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On the partition dimension of two-component graphs

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Abstract. In this paper, we continue investigating the partition dimension for disconnected graphs. We determine the partition dimension for some classes of disconnected graphs G consisting of two components. If $G = G_1 \cup G_2$, then we give the bounds of the partition dimension of G for $G_1 = P_n$ or $G_1 = C_n$ and also for $pd(G_1) = pd(G_2)$.

Keywords. Partition dimension; disconnected graph; component.

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1. Introduction

The study of the partition dimension for graphs was initiated by Chartrand *et al.* [2] aimed at finding a new way to solve the problem in metric dimensions of graphs. Many results in determining the partition dimension of some connected graphs have been obtained, see [2–5,9,10]. The graphs of order n with partition dimension 2, n, $n-1$ and $n-2$ have been characterized, see [3,10]. The partition dimension of graphs derived from some graph operations such as corona product, Cartesian product and strong product has also been determined, see [1,8,11,12]. However, this study was only limited for connected graphs.

In [6], Haryeni *et al.* extended the definition of partition dimension for a graph, such that it can be applied for any graph (including disconnected graphs). Let $G = (V, E)$ be an arbitrary (connected or disconnected) graph. The *distance* $d(u, v)$ between $u, v \in V(G)$ is defined as the number of edges in a shortest path connecting u and v in G. If there is no $u - v$ path in G, then define $d(u, v) = \infty$. A vertex $x \in V(G)$ *resolves* G if $d(w, x) \neq d(y, x)$ for any pair of distinct vertices $w, y \in V(G)$. For a subset of vertices $R \subset V(G)$ and $v \in V(G)$, the distance $d(v, R)$ between v and R is defined as $\min\{d(v, x) : x \in R\}$. Let $\Gamma = \{R_1, R_2, \ldots, R_k\}$ be an ordered k-partition of $V(G)$. Then we call R_i as the *partition class* of Γ for any $i \in [1, k]$. If all $d(v, R_i)$ are finite for all vertices $v \in V(G)$, then the *representation* of v with respect to Γ is defined as $r(v|\Gamma) = (d(v, R_1), d(v, R_2), \ldots, d(v, R_k))$. The partition Γ is a *resolving partition* if any two distinct vertices in G are resolved by some R_i. The least integer k such that G has a resolving k-partition is called the *partition dimension* of G and denoted by $pd(G)$ or $pdd(G)$ for connected or disconnected G, respectively. Otherwise, $pdd(G) = \infty$ if no integer k satisfies the above condition.

Some results in finding the partition dimension for disconnected graph have been obtained, see [6,7]. In this paper, we continue studying the partition dimension of disconnected graphs. In particular, we determine the partition dimension of $G = G_1 \cup G_2$ where $G_1 = P_n$ or C_n and G_2 is any connected graph, or for some graphs G_1 and G_2 in which $pd(G_1) = pd(G_2)$.

The following results are useful in proving our main theorem in the next section.

Lemma 1.1 [3]. *Let Γ be a resolving partition of $V(G)$ and $u, v \in V(G)$. If $d(u, w) = d(v, w)$ for all $w \in V(G)\backslash\{u, v\}$, then u and v belong to distinct partition classes of Γ.*

Theorem 1.2 **[6]**. *Let $G = \bigcup_{i=1}^{m} G_i$. If $pdd(G) < \infty$, then* $\max\{pd(G_i) : i \in [1, m]\} \leq pdd(G) \leq \min\{|V(G_i)| : i \in [1, m]\}$.

For integers $k, l_1, l_2, \ldots, l_k \geq 2$, define a caterpillar $C(k; l_1, l_2, \ldots, l_k)$ as a graph obtained by attaching l_i vertices to each vertex v_i of the path P_k for $i \in [1, k]$. If $l_1 = l_2 = \ldots = l_k = l$, then it is called a homogeneous caterpillar and is denoted by $C(k, l)$. Darmaji *et al.* [4] gave the partition dimension of a homogeneous caterpillar as follows.

Theorem 1.3 **[4]**. *Let $C(k, l)$ be a homogeneous caterpillar with $l \geq 3$ and $k \geq 2$. Then,*

$$pd(C(k, l)) = \begin{cases} l, & \text{if } k \leq l, \\ l + 1, & \text{otherwise.} \end{cases}$$

2. Path or cycle components

For $m \geq 1$, let $G = \bigcup_{i=1}^{m} G_i$ and $\Gamma = \{R_1, R_2, \ldots, R_k\}$ be a partition of $V(G)$. Following [6], for any $t \geq 1$, define a vertex v of G as a *t-distance vertex* if $d(v, R_j) = 0$ or t for any $R_j \in \Gamma$. Such a partition Γ is called as a *connected partition* if every subgraph induced by $R_j \cap V(G_i)$ is connected for every $j \in [1, k]$ and $i \in [1, m]$. In this section we consider graph G with only two components. In particular, we will determine the partition dimension of $G = G_1 \cup G_2$ where $G_1 = P_n$ or C_n. If $G_2 = K_m$ where $|V(G_1| \geq m$ and $m \geq 4$, then $pdd(G) = m$. Otherwise, $pdd(G) = \infty$. For now on, we consider G_2 is not complete.

The following lemma shows that any connected k-partition is a resolving partition of a path P_m or a cycle C_m.

Lemma 2.1. *For $k \in [3, m]$, any connected k-partition of $V(P_m)$ or $V(C_m)$ is a resolving partition.*

Proof. Let $G = P_m$ or C_m, where $V(G) = \{u_i : i \in [1, m]\}$, and $\Gamma = \{R_1, R_2, \ldots, R_k\}$ be a connected k-partition of G where $k \in [3, m]$. Then, without loss of generality there are integers $s_1 < s_2 < \ldots < s_k = m$ such that

$$\begin{aligned} R_1 &= \{u_1, u_2, \ldots, u_{s_1}\}, \\ R_2 &= \{u_{s_1+1}, u_{s_1+2}, \ldots, u_{s_2}\}, \\ &\vdots \\ R_k &= \{u_{s_{k-1}+1}, u_{s_{k-1}+2}, \ldots, u_{s_k}\}. \end{aligned}$$

Now, consider any two vertices $x, y \in V(G)$ in R_t for some $t \in [1, k]$. If $x = u_i$ and $y = u_j$ where $i < j$, then $d(x, R_{t+1}) > d(y, R_{t+1})$ or $d(x, R_{t-1}) < d(y, R_{t-1})$. Therefore, $r(x|\Gamma) \neq r(y|\Gamma)$ and so Γ is a resolving partition of G. □

In the following theorem, we give the necessary condition such that $pdd(H \cup P_n) = pd(H)$ for any connected graph H where $pd(H) \geq 3$.

Theorem 2.2. *Let H be a connected graph of order at least 3 and $pd(H) \geq 3$. If $n \geq \max\{2 \times \mathrm{diam}(H) + 2, pd(H)\}$, then $pdd(H \cup P_n) = pd(H)$.*

Proof. For $t \geq 3$ and $k \geq 1$, let $pd(H) = t, \mathrm{diam}(H) = k$ and $G = H \cup P_n$. If $n \geq \max\{2k + 2, t\}$, then we assume that $pdd(G) < \infty$ and thus $pdd(G) \geq t$ by using Theorem 1.2. To prove that $pdd(G) \leq t$, let $\Gamma = \{R_1, R_2, \ldots, R_t\}$ be any resolving partition of H and $V(P_n) = \{v_i : i \in [1, n]\}$. Now we consider the following two cases.

Case 1. $2k + 2 > t$. Let $\Gamma'_1 = \{R'_1, R'_2, \ldots, R'_t\}$ be a partition of G, where

$$\begin{aligned} R'_i &= R_i \cup \{v_i\} \text{ for any } i \in [1, t-2], \\ R'_{t-1} &= R_{t-1} \cup \{v_i \,:\, i \in [t-1, n-1]\} \text{ and} \\ R'_t &= R_t \cup \{v_n\}. \end{aligned}$$

Now, we will show that Γ'_1 is a resolving partition of G. We consider any two vertices $x, y \in V(G)$ in R'_p for some $p \in [1, t]$. If $x, y \in V(H)$, then $r(x|\Gamma'_1) = r(x|\Gamma) \neq r(y|\Gamma) = r(y|\Gamma'_1)$. If $x = v_i$ and $y = v_j$ in P_n where $t-1 \leq i < j \leq n-1$, then $r(x|\Gamma'_1) \neq r(y|\Gamma'_1)$ since Γ'_1 is a connected partition in P_n by Lemma 2.1. Now assume for $x \in V(H)$ and $y = v_i \in V(P_n)$. If $i \geq k+2$, then $d(x, R'_1) \leq k < k+1 \leq i-1 = d(y, v_1) = d(y, R'_1)$. Otherwise, $d(x, R'_n) \leq k = (2k+2)-(k+2) < n-i = d(y, v_n) = d(y, R'_n)$. Therefore, $r(x|\Gamma'_1) \neq r(y|\Gamma'_1)$ for any two vertices $x, y \in V(G)$ in R'_p and thus Γ'_1 is a resolving partition of G.

Case 2. Let $t \geq 2k + 2$. Let $\Gamma'_2 = \{R'_1, R'_2, \ldots, R'_t\}$ be a partition of G, where

$$\begin{aligned} R'_i &= R_i \cup \{v_i\} \text{ for any } i \in [1, t-1] \text{ and} \\ R'_t &= R_t \cup \{v_i \,:\, i \in [t, n]\}. \end{aligned}$$

We investigate any two vertices $x, y \in V(G)$ in R'_l for some $l \in [1, t]$. As in Case 1, if $x, y \in V(H)$ or $x, y \in V(P_n)$, then $r(x|\Gamma'_2) \neq r(y|\Gamma'_2)$. Now we assume $x \in V(H)$ and $y = v_i \in V(P_n)$ for some $i \in [1, n]$. If $i \leq \frac{t}{2}$, then $d(x, R'_t) \leq k < k+1 \leq \frac{t}{2} \leq t - i = d(y, v_t) = d(y, R'_t)$. Otherwise, $d(x, R'_1) \leq k = \frac{t}{2} - 1 < i - 1 = d(y, v_1) = d(y, R'_1)$. Therefore, $r(x|\Gamma'_2) \neq r(y|\Gamma'_2)$ for any two vertices $x, y \in V(G)$ and so Γ'_2 is a resolving partition of G.

This concludes the proof that $pdd(G) = pd(H)$. □

We give the condition for a graph $H \cup C_m$ such that $pdd(H \cup C_m) = pd(H)$ for some values of $pd(H)$, as follows.

Theorem 2.3. *Let H be a connected graph of order ≥ 3. If $(pd(H) = 3$ and $m \geq 6 \times \text{diam}(H))$ or $(pd(H) = 4$ and $m \geq 4 \times \text{diam}(H))$, then $pdd(H \cup C_m) = pd(H)$.*

Proof. For $k \geq 1$, let $\text{diam}(H) = k$ and $G = H \cup C_m$ with $V(C_m) = \{u_i : i \in [1, m]\}$. Assume that $pdd(G) < \infty$. We consider the following two cases:

Case 1. Let $pd(H) = 3$ and $m \geq 6k$. Then, $pdd(G) \geq 3$ by Theorem 1.2. Now, we will show that $pdd(G) \leq 3$. Let $\Gamma_1 = \{R_1, R_2, R_3\}$ be a resolving partition of H. Let us show that $\Gamma'_1 = \{R'_1, R'_2, R'_3\}$, where $R'_1 = R_1 \cup \{u_1, u_2, \ldots, u_{2k}\}$, $R'_2 = R_2 \cup \{u_{2k+1}, u_{2k+2}, \ldots, u_{4k}\}$ and $R'_3 = R_3 \cup \{u_{4k+1}, u_{4k+2}, \ldots, u_m\}$ is a resolving partition of G. We consider two vertices $x, y \in V(G)$ in R'_j for some $j \in [1, 3]$. Note that if $x, y \in V(H)$, then $r(x|\Gamma'_1) = r(x|\Gamma_1) \neq r(y|\Gamma_1) = r(y|\Gamma'_1)$. For two vertices $x, y \in V(C_m)$, since Γ'_1 is a connected 3-partition of C_m, we have $r(x|\Gamma'_1) \neq r(y|\Gamma'_1)$ by Lemma 2.1. Now assume for $x \in V(H)$ and $y = u_i \in V(C_m)$. We consider three subcases.

Subcase 1.1. Let $x, y \in R_1$. If $i \in [1, k]$, then $d(y, R'_2) = d(u_i, u_{2k+1}) = 2k + 1 - i \geq 2k + 1 - k = k + 1 > k \geq d(x, R'_2)$. Otherwise, $d(y, R'_3) = d(u_i, u_{6k}) = i \geq k + 1 > k \geq d(x, R'_3)$.
Subcase 1.2. Let $x, y \in R_2$. If $i \in [2k + 1, 3k]$, then $d(y, R'_3) = d(u_i, u_{4k+1}) = 4k + 1 - i \geq 4k + 1 - 3k = k + 1 > k \geq d(x, R'_3)$. Otherwise, $d(y, R'_1) = d(u_i, u_{2k}) = i - 2k \geq 3k + 1 - 2k > k \geq d(x, R'_1)$.
Subcase 1.3. Let $x, y \in R_3$. If $i \in [4k + 1, 5k]$, then $d(y, R'_1) = d(u_i, u_1) = 6k - i + 1 \geq 6k - 5k + 1 > k \geq d(x, R'_1)$. Otherwise, $d(y, R'_2) = d(u_i, u_{4k}) = i - 4k \geq 5k + 1 - 4k > k \geq d(x, R'_2)$.

Hence we conclude that $r(x|\Gamma'_1) \neq r(y|\Gamma'_1)$ for any two distinct vertices $x, y \in V(G)$. This implies that Γ'_1 is a resolving partition of $G = H \cup C_m$, where $pd(H) = 3$ and $m \geq 6k$.

Case 2. Let $pd(H) = 4$ and $m \geq 4k$. By Theorem 1.2, we have $pdd(G) \geq 4$. Let $\Gamma_2 = \{R_1, R_2, R_3, R_4\}$ be a resolving partition of H. Let us show that $\Gamma'_2 = \{R'_1, R'_2, R'_3, R'_4\}$, where $R'_i = R_i \cup \{u_{(i-1)k+1}, u_{(i-1)k+2}, \ldots, u_{ik}\}$ for each $i \in [1, 3]$ and the remaining vertices are in R_4, is a resolving partition of G. Consider two vertices $x, y \in R'_j$ for some $j \in [1, 4]$. For two vertices $x, y \in V(H)$ or $x, y \in V(C_m)$, $r(x|\Gamma'_2) \neq r(y|\Gamma'_2)$ by a similar reason to Case 1. Now, for two vertices $x \in V(H)$ and $y \in V(C_m)$ in R'_j, $d(y, R'_a) \geq k + 1 > d(x, R'_a)$ for some integer $a \in \{j + 2, j - 2\}$. Therefore, Γ'_2 is a resolving partition of $G = H \cup C_m$ for $pd(H) = 4$ and $m \geq 4k$. □

In the next result, we give the partition dimension for a graph containing a tree. For a tree T, a vertex of degree at least 3 in T is called a *major vertex* of T. Any pendant vertex u of T is said to be a *terminal vertex* of a major vertex v if $d(u, v) < d(u, w)$ for every other major vertex w of T. The *terminal degree* of a major vertex v is the number of terminal vertices of v. A major vertex v is an *exterior major vertex* if it has positive terminal degree. Furthermore, a *fibre tree* is a tree with all vertices as pendant or exterior major vertices and each exterior major vertices has terminal degree at least 5.

Theorem 2.4. *Let T be a fibre tree of order ≥ 6 and $H = P_m$ or C_m. If $m \geq pd(T)$, then $pdd(H \cup T) = pd(T)$.*

Proof. For $t \geq 5$, let $\Gamma = \{R_1, R_2, \ldots, R_t\}$ be any minimum resolving partition of a fibre tree T and $x \in V(T)$. By the properties of a fibre tree T, if x is an exterior major vertex of T, then there are at least 4 distinct partition classes $R_i \in \Gamma$ satisfying $d(x, R_i) = 1$. Otherwise, there are at least 3 distinct partition classes $R_i \in \Gamma$ such that $d(x, R_i) = 2$. By doing any connected t-partition Γ in $H = P_m$ or C_m, we always have that Γ is also a resolving partition of H by Lemma 2.1. Since for $v \in V(H)$, $d(v, R_i) = 1$ or 2 for at most two distinct partition classes $R_i \in \Gamma$ in H, we immediately have $r(x|\Gamma) \neq r(y|\Gamma)$ for any $x, y \in V(T) \cup V(H)$. □

In the following theorem, we give the partition dimension of a disjoint union of a path P_m and a cycle C_n.

Theorem 2.5. *For $m \geq 3$ and $n \geq 4$, $pdd(P_m \cup C_n) = 3$.*

Proof. Let $G = P_m \cup C_n$ and $V(G) = V(P_m) \cup V(C_n) = \{u_i : i \in [1, m]\} \cup \{v_j : j \in [1, n]\}$. By using Theorem 1.2, we have $pdd(G) \geq 3$. To prove that $pdd(G) \leq 3$, we define a partition $\Gamma = \{R_1, R_2, R_3\}$ of G such that $R_1 = \{u_1, v_1\}$, $R_2 = \{u_2\} \cup \{v_i : i \in [2, n-1]\}$, and $R_3 = \{u_i : i \in [3, m]\} \cup \{v_n\}$. To prove that such a partition Γ is a resolving partition, we consider any two distinct vertices $x, y \in R_t$ for some $t \in [1, 3]$. Note that Γ is a connected partition of G. Hence if ($x = v_i$ and $y = v_j$ where $2 \leq i < j \leq n-1$), or ($x = u_i$ and $y = u_j$ where $3 \leq i < j \leq m$), then $r(v_i|\Gamma) \neq r(v_j|\Gamma)$ by Lemma 2.1. If $x = u_1$ and $y = v_1$, then $d(x, R_3) = 2 \neq 1 = d(y, R_3)$. If $x = u_2$ and $y = v_i$ where $i \in [2, n-1]$ or $x = v_n$ and $y = u_j$ where $j \in [3, m]$, then $d(x, R_a) = 1 < d(y, R_a)$ for some $a \neq t$. Thus we can conclude that $r(x|\Gamma) \neq r(y|\Gamma)$ for any $x, y \in V(G)$ which implies that Γ is a resolving partition of G. □

In the next theorem, we give the partition dimension of a disconnected graph consisting of a star $K_{1,n}$ and a cycle C_m.

Theorem 2.6. *For $n \geq 3$ and $m \geq 4$,*

$$pdd(K_{1,n} \cup C_m) = \begin{cases} n, & \textit{if } (n = 3 \textit{ and } m \geq 4, m \neq 5), \textit{ or } (m \geq n \geq 4), \\ n+1, & \textit{if } (n = 3 \textit{ and } m = 5), \\ \infty, & \textit{otherwise.} \end{cases}$$

Proof. For $n \geq 3$ and $m \geq 4$, let $G = K_{1,n} \cup C_m$ and $V(G) = V(K_{1,n}) \cup V(C_m) = \{v, v_i : i \in [1, n]\} \cup \{u_j : j \in [1, m]\}$. For ($n = 3$ and $m \geq 4, m \neq 5$), or ($m \geq n \geq 4$), or ($n = 3$ and $m = 5$), we claim that $pdd(G) < \infty$. Hence $pdd(G) \geq n$ by Theorem 1.2. We consider two cases.

Case 1. Let ($n = 3$ and $m = 4, 6, 7, \ldots$) or ($m \geq n \geq 4$). We will prove that $pdd(G) \leq n$. Let $\Gamma_1 = \{R_1, R_2, \ldots, R_n\}$ be a partition of G, where $R_i = \{v_i, u_i\}$ for any $i \in [2, n]$ and the remaining vertices are in R_1. Let x, y be any two distinct vertices of G in R_t for some $t \in [1, n]$. If $x = v_i$ and $y = u_i$ where $i \in [2, n]$, then $d(x, R_a) = 2 \neq 1 = d(y, R_a)$ for some integer $a \in \{i+1, i-1\} \backslash \{1\}$. If $x = v$ and $y \in \{v_1, u_1, u_j : j \in [n+1, m]\}$, then $d(x, R_a) = 1 < d(y, R_a)$ for some $a \in \{2, 3\}$. If $x = v_1$ and $y \in \{u_1, u_j : j \in [n+1, m]\}$, then $d(x, R_a) = 2 \neq d(y, R_a)$ for some $a \neq t$. If $x = u_j$ and $y = u_k$ where $n+1 \leq j < k \leq m$, by Lemma 2.1 since Γ_1 is connected, $r(x|\Gamma_1) \neq r(y|\Gamma_1)$. Therefore any two distinct vertices $x, y \in V(G)$ have distinct representation under Γ_1 and thus Γ_1 is a resolving partition of G.

Case 2. Let $n = 3$ and $m = 5$ and so $G = K_{1,3} \cup C_5$. It is easy to check that for any resolving 3-partition Γ of $K_{1,3}$ and C_5, there always exist $x \in V(K_{1,3})$ and $y \in V(C_5)$ such that $r(x|\Gamma) = r(y|\Gamma)$. Therefore, $pdd(G) \geq 4$. To prove that $pdd(G) \leq 4$, it is routine to verify that a partition $\Gamma_2 = \{R_1, R_2, R_3, R_4\}$ of G where for $i \in [2, 4]$, $R_i = \{v_{i-1}, u_i\}$ and the remaining vertices are in R_1 is a resolving partition of G.

Furthermore, for $m < n$, we suppose for the contrary that $pdd(G) < \infty$. Then, $n \leq pdd(G) \leq m$ by Theorem 1.2, a contradiction. □

In the following result, we give partition dimension of a graph containing a path P_n and a homogeneous caterpillar $C(k, l)$.

Theorem 2.7. *For $n, l \geq 3$ and $k \geq 2$,*

$$pdd(P_n \cup C(k,l)) = \begin{cases} l, & \text{if } (n \geq l = 3 \text{ and } k = 2), \text{ or} \\ & \quad (n \geq 6 \text{ and } l = k = 3), \\ & \text{or } (n \geq l \geq 4 \text{ and } k \leq l), \\ l+1, & \text{if } (n = 4, 5 \text{ and } l = k = 3), \text{ or} \\ & \quad (n > l \geq 3 \text{ and } k \geq l+1), \\ \infty, & \text{otherwise.} \end{cases}$$

Proof. Let $G = P_n \cup C(k, l)$ where $n, l \geq 3$, $k \geq 2$ and $V(G) = V(P_n) \cup V(C(k, l)) = \{v_i : i \in [1, n]\} \cup \{u_i, u_{i,j} : i \in [1, k], j \in [1, l]\}$. In the following Cases 1–4, we claim that $pdd(G) < \infty$.

Case 1. Let $n \geq l = 3$ and $k = 2$. Then, $pdd(G) \geq pd(C(k, l)) \geq 3$ by Theorem 1.2. Now we will show that $pdd(G) \leq 3$. Define a partition $\Gamma_1 = \{R_1, R_2, R_3\}$ of G where $R_1 = \{v_1, u_1, u_{1,1}, u_{2,1}\}$, $R_2 = \{v_2, u_{1,2}, u_{2,2}\}$ and the remaining vertices are in R_3. It is routine to verify that any two vertices $x, y \in V(G)$ in R_t for some $t \in [1, 3]$ have distinct representation under Γ_1. Therefore, Γ_1 is a resolving partition of G.

Case 2. Let $n \geq 6$ and $l = k = 3$. By Theorem 1.2, $pdd(G) \geq 3$. Let $\Gamma_2 = \{R_1, R_2, R_3\}$ be a partition of $V(G)$ where for each $j \in [1, 3]$, $R_1 = \{v_1, u_1, u_{j,1}\}$, $R_2 = \{v_2, v_3, v_4, v_5, u_2, u_{j,2}\}$ and the remaining vertices are in R_3. Now, consider any two distinct vertices $x, y \in V(G)$ in R_t for some $t \in [1, 3]$. Clearly, if $x = v_a$ and $y = v_b$ where $a < b$, then $r(x|\Gamma_2) \neq r(y|\Gamma_2)$ by Lemma 2.1, since Γ_2 is a connected partition of P_n. If $x = u_i$ and $y \in \{u_{j,i}, v_a\}$ for some a, then $d(x, R_p) = 1 < d(y, R_p)$ for some $p \neq t$. If $x = u_{i,j}$ and $y \in \{u_{k,j}, v_a\}$, then $d(x, R_p) = 2 \neq d(y, R_p)$ for some $p \neq t$. Therefore, $r(x|\Gamma_2) \neq r(y|\Gamma_2)$ and so Γ_2 is a resolving partition of G.

Case 3. Let $n \geq l \geq 4$ and $k \leq l$. Then, $pdd(G) \geq l$ by Theorem 1.2. Let $\Gamma_3 = \{R_1, R_2, \ldots, R_l\}$ be a partition of G induced by the function $f_3 : V(G) \to \{1, 2, \ldots, l\}$ as follows.

$$f_3(v_j) = f_3(u_{i,j}) = j, \text{ for each } i \in [1, k],\ j \in [1, l-1],$$
$$f_3(v_i) = l, \text{ for each } i \in [l, n],$$
$$f_3(u_i) = i, \text{ for each } i \in [1, k],$$

where $f_3(x) = i$ means $x \in R_i$. By a similar reason to Case 2, for any two vertices $x, y \in V(G)$ in R_t, we have $r(x|\Gamma_3) \neq r(y|\Gamma_3)$. Hence Γ_3 is a resolving partition of G.

Case 4. Let ($n = 4, 5$ and $l = k = 3$) or ($n > l \geq 3$ and $k \geq l + 1$). For $n = 4, 5$ and $l = k = 3$, we can check easily that for any 3-partition of G there always exists at least a vertex $x \in V(P_n)$ and $y \in V(C(k, l))$ with the same representation under such 3-partition. Thus, $pdd(G) \geq 4$. For $n > l \geq 3$ and $k \geq l + 1$, we have $pdd(G) \geq l + 1$ by Theorem 1.3. For these two conditions, we will show that $pdd(G) \leq l + 1$. Let $\Gamma_4 = \{R_0, R_1, R_2, \ldots, R_l\}$ be a partition of G induced by the function f_4 as follows.

$$f_4(v_i) = i, \text{ for any } i \in [1, l],$$
$$f_4(v_j) = 0, \text{ for any } j \in [l + 1, n]$$
$$f_4(u_1) = l,$$
$$f_4(u_i) = (i - 1) \mod l, \text{ for any } i \in [2, k],$$
$$f_4(u_{i,j}) = j - 1, \text{ for any } i \in [1, k],\ j \in [1, l].$$

We consider any two vertices $x, y \in V(G)$ in R_t for some $t \in [0, l]$. If $x = v_i$ and $y = v_j$ where $i, j \in [l + 1, n]$, then clearly $r(x|\Gamma_4) \neq r(\Gamma_4)$ by Lemma 2.1. If $x \in \{u_1, u_2\}$ and $y \in \{v_i, u_j, u_{k,l}\}$, then $d(x, R_p) = 1 < d(y, R_p)$ for some $p \neq t$. If $x = u_i$ and $y = u_j$ for some $i, j \in [3, k]$, then $d(x, R_l) < d(y, R_l)$ for $i < j$. If $x = u_i$ and $y \in \{u_{a,b}, v_c\}$ for $i \geq 3$, then $d(x, R_p) = 1 < d(y, R_p)$ for some $p \notin \{t, a - 1(\mod l), c - 1, c + 1\}$. If $x = u_{i,a}$ and $y \in \{u_{j,a}, v_b\}$, then $d(x, R_p) \neq d(y, R_p)$ for some $p \in \{l, 0\}$. Thus $r(x|\Gamma_4) \neq r(y|\Gamma_4)$ for any two distinct vertices $x, y \in V(G)$. Therefore, Γ_4 is a resolving partition of G.

Case 5. Let $n < l$ or $n = l = k = 3$ or $n = l < k$. For $n < l$ or $n = l = k = 3$, it is easy to see that $pdd(G) = \infty$. For $n = l < k$, we suppose for the contrary that $pdd(G) < \infty$. By using Theorem 1.3, we have $pd(C(k, l)) = l + 1$. By Theorem 1.2, we have $l + 1 \leq pdd(G) \leq n = l$, a contradiction. □

By observing the previous results, we have the following conjecture.

Conjecture 2.8. Let $G = F \cup H$ where $F = P_n$ or C_n, $pd(H) \geq pd(F)$ and $H \neq K_m$. If $pdd(G) < \infty$, then $pd(H) \leq pdd(G) \leq pd(H) + 1$.

The upper bound of Conjecture 2.8 is tight, since $pdd(P_4 \cup C(3, 3)) = pd(C(3, 3)) + 1 = 4$ and $pdd(C_5 \cup K_{1,3}) = pd(K_{1,3}) + 1 = 4$.

3. Components with the same partition dimension

We know that the only graph of order n having the partition dimension $n - 1$ are $K_{1,n-1}$, $K_n - e$, or $K_1 + (K_1 \cup K_{n-2})$ [3]. In this section, we determine $pdd(G_1 \cup G_2)$, where $|V(G_1)| = |V(G_2)| = n$ and $pd(G_1) = pd(G_2) = n - 1$. We also derive the upper bound of $pdd(2T)$ where T is a tree. We begin this part with the following lemma.

Lemma 3.1. *For $n \geq 4$, let Γ be a resolving partition of $K_n - e$ and $\alpha(\Gamma)$ be the number of 1-distance vertex of $K_n - e$ with respect to Γ. Then,*

$$\alpha(\Gamma) = \begin{cases} n-1, & \text{if } \Gamma \text{ is a minimum resolving partition,} \\ n-2, & \text{otherwise.} \end{cases}$$

Proof. For $n \geq 4$, let $G = K_n - e$, where $V(G) = \{v_i : 1 \leq i \leq n\}$ and $E(G) = \{v_i v_j : i \neq j\} \backslash \{v_1 v_n\}$. Since $pd(G) = n - 1$, let $\Gamma_1 = \{R_1, R_2, \ldots, R_{n-1}\}$ be any minimum resolving partition of G. Note that $d(v_i, v_k) = d(v_j, v_k)$ for all $v_k \in V(G) \backslash \{v_i, v_j\}$ for all $i, j \in [2, n-1]$. Therefore, v_i and v_j for each $i, j \in [2, n-1]$ are belong to distinct partition class of Γ by Lemma 1.1. Without loss of generality, assume that $v_i \in R_{i-1}$ for all $i \in [2, n-1]$. Since v_1 and v_n have the same distance to the other vertices, these two vertices belong to distinct partition classes of Γ_1. Thus, assume that $v_1 \in R_{n-1}$ and $v_n \in R_k$ for some $k \in [1, n-2]$. Therefore, v_j are 1-distance vertices for all $j \in [1, n-1]$. On the other hand, if we have a resolving n-partition $\Gamma_2 = \{R_1, R_2, \ldots, R_n\}$ of $K_n - e$, then R_i is a singleton partition for all $i \in [1, n]$. So, we could assume that $R_i = \{v_i\}$ for each $i \in [1, n]$. It follows that v_k are 1-distance vertices for all $k \in [2, n-1]$. This concludes the proof. □

Theorem 3.2. *Let G_1 and G_2 be two graphs of order $n \geq 4$ and $pd(G_1) = pd(G_2) = n - 1$. Then,*

$$pdd(G_1 \cup G_2) = \begin{cases} n-1, & \text{if } (G_1 \neq K_n - e \text{ and } G_2 \neq K_n - e), \\ n, & \text{if } (G_1 = K_n - e, G_2 \neq K_n - e), \text{ or} \\ & \quad (n = 4 \text{ and } G_1 = G_2 = K_n - e), \\ \infty, & \text{if } (n \geq 5 \text{ and } G_1 = G_2 = K_n - e). \end{cases}$$

Proof. Let $G = G_1 \cup G_2$ where $pd(G_1) = pd(G_2) = n-1$. Then, G_1 and G_2 are one of these graphs: $K_{1,n-1}$, $K_n - e$ or $K_1 + (K_1 \cup K_{n-2})$. Therefore, to determine the partition dimension of G, we only need to consider two disjoint union of these graphs. Now, let $V(K_{1,n-1}) = \{v, v_i : i \in [1, n-1]\}$, $E(K_{1,n-1}) = \{vv_i : i \in [1, n-1]\}$, $V(K_n - e) = \{w_i : i \in [1, n]\}$, $E(K_n - e) = \{w_i w_j : i, j \in [1, n]\} \backslash \{w_1 w_n\}$, $V(K_1 + (K_1 \cup K_{n-2})) = \{x, y, u_i : i \in [1, n-2]\}$ and $E(K_1 + (K_1 \cup K_{n-2})) = \{xy, xu_i, u_i u_j : i, j \in [1, n-2]\}$. Notice that for $G_i \neq K_n - e$ for some $i \in [1, 2]$, we claim that $pdd(G) < \infty$ so that $pdd(G) \geq n - 1$ by Theorem 1.2. Now, we investigate the following three cases.

Case 1. Let $G_1 \neq K_n - e$ and $G_2 \neq K_n - e$. We will show that $pdd(G) \leq n - 1$. We consider three subcases.

Subcase 1.1. Let $G = K_{1,n-1} \cup (K_1 + (K_1 \cup K_{n-2}))$. Let $\Gamma_1 = \{R_1, R_2, \ldots, R_{n-1}\}$ be a partition of $V(G)$ where $R_1 = \{v, v_1, u_1\}$, $R_2 = \{x, v_2, u_2\}$, $R_j = \{v_j, u_j : j \in [3, n-2]\}$ and $R_{n-1} = \{y, v_{n-1}\}$. Let us consider any two vertices $p, q \in V(G)$ in R_t for some $t \in [1, n-1]$. If $p = v$ and $q \in \{v_1, u_1\}$ or $p = x$ and $q \in \{v_2, u_2\}$, then $d(p, R_{n-1}) = 1 \neq 2 = d(q, R_{n-1})$. If $p = v_j$ and $q = u_j$ for some $j \in [1, n-2]$, then $d(p, R_l) = 2 \neq 1 = d(q, R_l)$ for some $l \notin \{j, n-1\}$. If $p = y$ and $q = v_{n-1}$, then $d(p, R_1) = 1 \neq 2 = d(y, R_1)$. Therefore, $r(p|\Gamma_1) \neq r(q|\Gamma_1)$ for any two distinct vertices $p, q \in V(G)$, and thus Γ_1 is a resolving partition of $G = K_{1,n-1} \cup (K_1 + (K_1 \cup K_{n-2}))$.

Subcase 1.2. Let $G = 2K_{1,n-1}$. Define a partition $\Gamma_2 = \{R_1, R_2, \ldots, R_{n-1}\}$ of G by taking the center vertex v of the first copy of G in R_1 and the center vertex v of the second copy in R_2, while the other remaining vertices $v_i \in R_i$ for each $i \in [1, n-1]$. We can easily show that Γ_2 is a resolving partition of $G = 2K_{1,n-1}$.

Subcase 1.3. Let $G = 2(K_1 + (K_1 \cup K_{n-2}))$. Let $\Gamma_3 = \{R_1, R_2, \ldots, R_{n-1}\}$ be a partition of G in which x of the first copy of G in R_1, x of the second copy in R_{n-1}, y of the first copy in R_{n-1} while y of the second copy in R_1, and the remaining vertices $u_i \in R_i$ for all $i \in [1, n-2]$. We also can verify directly that Γ_3 is a resolving partition of $G = 2(K_1 + (K_1 \cup K_{n-2}))$.

This concludes the proof that $pdd(G_1 \cup G_2) = n-1$ for $G_1 \neq K_n - e$ and $G_2 \neq K_n - e$.

Case 2. Let ($G_1 = K_n - e$ and $G_2 \neq K_n - e$) or ($G_1 = G_2 = K_4 - e$). Note that for any resolving $(n-1)$-partition Γ of each $K_{1,n-1}$ or $K_1 + (K_1 \cup K_{n-2})$, $v \in V(K_{1,n-1})$ or $x \in V(K_1 + (K_1 \cup K_{n-2}))$ must be a 1-distance vertex with respect to such partition Γ. For any minimum resolving $(n-1)$-partition of $K_n - e$, we have that $K_n - e$ has $(n-1)$ 1-distance vertices under such partition, by Lemma 3.1. Therefore, if $G_1 = K_n - e$ and ($G_2 = K_{1,n-1}$ or $K_1 + (K_1 \cup K_{n-2})$), then $pdd(G) \geq n$. Now for $G = K_n - e \cup K_{1,n-1}$, let $\Gamma_4 = \{R_1, R_2, \ldots, R_n\}$ be a partition of $V(G)$ where $R_1 = \{w_1, v\}$ and $R_i = \{w_i, v_{i-1}\}$ for every $i \in [2, n]$, while for $G = K_n - e \cup (K_1 + (K_1 \cup K_{n-2}))$, let $\Gamma_5 = \{R_1, R_2, \ldots, R_n\}$ be a partition of $V(G)$ where $R_i = \{w_i, u_i\}$ for all $i \in [1, n-2]$, $R_{n-1} = \{w_{n-1}, y\}$ and $R_n = \{w_n, x\}$. It is routine to show that Γ_4 or Γ_5 are resolving partitions of $G = K_n - e \cup K_{1,n-1}$ or $G = K_n - e \cup (K_1 + (K_1 \cup K_{n-2}))$, respectively. Now, assume $G = 2(K_4 - e)$. By Lemma 3.1, for any minimum resolving 3-partition of $K_4 - e$, the number of 1-distance vertex of $K_4 - e$ is 3. Thus, $pdd(G) \geq 4$. Again by Lemma 3.1, the number of 1-distance vertex of $K_4 - e$ with resolving 4-partition is 2. This is easy to define a resolving 4-partition Γ of each component of G in which the two 1-distance vertices in the first copy and the two 1-distance vertices in the second copy of G are in distinct partition classes. Since each partition class of each component of G are singleton, Γ is a resolving partition of G. It follows that for ($G_1 = K_n - e$ and $G_2 \neq K_n - e$) or ($n = 4$ and $G_1 = G_2 = K_n - e$), we have $pdd(G) = n$.

Case 3. Let $G = 2(K_n - e)$ where $n \geq 5$. We suppose for the contrary that there exists a resolving partition Γ of G. By considering Lemma 3.1, G has at least $2(n-2)$ 1-distance vertices under Γ. This is a desired contradiction since $|\Gamma| \leq n < n+1 \leq 2(n-2)$. □

For a tree T, the maximum distance between a vertex $x \in V(T)$ to all other vertices in T is considered as *eccentricity* of x, and denoted by $\mathrm{ecc}(x)$. Furthermore, a vertex $x \in V(T)$ is called a *peripheral vertex* if $\mathrm{ecc}(x) = \mathrm{diam}(T)$. Now we are going to derive the upper bound of the partition dimension of $2T$. We begin with the following lemma.

Lemma 3.3. *Let T be a tree of order ≥ 3 and $\Gamma = \{R_1, R_2, \ldots, R_t\}$ be a minimum resolving partition of T. Let $xy \in E(T)$ and x be a peripheral vertex in R_i for some $i \in [1, t]$. If $|R_i| = 1$, then a new partition $\Gamma' = \{R'_1, R'_2, \ldots, R'_t\}$ where $R'_j = R_j \backslash \{y\}$ for all $j \neq i$ and $R'_i = \{x, y\}$ is also a resolving partition of T.*

Proof. Assume that $y \in R_j$ where $j \neq i$. If $|R_j| = 1$, then a partition $\Gamma^* = \{R^*_1, R^*_2, \ldots, R^*_t\} \backslash \{R^*_j\}$ where $R^*_l = R_l$ for all $l \neq i$ and $R^*_i = \{x, y\}$ is also a resolving partition of T. Contradict to the fact that Γ is a minimum resolving partition of T. Therefore, we have $|R_j| \geq 2$. Now, let us show that a partition $\Gamma' = \{R'_1, R'_2, \ldots, R'_t\}$ of T where $R'_j = R_j \backslash \{y\}$ for all $j \neq i$ and $R'_i = \{x, y\}$, is a resolving partition of T.

We consider any two vertices $u, v \in V(T)$ in R'_a for some $a \in [1, t]$. If $u = x$ and $v = y$, then $d(u, R'_b) = d(v, R'_b) + 1 > d(v, R'_b)$ for all $b \neq a$. Now consider two vertices $u, v \in V(T)\backslash\{x, y\}$. If $d(u, y) \neq d(v, y)$, then $d(u, R'_i) \neq d(v, R'_i)$. Otherwise, we distinguish two cases.

Case 1. Assume that u and v are resolved by R_k with respect to Γ for some $k \neq j$. This implies that $d(u, R'_k) = d(u, R_k) \neq d(v, R_k) = d(v, R'_k)$.

Case 2. Assume that u and v are only resolved by R_j with respect to Γ. Then, there exist two vertices $p, q \in R_j$ such that $d(u, R_j) = d(u, p) \neq d(v, q) = d(v, R_j)$. Note that at most one of p or q can be a vertex y. If neither p nor q are equal to y, then $d(u, R'_j) = d(u, p) \neq d(v, q) = d(v, R'_j)$. If $p = y$, then $d(u, R_j) = d(u, p) = d(u, y) = d(v, y) > d(v, q) = d(v, R_j)$. This implies that $d(u, R'_j) = d(u, y_1) \geq d(u, y) > d(v, q) = d(v, R'_j)$ for other vertex $y_1 \neq y$ in R_j. Similarly, if $q = y$, then $d(u, R_j) = d(u, p) < d(v, y) = d(v, q) = d(v, R_j)$ and hence $d(u, R'_j) = d(u, p) < d(v, y) \leq d(v, y_2) = d(v, R'_j)$ for other vertex $y_2 \neq y$ in R_j. Therefore, u and v are also resolved by R'_j. □

By Lemma 3.3, it is easy to verify the following result.

COROLLARY 3.4

Let x be any peripheral vertex of a tree T. Then, there exists a minimum resolving partition $\Gamma = \{R_1, R_2, \ldots, R_t\}$ of T such that $x \in R_i$ and $|R_i| \geq 2$ for some $i \in [1, t]$.

Furthermore, a peripheral vertex in T satisfying Corollary 3.4 is called an *ideal vertex* of T. Now we are ready to give the upper bound of partition dimension of two trees $2T$, as follows.

Theorem 3.5. *Let T be a tree of order $n \geq 3$. Then, $pdd(2T) \leq pd(T) + 1$.*

Proof. Let $V(2T) = V(T_1) \cup V(T_2)$ and $E(2T) = E(T_1) \cup E(T_2)$ where for $i \in [1, 2]$, $V(T_i) = \{u_i : u \in V(T)\}$ and $E(T_i) = \{u_i v_i : uv \in E(T)\}$. Let $\Gamma = \{R_1, R_2, \ldots, R_t\}$ be a minimum resolving partition of T. Thus, $\Gamma_i = \{R_{i1}, R_{i2}, \ldots, R_{it}\}$ where $R_{ij} = \{u_i : u \in R_j\}$ for all $j \in [1, t]$ and $i \in [1, 2]$, is also a minimum resolving partition of each T_i. Let $x \in V(T)$ be an ideal vertex in $R_s \in \Gamma$ and so that $|R_s| \geq 2$. Therefore, $x_i \in V(T_i)$ is an ideal vertex in R_{is} where $|R_{is}| \geq 2$ with respect to Γ_i for each $i \in [1, 2]$. Let $x_i y_i \in E(T_i)$ where $y_i \in R_{il}$ for some $l \in [1, t]$.

For $S_j = R_{1j} \cup R_{2j}$, we define a new partition $\Gamma' = \{S'_1, S'_2, \ldots, S'_t, S'_{t+1}\}$ of $T_1 \cup T_2$ where

$$
\begin{aligned}
S'_i &= S_i\backslash\{x_1, x_2\} \text{ for all } i \in [1, t]\backslash\{l\},\\
S'_l &= S_l \cup \{x_2\}\backslash\{y_2\},\\
S'_{t+1} &= \{x_1, y_2\}.
\end{aligned}
$$

To show that Γ' is a resolving partition, we consider any two distinct vertices $u_i, v_j \in V(T_1) \cup V(T_2)$ in S'_a for some $i, j \in [1, 2]$ and $a \in [1, t + 1]$. We consider three cases.

Case 1. Let $u_1, v_1 \in V(T_1)\backslash\{x_1\}$. We consider two subcases.

Subcase 1.1. Assume that u and v in T are resolved by R_b where $b \neq s$. Then, u_1 and v_1 in T_1 are also resolved by S'_b, since $d(u_1, S'_b) = d(u, R_b) \neq d(v, R_b) = d(v_1, S'_b)$.

Subcase 1.2. Assume that u and v in T are only resolved by R_s. Thus, there exist two vertices $p, q \in R_s$ such that $d(u, R_s) = d(u, p) \neq d(v, q) = d(v, R_s)$. If $d(u, y) \neq d(v, y)$, then $d(u_1, S'_{t+1}) = d(u_1, x_1) = d(u, x) \neq d(v, x) = d(v_1, x_1) = d(v_1, S'_{t+1})$. Otherwise, we distinguish three subcases.

Subcase 1.2.1. Neither p nor q are equal to x. Then, $d(u_1, S'_s) = d(u_1, p_1) = d(u, p) \neq d(v, q) = d(v_1, q_1) = d(v_1, S'_s)$.
Subcase 1.2.2. Let $p = x$. Then, $d(u_1, S'_s) = d(u_1, s_1) \geq d(u_1, p_1) = d(u, p) = d(u, x) = d(v, x) > d(v, q) = d(v_1, q_1) = d(v_1, S'_s)$ for some $s_1 \in S'_s$.
Subcase 1.2.3. Let $q = x$. By a similar way to Subcase 1.2.2, then $d(u_1, S'_s) = d(u_1, p_1) = d(u, p) < d(v, x) = d(v, q) = d(v_1, q_1) \leq d(v_1, t_1) = d(v_1, S'_s)$ for some $t_1 \in S'_s$.

Therefore, if u and v are resolved by R_s and $d(u, y) = d(v, y)$, then u_1 and v_1 are resolved by S'_s.

Case 2. $u_2, v_2 \in V(T_2)\backslash\{y_2\}$.

Subcase 2.1. Assume that u and v in T are resolved by R_b where $b \notin \{l, s\}$. Then, u_2 and v_2 in T_2 are also resolved by S'_b, since $d(u_2, S'_b) = d(u, R_b) \neq d(v, R_b) = d(v, S'_b)$.

Subcase 2.2. Assume that u and v in T are only resolved by R_s. Thus, there exist two vertices $p, q \in R_s$ such that $d(u, R_s) = d(u, p) \neq d(v, q) = d(v, R_s)$. If $d(u, y) \neq d(v, y)$, then $d(u_2, S'_{t+1}) = d(u_2, y_2) = d(u, y) \neq d(v, y) = d(v_2, y_2) = d(v_2, S'_{t+1})$. Otherwise, we distinguish in three subcases.

Subcase 2.2.1. Neither p nor q are equal to x. Then, $d(u_2, S'_s) = d(u_2, p_2) = d(u, p) \neq d(v, q) = d(v_2, q_2) = d(v_2, S'_s)$.
Subcase 2.2.2. Let $p = x$. Then, $d(u_2, S'_s) = d(u_2, s_2) \geq d(u_2, p_2) = d(u, p) = d(u, x) = d(v, x) > d(v, q) = d(v_2, q_2) = d(v_2, S'_s)$ for some $s_2 \in S'_s$.
Subcase 2.2.3. Let $q = x$. Similarly by Subcase 2.2.2, then $d(u_2, S'_s) = d(u_2, p_2) = d(u, p) < d(v, x) = d(v, q) = d(v_2, q_2) \leq d(v_2, t_2) = d(v_2, S'_s)$ for some $t_2 \in S'_s$.

Therefore, if u and v are resolved by R_s and $d(u, y) = d(v, y)$, then u_2 and v_2 are resolved by S'_s.

Subcase 2.3. Assume that u and v are only resolved by R_l. Thus, there exist two vertices $p, q \in R_l$ such that $d(u, R_l) = d(u, p) \neq d(v, q) = d(v, R_l)$. If $d(u, y) \neq d(v, y)$, then $d(u_2, S'_{t+1}) = d(u_2, y_2) = d(u, y) \neq d(v, y) = d(v_2, y_2) = d(v_2, S'_{t+1})$. Otherwise, we distinguish in three subcases.

Subcase 2.3.1. Neither p nor q are equal to y. Then, $d(u_2, S'_l) = d(u_2, p_2) = d(u, p) \neq d(v, q) = d(v_2, q_2) = d(v_2, S'_l)$.
Subcase 2.3.2. Let $p = y$. Then, $d(u_2, S'_l) = d(u_2, s_2) \geq d(u_2, p_2) = d(u, p) = d(u, y) = d(v, y) > d(v, q) = d(v_2, q_2) = d(v_2, S'_l)$ for some $s_2 \in S'_l$.
Subcase 2.3.3. Let $q = y$. By a similar way to Subcase 2.3.2, then $d(u_2, S'_l) = d(u_2, p_2) = d(u, p) < d(v, y) = d(v, q) = d(v_2, q_2) \leq d(v_2, t_2) = d(v_2, S'_l)$ for some $t_2 \in S'_l$.

Therefore, if u and v are resolved by R_l and $d(u, y) = d(v, y)$, then u_2 and v_2 are resolved by S'_l.

Case 3. Let $u_1 \in V(T_1)$ and $v_2 \in V(T_2)$. If $u_1 = x_1$ and $v_2 = y_2$, then $d(u_1, S'_b) = d(x_1, S'_b) = d(y_1, S'_b) + 1 = d(y_2, S'_b) + 1 > d(y_2, S'_b) = d(v_2, S'_b)$ for all $b \notin \{a, l\}$. If $u_1 = y_1$ and $v_2 = x_2$, then $d(u_1, S'_b) = d(y_1, S'_b) = d(y_2, S'_b) < d(y_2, S'_b) + 1 = d(x_2, S'_b) = d(v_2, S'_b)$ for all $b \notin \{a, t + 1\}$. Otherwise, $d(u_1, S'_{t+1}) = d(u_1, x_1) = d(u_1, y_1) + 1 = d(v_2, y_2) + 1 > d(v_2, y_2) = d(v_2, S'_{t+1})$.

This implies that $r(u_i|\Gamma') \neq r(v_j|\Gamma')$ for any two vertices $u_i, v_j \in V(T_1) \cup V(T_2)$. □

By considering the previous results, we have the following conjecture.

Conjecture 3.6. Let G_1 and G_2 be two graphs of order $n \geq 3$ and not isomorphic to complete graphs. If $pd(G_1) = pd(G_2) = k$ and $pdd(G_1 \cup G_2) < \infty$, then $pdd(G_1 \cup G_2) \leq k + 1$.

There are several graphs G with two components where $pdd(G)$ satisfying the upper bound above, such as for two copies of caterpillar $C(3, 3)$ where $pdd(2C(3, 3)) = pd(C(3, 3)) + 1 = 4$, and for $G = K_n - e \cup K_{1,n-1}$ where $pdd(G) = pd(K_n - e) + 1 = n$.

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