloring; Ramsey infinite; tree.

**2010 Mathematics Subject Classification.** Primary: 05C55; Secondary: 05D10.

## 1. Introduction

Let $G$ and $H$ be graphs. We write $F \rightarrow (G, H)$ if in any red–blue coloring on the edges of $F$, we will have either a red subgraph isomorphic to $G$ or a blue subgraph isomorphic to $H$. A graph $F$ is a *Ramsey $(G, H)$-minimal graph* if $F \rightarrow (G, H)$ but $F - e \nrightarrow (G, H)$, for every $e \in E(F)$. The class of all Ramsey $(G, H)$-minimal graphs is denoted by $\mathcal{R}(G, H)$. The pair $(G, H)$ will be called *Ramsey-finite* or *Ramsey-infinite* depending upon whether $\mathcal{R}(G, H)$ is finite or infinite, respectively. For other notations and terminologies, we refer to [2].

The problem of characterizing pairs of graphs $(G, H)$ that are Ramsey infinite was first addressed by Nešetřil and Rödl [4] in 1976. Yulianti et al. [6] determined some classes of graphs belonging to $\mathcal{R}(P_3, P_4)$. Faudree and Sheehan [3] determined all trees in $\mathcal{R}^T(P_3, P_4)$ and $\mathcal{R}^T(P_3, P_5)$. Rahmadani et al. [5] determined some families of graphs belonging to $\mathcal{R}(P_3, P_n)$, for $n \geq 6$. Furthermore, they gave an infinite class of trees in $\mathcal{R}(P_3, P_7)$.

 

It is known in [1] that the pair $(P_3, P_n)$, for $n \geq 6$ is Ramsey-infinite. It means that there is an infinite set of graphs that are in $\mathcal{R}(P_3, P_n)$, for each $n \geq 6$. In this paper, we give some infinite families of trees in $\mathcal{R}(P_3, P_n)$, for $n = 8$ and 9. We also give an algorithm to obtain some infinite families of trees belonging to $\mathcal{R}(P_3, P_n)$, for $n \geq 10$.

## 2. Main result

First, we will give a lemma that is useful in the proof of the main theorem. The following lemma gives some necessary conditions of Ramsey $(P_3, P_n)$-minimal graphs.

*Lemma* 2.1. *Let $F \in \mathcal{R}(P_3, P_n)$. Then,*

(*i*) *$P_n \subset F - M$ for any matching $M$ in $F$.*
(*ii*) *Every edge $e$ is in a path on $n$ vertices in $F$.*

*Proof.* We prove this lemma in the following:

(i) Consider any red–blue coloring of the edges of $F$. If $F$ contains no red $P_3$, then all red edges of $F$ will form a matching, say $M$, in $F$. Since $F \in \mathcal{R}(P_3, P_n)$ and $F$ contains no red $P_3$, there exists a blue $P_n$ in $F$. Therefore, $P_n \subset F - M$ for any matching $M$.
(ii) Suppose there exists one edge $e$ in $F$ such that $e$ is not in any $P_n$. Since $F \in \mathcal{R}(P_3, P_n)$, then there is a red–blue coloring such that $F - e$ contains neither a red $P_3$ nor a blue $P_n$. Extend this coloring for $F$ by coloring $e$ by blue. Since $e$ is not in any $P_n$, there is neither a red $P_3$ nor a blue $P_n$ in $F$. This is a contradiction to $F \in \mathcal{R}(P_3, P_n)$. Thus, every edge $e \in F$ is contained in some $P_n$.

□

We know in [1] that the Ramsey sets $\mathcal{R}(P_3, P_8)$ and $\mathcal{R}(P_3, P_9)$ are infinite. It means that $\mathcal{R}(P_3, P_8)$ and $\mathcal{R}(P_3, P_9)$ have infinite number of elements. The following theorems give an infinite family of trees in $\mathcal{R}(P_3, P_8)$ and $\mathcal{R}(P_3, P_9)$.

**Theorem 2.2.** *The tree $T_1^8$ belongs to $\mathcal{R}(P_3, P_8)$.*

*Proof.* Consider the tree $T_1^8$ as in Fig. 1. First, we prove that $T_1^8 \rightarrow (P_3, P_8)$. Consider any red–blue coloring on the edges of $T_1^8$. Suppose that there is no red $P_3$ in $T_1^8$. Then, there is at most one red edge from these three edges $v_1v_2, v_2v_3, c_0v_2$. If there is exactly one red edge among these three edges, say $x$, then consider the subtrees $A, B, C, D$ and $E$ in $T_1^8$. If there is no red edge among these three edges then the result follows immediately.

If $x = v_1v_2$, then there will be a blue path $P_8$ connecting a pendant vertex in $C$ to a pendant vertex in $D$ or $E$. If $x = v_2v_3$ then there will be a blue path $P_8$ connecting a pendant vertex in $C$ and to a pendant vertex in $A$ or $B$. If $x = c_0v_2$ then there will be a blue path $P_8$ connecting a pendant vertex in $A$ or $B$ to a pendant vertex in $D$ or $E$. In case all of the edges $v_1v_2, v_2v_3, c_0v_2$ are blue, then there will be a blue path $P_8$ connecting a pendant vertex in $C$ and to a pendant vertex in one of the other subtrees. Hence, $T_1^8 \rightarrow (P_3, P_8)$.

Second, we prove that $T_1^8 - e \nrightarrow (P_3, P_8)$, for every edge $e$ in $T_1^8$. Let $i, j \in \{1, 2\}$, where $i \neq j$. If $e = a_1a_{1i}$, then color the edges $v_2v_3, b_0v_1, a_0a_2, a_1a_{1j}$ by red and the remaining edges by blue. If $e = a_2a_{2i}$, then color $v_2v_3, b_0v_1, a_0a_1, a_2a_{2j}$ by red and the

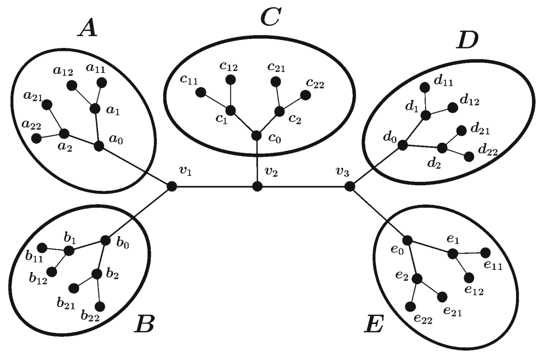


**Figure 1.** Tree $T_1^8$.

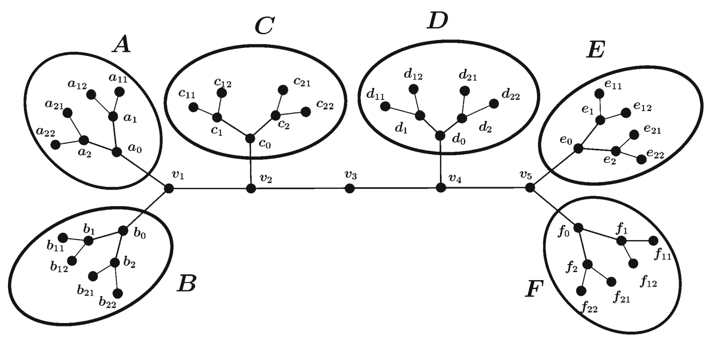


**Figure 2.** Tree $T_2^8$.

remaining edges by blue. If $e = a_0a_i$, then color edges $v_2v_3, b_0v_1, a_0a_j$ by red and the remaining edges by blue. Similarly, it also holds for every edge $e$ in $B$, $D$, and $E$.

If $e = c_1c_{1i}$, then color the edges $v_2v_3, b_0v_1, c_0c_2, c_1c_{1j}$ by red and the remaining edges by blue. If $e = c_2c_{2i}$, then color the edges $v_2v_3, b_0v_1, c_0c_1, c_2c_{2j}$ by red and the remaining edges by blue. If $e = c_0c_i$, then color the edges $v_2v_3, b_0v_1$ and $c_0c_j$ by red and the remaining edges by blue. If $e \in \{c_0v_2, b_0v_1, v_1v_2\}$, then color the edges $v_2v_3, a_0v_1, b_0b_1$ by red and the remaining edges by blue. If $e = a_0v_1$, then color the edges $v_2v_3, b_0v_1$ by red and the remaining edges by blue. If $e \in \{v_2v_3, d_0v_3\}$, then color the edges $e_0v_3, v_1v_2$ by red and the remaining edges by blue. If $e = e_0v_3$, then color the edges $d_0v_3, v_1v_2$ by red and the remaining edges by blue.

For any edge $e \in T_1^8$, we produce a coloring of $T_1^8 - e$ containing neither a red $P_3$ nor a blue $P_8$. Therefore, $T_1^8 - e \nrightarrow (P_3, P_8)$, for every $e$ in $T_1^8$. The proof is complete. □

**Theorem 2.3.** *The tree $T_2^8$ belongs to $\mathcal{R}(P_3, P_8)$.*

*Proof.* Consider the tree $T_2^8$ as in Fig. 2. We first prove that $T_2^8 \rightarrow (P_3, P_8)$. Consider any red–blue coloring on the edges of $T_2^8$. Suppose that there is no red $P_3$ in $T_2^8$. Then, there is at most one red edge from these two edges $v_2v_3, v_3v_4$ in $T_2^8$, say $x$. Next, we consider the subtrees $A, B, C, D, E$ and $F$ in $T_2^8$.

If $x = v_2v_3$, then the edges $v_1v_2$ and $c_0v_2$ must be blue. In addition, one of the edges $a_0v_1$ and $b_0v_1$ also must be blue. Consequently, there will be a blue path $P_8$ connecting a

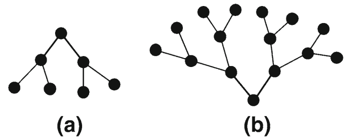


**Figure 3.** (**a**) Graph $J$ and (**b**) graph $L$.

pendant vertex in $C$ to a pendant vertex in $A$ or $B$. Similarly, it also holds if $x = v_3v_4$. If both of the edges $v_2v_3$, $v_3v_4$ are blue then we can extend these two blue edges into a blue path $P_8$ from two subtrees where the distance of their pendant vertex is at least 8. Hence, $T_2^8 \rightarrow (P_3, P_8)$.

Second, we prove that $T_2^8 - e \nrightarrow (P_3, P_8)$, for any edge $e$ in $T_2^8$. Let $i, j \in \{1, 2\}$, where $i \neq j$. If $e = a_1a_{1i}$, then color the edges $v_2v_3$, $v_4v_5$, $b_0v_1$, $a_0a_2$, $a_1a_{1j}$ by red and the remaining edges by blue. If $e = a_2a_{2i}$, then color the edges $v_2v_3$, $v_4v_5$, $b_0v_1$, $a_0a_1$, $a_2a_{2j}$ by red and the remaining edges by blue. If $e = a_0a_i$, then color the edges $v_2v_3$, $v_4v_5$, $b_0v_1$, $aa_j$ by red and the remaining edges by blue. Similarly, it also holds for every edge $e$ in $B$, $E$, and $F$.

If $e = c_1c_{1i}$, then color the edges $v_2v_3$, $v_4v_5$, $b_0v_1$, $c_0c_2$, $c_1c_{1j}$ by red and the remaining edges by blue. If $e = c_2c_{2i}$, then color edges $v_2v_3$, $v_4v_5$, $b_0v_1$, $c_0c_1$, $c_2c_{2j}$ by red and the remaining edges by blue. If $e = c_0c_i$, then color $v_2v_3$, $v_4v_5$, $b_0v_1$, $c_0c_j$ by red and the remaining edges by blue. Similarly, it also holds for every edge $e$ in $D$.

If $e = v_1v_2$, then color the edges $v_2v_3$, $v_4v_5$ by red and the remaining edges by blue. If $e \in \{v_2v_3, v_3v_4\}$, then color the edges $v_1v_2$, $v_4v_5$ by red and the remaining edges by blue. If $e = v_4v_5$, then color $v_1v_2$, $v_3v_4$ by red and the remaining edges by blue. If $e \in \{c_0v_2, d_0v_4\}$, then color $v_1v_2$, $v_4v_5$ by red and the remaining edges by blue. If $e = yv_1$ for $y \in \{a_0, b_0\}$, then color the edges $\{v_2v_3, v_4v_5\} \cup \{wv_1 | w \in \{a_0, b_0\}, w \neq y\}$ by red and the remaining edges by blue. If $e = zv_5$ for $z \in \{e_0, f_0\}$, then color the edges $\{v_1v_2, v_3v_4\} \cup \{wv_5 | w \in \{e_0, f_0\}, w \neq z\}$ by red and the remaining edges by blue.

For any edge $e \in T_2^8$, we produce a coloring of $T_2^8 - e$ containing neither a red $P_3$ nor a blue $P_8$. Therefore, $T_2^8 - e \nrightarrow (P_3, P_8)$, for every edge $e$ in $T_2^8$. The proof is complete. □

Consider the graph $J$ and $L$ as in Figures 3(a) and (b) respectively.

DEFINITION 2.4

For $m \geq 3$, the graph $T_m^8$ as in Fig. 4 is a tree constructed from a backbone path $P$ of order $2m + 1$ ($P \equiv v_1v_2 \ldots v_{2m+1}$) and subtrees $C_1, C_2, \ldots, C_{m-2}$, where $C_t \simeq K_{1,2}$ if $t$ is odd and $C_t \simeq J$ if $t$ is even.

For example, the trees $T_3^8$ and $T_4^8$ are shown in Fig. 5.

**Theorem 2.5.** *For any* $m \geq 3$, *the tree* $T_m^8 \in \mathcal{R}(P_3, P_8)$.

*Proof.* We first prove that $T_m^8 \rightarrow (P_3, P_8)$. Consider any red–blue coloring on the edges of $T_m^8$. Suppose that there is no red $P_3$. Then, there is at most one red edge from these three edges $\{v_1v_2, v_2v_3, v_2c_0\}$, say $x$ and at most one red edge from these edges $\{v_{2m-1}v_{2m}, v_{2m}v_{2m+1}, v_{2l}d\}$, say $y$. Next, we consider subtrees $A$, $B$, $C$, $D$, $E$ and $F$ in $T_m^8$ as in Fig. 4.

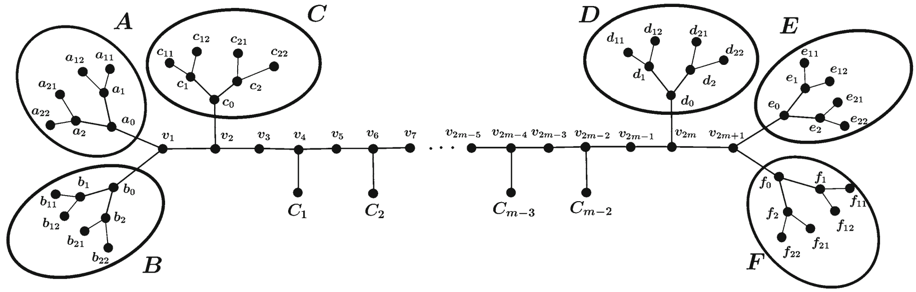


**Figure 4.** Tree $T_m^8$, where $C_t \cong K_{1,2}$ for $t$ is odd and $C_t \cong J$ for $t$ is even.

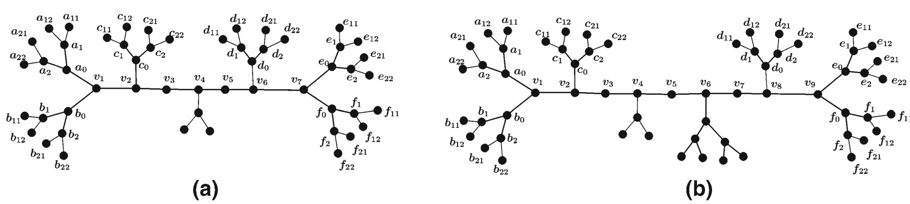


**Figure 5.** (**a**) Tree $T_3^8$ and (**b**) tree $T_4^8$.

Now, consider the three components of $T_m^8 - \{v_2v_3, v_{2m-1}v_{2m}\}$, denoted by $H_1$ for the component containing $v_2$; $H_2$ for the component containing $v_{2m}$; $H_3$ for the component containing $v_3$ and $v_{2m-1}$. If $x \in \{v_1v_2, v_2v_3, v_2c_0\}$ and $y = v_{2m-1}v_{2m}$, then there will be a blue path $P_8$ connecting a pendant vertex in $D$ to a pendant vertex in $E$ or $F$. If $x = v_1v_2$ and $y = v_{2m}d_0$, then there will be a blue path $P_8$ connecting a pendant vertex in $C$, $E$ or $F$ to a pendant vertex in $H_3$. If $x = v_1v_2$ and $y = v_{2m}v_{2m+1}$, then there will be a blue path $P_8$ connecting a pendant vertex in $C$ or $E$ or $F$ to a pendant vertex in $H_3$. If $x = v_2c_0$ and $y = v_{2m}d_0$, then there will be a blue path $P_8$ connecting a pendant vertex in $A$, $B$, $E$ or $F$ to a pendant vertex in $H_3$. Otherwise, this blue path $P_8$ connect two vertices in $H_3$. Similarly, it also holds for the other possibilities from the pair of $x$ and $y$. Hence, $T_m^8 \rightarrow (P_3, P_8)$, for $m \geq 3$.

Second, we prove that $T_m^8 - e \nrightarrow (P_3, P_8)$, for every edge $e$ in $T_m^8$. Let $i, j \in \{1, 2\}$, where $i \neq j$. We define $M_1 = \{v_2v_3, v_4v_5, \ldots, v_{2m}v_{2m+1}\}$ and $M_2 = \{v_1v_2, v_3v_4, \ldots, v_{2m-1}v_{2m}\}$.

If $e = a_1a_{1i}$, then color the edges $M_1 \cup \{b_0v_1, a_0a_2, a_1a_{1j}\}$ by red and the remaining edges by blue. If $e = a_2a_{2i}$, then color the edges $M_1 \cup \{b_0v_1, a_0a_1, a_2a_{2j}\}$ by red and the remaining edges by blue. If $e = a_0a_i$, then color the edges $M_1 \cup \{b_0v_1, a_0a_j\}$ by red and the remaining edges by blue. Similarly, it also holds for every edge $e$ in $B$, $E$ and $F$.

If $e = c_1c_{1i}$, then color the edges $M_1 \cup \{b_0v_1, c_0c_2, c_1c_{1j}\}$ by red and the remaining edges by blue. If $e = c_2c_{2i}$, then color the edges $M_1 \cup \{b_0v_1, c_0c_1, c_2c_{2j}\}$ by red and the remaining edges by blue. If $e = c_0c_i$, then color the edges $M_1 \cup \{b_0v_1, c_0c_j\}$ by red and the remaining edges by blue. Similarly, it also holds for every edge $e$ in $D$.

If $e = v_1v_2$, then color the edges $M_1$ by red and the remaining edges by blue. If $e = yv_1$ for $y \in \{a_0, b_0\}$, then color the edges $M_1 \cup \{wv_1 | w \in \{a_0, b_0\}, w \neq y\}$ by red and the remaining edges by blue. If $e = v_{2m}v_{2m+1}$, then color the edges $M_2$ by red and the remaining edges by blue. If $e = zv_{2m+1}$ for $z \in \{e_0, f_0\}$, then color the edges

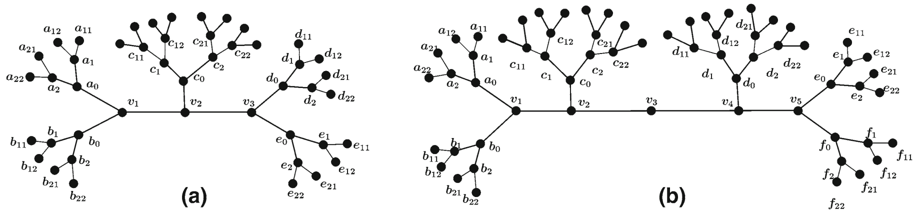


**Figure 6.** (**a**) Tree $T_1^9$ and (**b**) tree $T_2^9$.

$M_2 \cup \{pv_{2m+1} | p \in \{e_0, f_0\}, p \neq z\}$ by red and the remaining edges by blue. Each of these colorings of $T_m^8 - e$ contains neither a red $P_3$ nor a blue $P_8$.

Next, consider all edges in $H_3$. If $e$ is in $H_3$, then color edges $\{v_1v_2, v_{2m}v_{2m+1}\}$ by red and the remaining edges by blue. Consequently, there is no blue path $P_8$ in $H_1$ or $H_2$. In case $e$ in backbone path of $H_3$ color also one of two edges incident to every vertex that is connected to subtrees $C_1, C_2, \ldots, C_{m-2}$ such that there is no subtree in $H_3$ connected by two blue edges in backbone path. We color blue for the remaining edges. If $e$ is in a subtree in $H_3$, then use coloring as in case $e$ in the backbone path of $H_3$ and color an adjacent edge of $e$ by red. We color blue for the remaining edges.

For any edge $e \in T_m^8$, we produce a coloring of $T_m^8 - e$ containing neither a red $P_3$ nor a blue $P_8$. Therefore, $T_m^8 - e \nrightarrow (P_3, P_8)$, for every edge $e$ in $T_m^8$. The proof is complete. □

Next, consider the graph $T_1^9$ and $T_2^9$ as in Figures 6(a) and (b), respectively.

**Theorem 2.6.** *The tree $T_1^9$ belongs to $\mathcal{R}(P_3, P_9)$.*

*Proof.* First, we prove that $T_1^9 \rightarrow (P_3, P_9)$. Consider any red–blue coloring of all edges of $T_1^9$. Suppose that there is no red $P_3$. It is easy to check that $T_1^9 \supset T_1^8$. Now, consider all edges in $E(T_1^9) \setminus E(T_1^8)$ are pendant edges in $T_1^9$. Since, there are at least one of two pendant edges, it must be blue. Then, by using similar coloring as in $T_1^8$, we can extend a blue path $P_8$ into a blue path $P_9$. In any case, this component will contain a blue path $P_9$. Hence, $T_1^9 \rightarrow (P_3, P_9)$.

Second, we prove that $T_1^9 - e \nrightarrow (P_3, P_9)$ for every edge $e$. If $e \in E(T_1^9) \cap E(T_1^8)$, then use similar coloring as in $T_1^8 - e$ and color the remaining edges by blue. By this coloring in the graph $T_1^9 - e$, there is no red $P_3$ and no blue $P_9$ in $T_1^9 - e$. If $e \in E(T_1^9) \setminus E(T_1^8)$, then color a pendant edge adjacent to $e$ by red and use similar coloring as in $T_1^8 - e_1$ where $e_1$ is a pendant edge in $T_1^8$ adjacent to $e$ in $T_1^9$. Next, color $e_1$ in $T_1^9$ by blue. By this coloring, there is neither a red $P_3$ nor a blue $P_9$ in $T_1^9 - e$, for any edge $e \in E(T_1^9) \setminus E(T_1^8)$. Therefore, $T_1^9 - e \nrightarrow (P_3, P_9)$ for every edge $e$. The proof is complete. □

**Theorem 2.7.** *The tree $T_2^9$ belongs to $\mathcal{R}(P_3, P_9)$.*

*Proof.* It is easily shown that $T_2^9 \supset T_2^8$. Then, we can verify that graph $T_2^9$ is an element $\mathcal{R}(P_3, P_9)$. The main idea used in the proof of this theorem is similar to the one used in the proof of Theorem 2.6. □

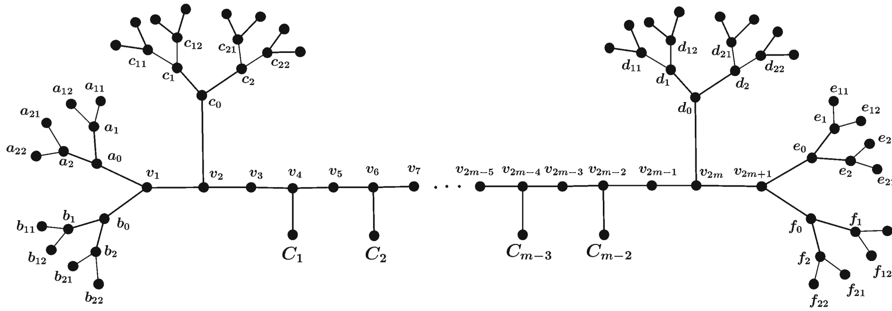


**Figure 7.** Graph $T_m^9$ where $C_t \cong K_{1,2}$ for $t$ is odd and $C_t \cong L$ for $t$ is even.

DEFINITION 2.8

For $m \geq 3$, the graph $T_m^9$ as in Fig. 7 is a tree constructed from a backbone path $P$ of order $2m+1$ ($P \equiv v_1 v_2 \ldots v_{2m+1}$) and subtrees $C_1, C_2, \ldots, C_{m-2}$, where $C_t \simeq K_{1,2}$ if $t$ is odd and $C_t \simeq L$ if $t$ is even.

**Theorem 2.9.** *For any $m \geq 3$, the tree $T_m^9 \in \mathcal{R}(P_3, P_9)$.*

*Proof.* Since $T_m^9 \supset T_m^8$, we can verify that the graph $T_m^9$ for $m \in \{3, 4, 5, \ldots\}$ is an element $\mathcal{R}(P_3, P_9)$. The main idea used in the proof of this theorem is similar to the one in the proof of Theorem 2.6. □

**Theorem 2.10.** *Let $F$ be a tree. For $n \geq 3$, if $F \in \mathcal{R}(P_3, P_n)$ then $\triangle(F) = 3$.*

*Proof.* By a contradiction, suppose that $\triangle(F) \neq 3$. If $F$ is a tree and $\triangle(F) < 3$ then $F \cong P_m$, for $m \geq 2$. It is easy to check that $P_m \notin \mathcal{R}(P_3, P_n)$, for $m \geq 2$.

Next, if $\triangle(F) > 3$, then without loss of generality we assume $\triangle(F) = 4$. Consider any red–blue coloring in $F$. Suppose that there is no red $P_3$ in $F$. Since $F \in \mathcal{R}(P_3, P_n)$, for $n \geq 8$, there exists a blue $P_n$ in $F$. Consider a vertex $v$ in $F$, where $d(v) = 4$. From Lemma 2.1(ii), we know that there exists a red–blue coloring such that $v$ is contained in a blue $P_n$ and $v$ is not the end vertex of a blue $P_n$. Since at most one of four incident edges with $v$ can be red, then the remaining edges must be blue. Let $e$ be one of the incident edges to $v$. Consider the minimality property of $F$ by deleting edge $e$. Now, suppose that there is no red $P_3$ in $F - e$. Then, we have at least two blue edges incident to $v$. However, we can extend two blue incident edges to $v$ into a blue $P_n$ in $F - e$. Consequently, there will be a blue $P_n$ in $F - e$. It means that $F$ is not minimal. This contradicts with $F \in \mathcal{R}(P_3, P_n)$. Therefore, $\triangle(F) = 3$. □

By the following algorithm, we can construct an infinite family of tree in $\mathcal{R}(P_3, P_n)$, for any $n \geq 10$.

*Algorithm* 2.11. To construct the graph $T_k^n \in \mathcal{R}(P_3, P_n)$, add two pendants to all vertices of degree one in graph $T_k^{n-2} \in \mathcal{R}(P_3, P_{n-2})$, for $n \geq 10$ and $k \geq 1$.

*Proof.* We prove by induction on $n$.

- From Theorems 2.2–2.9, we know that $T_k^8 \in \mathcal{R}(P_3, P_8)$ and $T_k^9 \in \mathcal{R}(P_3, P_9)$, for $k \geq 1$.
- Assume that $T_k^{n-2} \in \mathcal{R}(P_3, P_{n-2})$. Now, we prove that $T_k^n \in \mathcal{R}(P_3, P_n)$.
- By adding two pendants in every vertex of degree one in $T_k^{n-2}$, we obtain $T_k^n$. Consider any red–blue coloring of the edges of $T_k^n$. Assume that there is no red $P_3$. Since $T_k^{n-2} \subset T_k^n$ and $T_k^{n-2} \in \mathcal{R}(P_3, P_{n-2})$, there exists a blue $P_{n-2}$ in $T_k^n$. Let $u$ and $v$ be end vertices in a blue $P_{n-2}$. If $d(u) = d(v) = 1$ in $T_k^{n-2}$, then $d(u) = d(v) = 3$ in $T_k^n$. Since, at least one of the two incident edges to $u$ and $v$ in $T_k^n$ must be blue, we can extend a blue path $P_{n-2}$ into a blue path $P_n$. If $d(u) = 1$ (or $d(u) = 3$) and $d(v) = 2$ in $T_k^{n-2}$, then $d(u) = 3$ and $d(v) = 2$ in $T_k^n$. Since, at least one of the two edges incident to $u$ in $T_k^n$ must be blue, then this blue path can be extended into a blue path $P_{n-1}$.

  To avoid a blue path $P_n$, color an edge incident to $v$, say $e_1$ by red. However, this coloring will obtain a blue path $P_n$ which contain two edges adjacent to $e_1$. If $d(u) = d(v) = 2$ (or $d(u) = d(v) = 3$) in $T_k^{n-2}$ then $d(u) = d(v) = 2$ (or $d(u) = d(v) = 3$) in $T_k^n$. Since two incident edges with $u$ and $v$ are blue, then a blue path $P_{n-2}$ can be extended into a blue path $P_n$ in any coloring. Hence, $T_k^n \to (P_3, P_n)$.

  Second, we prove that $T_k^n - e \nrightarrow (P_3, P_n)$, for every $e$ in $T_k^n$. If $e \in E(T_k^n) \cap E(T_k^{n-2})$, we can use a similar coloring as in $T_k^{n-2} - e$ for every $e \in T_k^{n-2}$ and we color blue for the remaining edges. If $e \in E(T_k^n) \setminus E(T_k^{n-2})$, then we color by red a pendant edge adjacent to $e$ and use similar coloring as in $T_k^{n-2} - e_1$, where $e_1$ is a pendant edge in $T_k^{n-2}$ but $e_1$ is not a pendant edge that is adjacent to $e$ in $T_k^{n-2}$. Next, color $e_1$ in $T_k^{n-2}$ by blue. These colorings of $T_k^n - e$ contain neither a red $P_3$ nor a blue $P_n$. Therefore, $T_k^n - e \nrightarrow (P_3, P_n)$, for every $e$ in $T_k^n$. The proof is complete.

□

## Acknowledgements

This research was supported by Research Grants "Program Riset dan Inovasi KK ITB", "Program Hibah PMDSU ITB-DIKTI", Ministry of Research, Technology and Higher Education, Indonesia.

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Communicating Editor: Sharad S Sane