Proc. Indian Acad. Sci. (Math. Sci.) Vol. 127, No. 4, September 2017, pp. 737–751.
DOI 10.1007/s12044-017-0356-1

Hamiltonian cycles in polyhedral maps

DIPENDU MAITY* and ASHISH KUMAR UPADHYAY

Department of Mathematics, Indian Institute of Technology Patna, Bihta, Patna 801 103, India
*Corresponding author.
E-mail: dipendumaity@gmail.com; upadhyay@iitp.ac.in

MS received 24 December 2014; revised 14 September 2016; published online 24 August 2017

Abstract. We present a necessary and sufficient condition for existence of a contractible, non-separating and non-contractible separating Hamiltonian cycle in the edge graph of polyhedral maps on surfaces. We also present algorithms to construct such cycles whenever it exists where one of them is linear time and another is exponential time algorithm.

Keywords. Contractible Hamiltonian cycles; non-separating Hamiltonian cycles; non-contractible separating Hamiltonian cycles; proper graphs in polyhedral maps.

Mathematics Subject Classification. 57M20, 57N05, 05C38.

1. Introduction and definitions

By a topological graph we mean a representation of a graph in the plane or surfaces, where the vertices are represented by distinct points and the edges are represented by joining the corresponding pairs of points by arcs. A cycle in a topological graph is said to be a Hamiltonian cycle if it is a spanning cycle. In the year 1931, Whitney [18] studied Hamiltonian cycles in planar graph. He proved that every planar triangulation with no separating triangles has a Hamiltonian cycle. In 1956, Tutte [16] extended Whitney's result to all 4-connected planar graphs. In the year 1970, Grünbaum [9] conjectured that every 4-connected graph which admits an embedding in the torus has a Hamiltonian cycle. We summarize the known partial results related to the solution of Grünbaum's conjecture. Altshuler [1] has shown that every 4 and 6-connected equivelar map (defined later) of types $\{4, 4\}$ and $\{3, 6\}$, respectively, on the torus has Hamiltonian cycles. Brunet and Richter [6] have shown that every 5-connected triangulations on torus is Hamiltonian and then, Thomas and Yu [15] improved this result for any 5-connected graph on the torus. Brunet, Nakamoto and Negami [5] have shown that every 5-connected triangulated Klein bottle is Hamiltonian. In [8] it is shown that a 3-connected bipartite graph embeddable in torus has a Hamiltonian cycle if it is balanced and each vertex of one of its partite sets has degree four. Recently, Kawarabayashi and Ozeki [10] have shown that every 4-connected triangulation of torus is Hamiltonian. We are thus led to think about vertex degree considerations in graphs while looking for such topological cycles. Topologically, cycles in maps may or may not be homotopic to the generators of fundamental group of the surface on which they lie. The cycles which are homotopic to a generator are called *essential or non-separating cycles*. Those which are homotopic to a point are called *inessential or contractible cycles*. Those cycles which are not homotopic to a point and a generator are called *noncontractible separating cycles*. Among these, contractible Hamiltonian cycles have been investigated in [12,17]. We have given a necessary and sufficient condition for existence of contractible Hamiltonian cycles by introducing *proper trees* in the dual map (defined later) of polyhedral maps which is also defined at a later stage in this article. Separating cycles in triangulations of the double torus studied in [11]. Archdeacon [2] has given a survey of such cycles in maps. In this article our focus is on finding such cycles in polyhedral maps. We accomplish this by introducing *proper graph* (see Definition 1). We give two algorithms to search such Hamiltonian cycles in equivelar maps where one algorithm is for finding out Hamiltonian cycles of types contractible and noncontarctible in linear time and another algorithm is for finding out Hamiltonian cycles of all three types in exponential time. In this context, Cabello *et al.* [7] have studied about whether cellular maps contain cycles of types contractible, noncontractible and nonseparating in linear time.

A *p-gonal 2-disc* is a polygon which is bounded by a cycle of length p. A *surface* S is a 2-dimensional manifold which is connected, compact and without boundary. A *map* on a surface S is an embedding of a graph G such that the closure of each component of $S \setminus G$ is a p-gonal 2-disc for $p \geq 3$. The components are called *faces* of the map and the vertices and edges of the embedded graph G are called vertices and edges of the map. The map M is called a *polyhedral map* if intersection of any two faces of M is either empty, a vertex or an edge, see [4]. A polyhedral map M is said to be a *triangulation* of the surface if each face of the map is a 3-gon. We call G the edge graph of M and denote it by $EG(M)$. A polyhedral map is called *equivelar triangulation* if each vertex of M has same degree. A polyhedral map M for which each face is topologically a p-gon and each vertex belongs to exactly q faces is called $\{p, q\}$-*equivelar* or (p^q)-*equivelar map*. Let F_n denote a face with n-gonal boundary and $(a^p, b^q, \ldots, m^r)$ denote a finite cyclically ordered face sequence where $a, b, \ldots, m \geq 3$, $p, q, \ldots, r \geq 1$ and x^m represents m numbers of F_x. A polyhedral map M is said to be a *semi-equivelar map* of type $(a^p, b^q, \ldots, m^r)$ if face sequence of each vertex of M is $(a^p, b^q, \ldots, m^r)$. We will use the terms map and polyhedral map interchangeably to mean a polyhedral map. See [14] for details about graphs on surfaces and [3] for graph theory related terminology.

We begin with some definitions which will be needed in the course of proof of Theorems 1, 2 and 3. For more details on these topics one may also refer to [13].

Let v be a vertex of a map K. The *degree* of a vertex v is denoted by $\deg(v)$ and it is the number of edges incident with v. If we denote the number of vertices, edges and faces of K by $f_0(K)$, $f_1(K)$ and $f_2(K)$ respectively, then the *Euler characteristic of* K is the integer $\chi(K) = f_0(K) - f_1(K) + f_2(K)$. The *dual map* of a map M is by definition a map on same surface S as M which has a vertex corresponding to each face of M, and an edge joining two adjacent faces for each edge in M. Let K be a map on a surface S. Let M denote the dual map of K and $C(u_1, \ldots, u_r)$ be a cycle in K. We consider the edges of M which are dual to edges of C and form a set E by putting these dual edges in E. Let $V = \cup_{e \in E} V(e)$, that is V is union of all vertices which form edges lying in E. The graph $G := (V, E)$ is said to be *dual graph* corresponding to the cycle C in M. In § 2, we give examples of maps on torus. These are well known maps of type $\{3, 6\}$ and $\{6, 3\}$ and are examples of mutually dual maps. Let K' be a subset of set of faces of K and

$D = \cup_{\sigma \in K'} \sigma$. If D is topologically a 2-disc then we will call D a *2-disc* in K. A vertex v of $EG(M)$ is called *cut vertex* if on its removal (deletion) graph becomes disconnected. Similarly, we call a cycle C in M is *cut cycle* if it divides F_M into disjoint set of faces. Similarly, *cut graph* is defined. Consider a map K on a surface S that has n vertices. Let M denote the dual map of K. Let $T := (V, E)$ denote a tree in the edge graph $EG(M)$ of M. We say that T is a *proper tree* (see also [12]) if the following conditions hold: (1) $\sum_{i=1}^{k} \deg(v_i) = n + 2(k-1)$, where $V = \{v_1, v_2, \ldots, v_k\}$ and $\deg(v)$ denotes degree of v in $EG(M)$,

(2) whenever two vertices v_1 and v_2 of T lie on a face F in M, a path $P[v_1, v_2]$ joining v_1 and v_2 in the boundary ∂F of F is a subtree of T and
(3) any path P in T which lies in a face F of M is of length at most $q-2$, where q denotes the length of (∂F).

Let M be a map which is the dual map of a map K on n vertices. We call (V_M, E_M, F_M) the face vector of the map, where V_M, E_M and F_M denote the set of vertices, edges and faces of M, respectively.

DEFINITION 1

Let $G = (V, E)$ denote a subgraph of $EG(M) := (V_M, E_M)$. We say that G is an *admissible graph* if the following conditions hold:

1. $|E| = n$, that is, G consists of n edges.
2. For each $F \in F_M$, the set $F \cap E$ contains exactly two edges of E.
3. There exists a finite sequence of n faces, namely, $F_1, F_2, \ldots, F_n$ such that $F_i \cap F_{i+1} \in E$ for $1 \leq i \leq n-1$ and $F_1 \cap F_n \in E$.

We say that G is a proper graph of type-I if the graph $H = (V_M, E_M \backslash E)$ is connected and call G a proper graph of type-II if H has exactly two components one of which is a proper tree. If the graph $G(V_M, E_M \backslash E)$ has two components where none of them is proper tree then G is called a proper graph of type-III.

The main results of this article are:

Theorem 1. *The edge graph $EG(M)$ of a map M has a non-separating Hamiltonian cycle if and only if the edge graph of the dual map of M has a proper graph of type-I.*

In [12], we have presented a necessary and sufficient condition for existence of a contractible Hamiltonian cycle in a polyhedral map using a tree. Here we present the criterion in terms of a more general graph and show:

Theorem 2. *The edge graph $EG(M)$ of a map M has a contractible Hamiltonian cycle if and only if the edge graph of the dual map of M has a proper graph of type-II.*

Theorem 3. *The edge graph $EG(K)$ of a map K has a non-contractible separating Hamiltonian cycle if and only if the edge graph of corresponding dual map of K has a proper graph of type-III.*

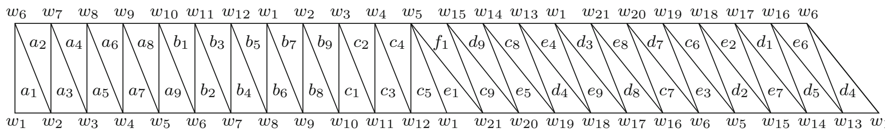

Figure 1. A triangulation of the genus two surface.

In § 2 and § 3, we give examples of proper graphs of types-I, II, III and their corresponding cycles and proper trees and their corresponding cycles respectively. In § 4, we present some properties of proper graphs and proceed to prove the main result of this article in § 5.

2. Example of proper graphs and their corresponding cycles

We consider two well known maps of types $\{3, 6\}$ and $\{6, 3\}$ on the torus. Let $M_1 := \{u_{11}u_{12}u_{14}, u_{11}u_{13}u_{14}, u_{12}u_{13}u_{15}, u_{12}u_{14}u_{15}, u_{13}u_{14}u_{16}, u_{13}u_{15}u_{16}, u_{14}u_{15}u_{17}, u_{14}u_{16}u_{17}, u_{15}u_{16}u_{11}, u_{15}u_{17}u_{11}, u_{16}u_{17}u_{12}, u_{16}u_{11}u_{12}, u_{17}u_{11}u_{13}, u_{17}u_{12}u_{13}\}$ and $K_1 := \{[v_1, v_2, v_3, v_8, v_7, v_6], [v_3, v_4, v_5, v_{10}, v_9, v_8], [v_5, v_6, v_7, v_{12}, v_{11}, v_{10}], [v_7, v_8, v_9, v_{14}, v_{13}, v_{12}], [v_9, v_{10}, v_{11}, v_2, v_1, v_{14}], [v_{11}, v_{12}, v_{13}, v_4, v_3, v_2], [v_{13}, v_{14}, v_1, v_6, v_5, v_4]\}$. The M_1 and K_1 are dual to each other. We consider proper graphs of type-I, II in K_1. Let $G_1 = (V_1, E_1)$ where $V_1 = \{v_i \mid i \in \{1, \ldots, 14\}\}$ and $E_1 = \{v_i v_{5+i} \mid i \in \{1, 3, 5, 7, 9\}\} \cup \{v_i v_{9+i} \mid i \in \{2, 4\}\}$ and $G_2 = (V_2, E_2)$ where $V_2 = \{v_i \mid i \in \{1, 2, 3, 4, 5, 8, 10, 11, 12, 13, 14\}\}$ and $E_2 = \{v_1v_2, v_3v_8, v_4v_5, v_{10}v_{11}, v_{11}v_{12}, v_{12}v_{13}, v_{13}v_{14}\}$. The graph G_1 is of type-I and G_2 is of type-II. Now consider dual of G_1 and G_2 in M_1. We get the cycle $C_1(u_{11}, u_{12}, \ldots, u_{17})$ which is corresponding to G_1 and non-contractible. Also, the cycle $C_2(u_{11}, u_{14}, u_{15}, u_{13}, u_{17}, u_{12}, u_{16})$ which is corresponding to disconnected graph G_2 and contractible.

Let $K_2 := \{w_1w_2w_6, w_2w_6w_7, w_2w_3w_7, w_3w_7w_8, w_3w_4w_8, w_4w_8w_9, w_4w_5w_9, w_5w_9w_{10}, w_5w_6w_{10}, w_6w_{10}w_{11}, w_6w_7w_{11}, w_7w_{11}w_{12}, w_7w_8w_{12}, w_1w_8w_{12}, w_1w_8w_9, w_1w_2w_9, w_2w_9w_{10}, w_2w_3w_{10}, w_3w_{10}w_{11}, w_3w_4w_{11}, w_4w_{11}w_{12}, w_4w_5w_{12}, w_1w_5w_{12}, w_1w_6w_{13}, w_6w_{13}w_{16}, w_{13}w_{14}w_{16}, w_{14}w_{16}w_{17}, w_{14}w_{15}w_{17}, w_{15}w_{17}w_{18}, w_5w_{15}w_{18}, w_5w_{18}w_{19}, w_5w_6w_{19}, w_6w_{19}w_{20}, w_6w_{16}w_{20}, w_{16}w_{20}w_{21}, w_{16}w_{17}w_{21}, w_1w_{17}w_{21}, w_1w_{17}w_{18}, w_1w_{13}w_{18}, w_{13}w_{18}w_{19}, w_{13}w_{14}w_{19}, w_{14}w_{19}w_{20}, w_{14}w_{15}w_{20}, w_{15}w_{20}w_{21}, 5w_{15}w_{21}, w_1w_5w_{21}\}$ be a triangulation of double torus in figure 1. Let $M_2 := \{[a_1, b_7, b_6, b_5, c_5, e_1, d_3, e_9, e_4, d_4], [a_1, a_2, a_3, b_9, b_8, b_7], [a_3, a_4, a_5, c_2, c_1, b_9], [a_5, a_6, a_7, c_4, c_3, c_2], [a_7, a_8, a_9, e_3, e_6, d_2, f_1, e_1, c_5, c_4], [a_9, b_1, b_2, a_2, a_1, d_4, e_6, c_7, d_7, e_3], [b_2, b_3, b_4, a_4, a_3, a_2], [b_4, b_5, b_6, a_6, a_5, a_4], [b_6, b_7, b_8, a_8, a_7, a_6], [b_8, b_9, c_1, b_1, a_9, a_8], [c_1, c_2, c_3, b_3, b_2, b_1], [c_3, c_4, c_5, b_5, b_4, b_3], [d_4, e_6, d_5, c_8, d_4, e_4], [d_5, d_1, e_7, d_9, e_5, c_8], [e_7, e_2, d_2, f_1, c_9, d_9], [c_7, e_8, d_8, d_1, d_5, e_6], [d_8, d_3, e_9, e_2, e_7, d_1], [e_9, e_4, d_4, e_6, d_2, e_2], [d_4, c_8, e_5, d_7, e_3, e_6], [e_5, d_9, c_9, e_8, c_7, d_7], [c_9, f_1, e_1, d_3, d_8, e_8]\}$ in figure 2. The map M_2 is dual of K_2. Let $E_3 = \{a_1d_4, a_9b_1, b_1c_1, b_2b_3, b_3b_4, c_4c_5, f_1e_1, c_9e_8, c_7e_8, e_6d_5, d_4c_8, c_8e_5, e_7d_9, e_2d_2, e_2e_9, d_3e_9\}$ and $V_3 := \{a_1, d_4, a_9, b_1, c_1, b_2, b_3, b_4, c_4, c_5, f_1, e_1, c_9, e_8, c_7, e_6, d_5, d_4, c_8, e_5, e_7, d_9, e_2, d_2, e_2, e_9, d_3\}$. The graph $G_3 = (V_3, E_3)$ satisfies all the properties of proper graph of type-III. So, graph G_3 is of type-III. Consider dual of G_3 in K_2. We get the cycle $C_3(w_1, w_9, w_2, w_6, w_{10}, w_3, w_4, w_{11}, w_7, w_8, w_{12}, w_5, w_{21}, w_{20}, w_{16}, w_{13}, w_{19}, w_{14}, w_{15}, w_{18}, w_{17})$. The cycle C_3 is non-contractible separating Hamiltonian cycle.

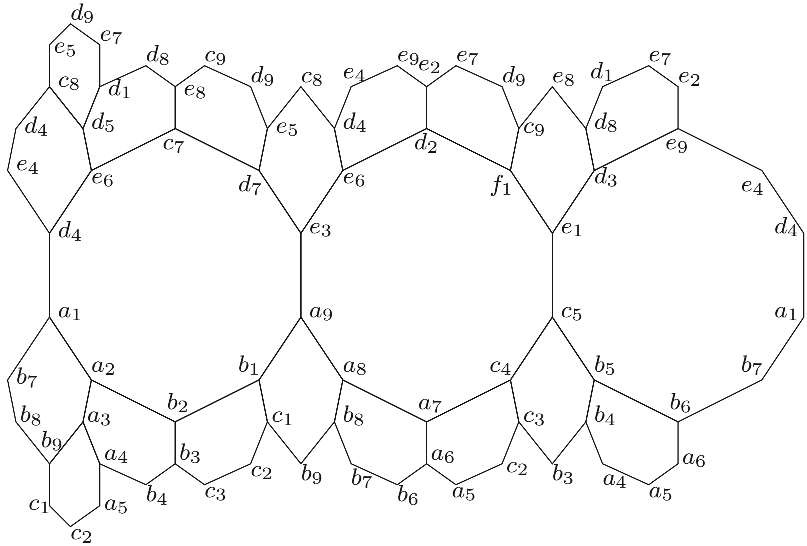

Figure 2. The dual of the map in figure 1.

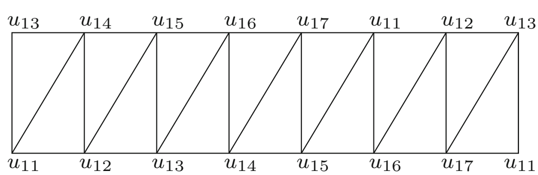

Figure 3. Map of type {3, 6}.

3. Proper trees and corresponding cycles

We give examples of proper trees and their corresponding contractible Hamiltonian cycles in polyhedral maps of types {3, 6}, {6, 3} and {4, 4} on the torus. Figure 3 is an example of {3, 6}-equivelar map on the torus with 7 vertices. Figure 4 is an example of {6, 3}-equivelar map on the torus which is dual of figure 3 with 14 vertices. Let $G_1 = (V_1, E_1)$ be a graph in figure 4 where $V_1 = \{v_i \mid i \in \{2, 3, 4, 11, 13\}\}$ and $E_1 = \{v_2v_{11}, v_2v_3, v_3v_4, v_4v_{13}\}$. The graph G_1 satisfies all the properties of proper tree. So, the graph G_1 is an example of a proper tree. Now, consider dual of G_1 and hence, we get a cycle $C_1(u_{11}, u_{14}, u_{15}, u_{13}, u_{17}, u_{12}, u_{16})$. The cycle C_1 is contractible as it is boundary of 2-disk $|D_1|$ where $D_1 = \{u_{11}u_{12}u_{16}, u_{11}u_{12}u_{14}, u_{12}u_{14}u_{15}, u_{12}u_{13}u_{15}, u_{12}u_{13}u_{17}\}$.

We consider quadrangulated maps of type {4, 4} on torus. In figure 5, there are two examples of {4, 4}-equivelar maps on the torus with 16 vertices which are dual to each other. Let $G_2 = (V_2, E_2)$ be a graph in figure 5 where $V_2 = \{v_i \mid i \in \{1, 2, 3, 5, 7, 9, 11\}\}$ and $E_2 = \{v_1v_2, v_2v_3, v_3v_7, v_7v_{11}, v_1v_5, v_5v_9\}$. The graph G_2 satisfies all the properties of proper tree. So, the graph G_2 is an example of proper tree. Consider dual of G_2. Here, we get the cycle $C_2(u_1, u_2, u_3, u_4, u_8, u_{12}, u_{16}, u_{15}, u_{11}, u_7, u_6, u_{10}, u_{14}, u_{13}, u_9, u_5)$. The cycle C_2 is contractible as it is boundary of 2-disk $|D_2|$ where $D_2 = \{[u_1, u_2, u_6, u_5], [u_2, u_3, u_7, u_6], [u_3, u_4, u_8, u_7], [u_7, u_8, u_{12}, u_{11}], [u_{11}, u_{12}, u_{16}, u_{15}], [u_5, u_6, u_{10}, u_9], [u_9, u_{10}, u_{14}, u_{13}]\}$.

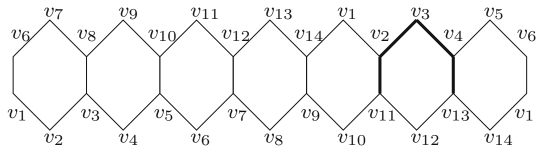

Figure 4. Map of type {6, 3}.

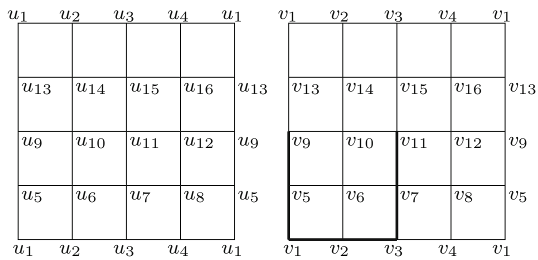

Figure 5. Maps of type {4, 4}.

4. Properties of proper graphs

4.1 *Proper graphs of type-I*

Let K be a polyhedral map on a surface S with n vertices and M denote its dual. Let $C(u_1, \ldots, u_n)$ be a non-separating Hamiltonian cycle in K. Let $G(V, E)$ denote the dual graph corresponding to the cycle C. Then we have:

Lemma 4.1.1. *The graph G contains n edges.*

Proof. The cycle C consists of n edges. By the definition of duality, we get an edge in M for an edge of C. So, we get n distinct edges corresponding to n edges of C. Therefore, G contains n edges. This completes the proof Lemma 4.1.1. □

Lemma 4.1.2. *Let F be a face of M. Then, $F \cap E$ contains exactly two edges of $EG(M)$.*

Proof. Let v be a vertex of K. The link of v is a cycle, say $C_1(v_1, \ldots, v_r)$. Since the cycle C is Hamiltonian cycle in $EG(K)$, so, it contains all the vertices of K and hence, $v \in V(C)$. Also, the cycle C passes through two vertices, say v_i, v_j of C_1 where the edges vv_i and vv_j are in C. Since the degree of each vertex in C is 2 therefore, there is no edge vv_k for which $v_k \in V(C_1)\setminus\{v_i, v_j\}$ in C. Now consider the dual face corresponding to v in M and denote it by F. Also, consider the edges which are dual of vv_i and vv_j. We get that exactly two edges are common in G and F and that there is no other edge of G which also belongs to F. Since, choice of v was arbitrary this is true for all the faces of M. Therefore, any face of M contains exactly two edges of G. This completes the proof of Lemma 4.1.2. □

Lemma 4.1.3. *There exists a face sequence $F_1, F_2, \ldots, F_n$ in M such that $F_i \cap F_{i+1} \in E$ for $1 \leq i \leq n-1$ and $F_1 \cap F_n \in E$.*

Proof. Let F_i denote the faces in M which are the dual faces corresponding to vertices u_i of $C(u_1, \ldots, u_n)$ in K. Then F_i and F_{i+1} have an edge in common for $1 \leq i \leq n-1$ and so does F_1 and F_n. Since u_{i-1} and u_{i+1} lie in the link of u_i, by the argument in proof of previous Lemma 4.1.2 it follows that the common edges lie in E. Hence we get a sequence of faces $F_1, \ldots, F_n$ where $F_1 \cap F_n \in E$ and $F_i \cap F_{i+1} \in E$ for $1 \leq i \leq n-1$. This completes the proof of Lemma 4.1.3. □

Lemma 4.1.4. *The graph $G(V_M, E_M \backslash E)$ is connected.*

Proof. Let u_{t_1} and u_{t_2} be two vertices of $G_1 := G(V_M, E_M \backslash E)$. We show the existence of a path between u_{t_1} and u_{t_2} in G_1. Let $FS(A, B) := \{A = F_1, F_2, \ldots, F_r = B \mid F_i \cap F_{i+1}$ is an edge for $1 \leq i \leq r-1\}$. We call $FS(A, B)$ a face sequence between two faces A and B. Let $E(FS(A, B)) := \{e_i \mid e_i = F_i \cap F_{i+1}$ and $F_i, F_{i+1} \in FS(A, B)\}$ denote the set of common edges between successive faces contained in $FS(A, B)$. Let $F_{u_{t_1}}$ and $F_{u_{t_2}}$ be two faces(dual) in K corresponding to u_{t_1} and u_{t_2} respectively. The map K is a connected polyhedral map. So, we get a face sequence $FS(F_{u_{t_1}}, F_{u_{t_2}})$. We claim that there exists a face sequence $FS(F_{u_{t_1}}, F_{u_{t_2}})'$ such that $E(FS(F_{u_{t_1}}, F_{u_{t_2}})') \cap E(C) = \emptyset$. Suppose, there is no such face sequence in K. That is, for every $FS(F_{u_{t_1}}, F_{u_{t_2}})$, $E(FS(F_{u_{t_1}}, F_{u_{t_2}})) \cap E(C) \neq \emptyset$. This implies that the cycle C is cut cycle. Let C divide $F(K)$, the set of faces of K, into disjoint set of faces, namely, H_1 and H_2. Then, $H_1 \cup H_2 = F(K)$ and $\partial|H_1| \cap \partial|H_2| = C$. Thus, the cycle C is separating. This is a contradiction as C is non-separating in K. Therefore, there exits a face sequence $FS(F_{u_{t_1}}, F_{u_{t_2}})'$ with $E(FS(F_{u_{t_1}}, F_{u_{t_2}})') \cap E = \emptyset$. Let G' denote the dual graph corresponding to the face sequence $FS(F_{u_{t_1}}, F_{u_{t_2}})'$. Then the graph G' is connected because between any two consecutive faces of $FS(F_{u_{t_1}}, F_{u_{t_2}})'$ there is a common edge. So, the graph G' contains a path $(u_{t_1} \to u_{t_2})$ in $G(V_M, E_M \backslash E)$. Since, u_{t_1} and u_{t_2} are arbitrary, therefore, the graph $G(V_M, E_M \backslash E)$ is connected. This completes the proof of Lemma 4.1.4. □

Lemma 4.1.5. *Let G_1 be a proper graph of type-I in M and C_1 be its dual in K. Suppose, $G(V_M, E_M \backslash E(G_1))$ is connected. Then, C_1 is non-separating.*

Proof. Suppose C_1 is separating. We proceed as in previous lemma and use some arguments and definitions of Lemma 4.1.4. The cycle C_1 divides $F(K)$ into two sets, say N_1 and N_2 where $N_1 \cap N_2 = \emptyset$, $N_1 \cup N_2 = F(K)$ and $|N_1| \cap |N_2| = C_1$. Let $F_i \in N_1$ and $F_j \in N_2$ be two faces of K and v_k be dual vertices corresponding to F_k. Consider a face sequence $FS(F_i, F_j)$ between F_i and F_j in K. By assumption, $F_i \in N_1$, $F_j \in N_2$ and $|N_1| \cap |N_2| = C_1$. So, $E(FS(F_i, F_j)) \cap E(C_1) \neq \emptyset$ and this is true for any face sequence $FS(F_i, F_j)'$ between F_i and F_j. Let $E(FS(F_i, F_j)) \cap E(C_1) = \{e_1, e_2, \ldots, e_s\}$. Let H denote the dual of the face sequence $FS(F_i, F_j)$. The graph H is connected. The vertices v_i, v_j corresponding to F_i and F_j lie in H. So, there is a path $P := P(v_i, \ldots, v_j)$ in H. Let ϵ_i be dual edges corresponding to e_i. So, path P contains some ϵ_i. Thus, for arbitrary path P in H, $E(P) \cap E(G_1) \neq \emptyset$. We restrict the path P in $G(V_M, E_M \backslash E(G_1))$, then we get a disconnection between v_i and v_j. So, there is no path between v_i and v_j in $G(V_M, E_M \backslash E(G_1))$. This gives a contradiction as $G(V_M, E_M \backslash E(G_1))$ is connected. Therefore C_1 is non-separating. This completes the proof of Lemma 4.1.5. □

4.2 *Proper graph of type-II*

Let K be a polyhedral map on n vertices and M denote its dual. Let $C(u_1, \ldots, u_n)$ be a contractible Hamiltonian cycle in K and $G := (V, E)$ be its dual graph. Thus, we have

Lemma 4.2.1. *The graph $G_1 := (V_M, E_M \backslash E)$ has two components where one of them is a proper tree.*

Proof. The cycle C is contractible. So, cycle C bounds a 2-disk. Let $D \subset F(K)$ such that $\partial|D| = C$ and $|D|$ is homeomorphic to 2-disk. Let $K_1 = D$ and $K_2 = F(K)\backslash D$. Then, $\partial|K_1| = C$, $\partial|K_2| = C$ and $\partial|K_1| \cap \partial|K_2| = C$. Thus, C is cut cycle. Let M_i be dual of K_i. Then, $EG(M) = EG(M_1) \cup EG(M_2) \cup G$. Thus, the set E is cut set. The set E divides $EG(M)$ into two components $EG(M_1)$ and $EG(M_2)$ and $G_1 = EG(M_1) \cup EG(M_2)$. So, G_1 has two components $EG(M_1)$ and $EG(M_2)$. Again, by assumption, D is 2-disk, $\partial|D| = C$ and M_1 is dual of D. So, by the Lemma 5.2 [12], $EG(M_1)$ is proper tree. Therefore, one of the component of G_1 is a proper tree. This completes the proof of the Lemma 4.2.1. □

Lemma 4.2.2. *Let H be a proper graph of type-II in M and C be its dual in K. If the graph $G := G(V_M, E_M \backslash E(H))$ has two components one of which is a proper tree then C is contractible.*

Proof. Let G consist of components G_1 and G_2. We may assume without loss of generality that the graph G_1 is a proper tree. Let D be the 2-disk which is the dual of G_1 and bounded by a Hamiltonian cycle (see Lemma 5.2 in [12]). Let $\partial|D| = C_2$. By remark 1, G_1 defines H and H is proper graph of type-II. By similar arguments as in Lemma 4.2.3, H and G_1 define same graphs. Thus, $\partial|D| = C_1 = C_2$. Therefore, the cycle C_1 is contractible Hamiltonian cycle. This completes the proof of Lemma 4.2.2. □

Lemma 4.2.3. *Let $EG(K)$ denote the edge graph of K. $EG(K)$ contains a proper tree if and only if it contains a proper graph of type-II.*

Proof. Let T be a proper tree in $EG(K)$ and $V(T)$, $V(e)$ denote the set of vertices of T and e respectively. Let $G = (V, E)$ denote a graph where $E = \{e \in EG(K) \mid V(e) \cap V(T) \neq \emptyset$ and $e \notin E(T)\}$ and $V = \cup_{e \in E} V(e)$. We show that G is a proper graph of type-II. Let D denote the dual of T which is a 2-disk. The cycle ∂D is a Hamiltonian cycle. Let $C := \partial D$. Let $E(C)$ denote the set of edges of C. Let $e \in E(C)$ and $\tilde{e}$ denote the dual edge of e. Let $\tilde{e} = uv$. We show that the edge $\tilde{e}$ belongs to E. Let $F(D)$ denote the set of faces of D. Let $F \in F(D)$ where F contains the edge e. Let v_F denote the dual vertex corresponding to face F. Then, $v_F \in V(\tilde{e})$ and $v_F \in V(T)$. Hence, $V(\tilde{e}) \cap V(T) \neq \emptyset$. Let F_u and F_v denote two dual faces corresponding to the vertices u and v respectively. Let $E(F)$ denote the set edges of the face F. The edge $e \in E(F)$ and by assumption $e = F_u \cap F_v$. So, F is either F_u or F_v. Let $F_u = F$. If $F_v \in F(D)$. Then, $e \notin E(C)$. But, by assumption $e \in E(C)$. So, $F_v \notin F(D)$. This implies, $\tilde{e} \notin E(T)$. Since $V(\tilde{e}) \cap V(T) \neq \emptyset$ and $\tilde{e} \notin E(T)$, so, $\tilde{e} \in E$. Let $\tilde{E}$ denote a set of dual edges corresponding to edges of C. This implies, $\tilde{E} \subset E$. We show that $E\backslash\tilde{E}$ is empty set. Let $E\backslash\tilde{E} \neq \emptyset$ and $e1 \in E\backslash\tilde{E}$. The edge $e1 \notin \tilde{E}$. Let $\tilde{e1}$ be dual edge corresponding to the edge $e1$. By assumption, $e1 \in E$, $e1 \notin \tilde{E}$ and $V(e1) \cap V(T) \neq \emptyset$. Hence, $\tilde{e1}$ belongs to interior of D. This implies that there are two faces in D whose common edge is $e1$. This implies $e1 \in E(T)$. This is a contradiction as $e1 \in E$ and $e1 \notin E(T)$. Thus, $E = \tilde{E}$. Therefore, G is dual graph corresponding to C. Again, $C = \partial D$. So, G and T define the same cycle C. The graph G is dual of contractible Hamiltonian cycle C. By Lemmas 4.1.1, 4.1.2 and 4.1.3, G is an admissible graph. By Lemma 4.2.1, $G(V_M, E_M \backslash E)$ consists of two components where one of them is proper tree. Thus, G is a proper graph of type-II.

Let $G_1 := G(V_1, E_1)$ be a proper graph of type-II. By the third property of G_1, let $F_1, F_2, \ldots, F_n$ be a face sequence in M where $F_1 \cap F_n \in E$ and $F_i \cap F_{i+1} \in E$ for $1 \leq i \leq n-1$. Let u_i denote dual vertex corresponding to face F_i. This gives an edge set $\{u_1u_2, u_2u_3, \ldots, u_iu_{i+1}, \ldots, u_{n-1}u_n, u_nu_1\}$ in K. Define $C_1 := C(u_1, u_2, \ldots, u_i, u_{i+1}, \ldots, u_n)$. The cycle C_1 contains all the vertices of K. So, C_1 is Hamiltonian cycle. Since the graph $G(V_M, E_M \backslash E)$ is of type-II, so, $G(V_M, E_M \backslash E)$ consists of two components where one of them is proper tree. Thus, the cycle C_1 is contractible Hamiltonian cycle. Let D_1 be 2-disk which is bounded by C_1. Let T_1 denote the dual of D_1. We show that T_1 is a proper tree. We know that every contractible Hamiltonian cycle gives a proper tree in K (see [12], Theorem 2). Thus T_1 is a proper tree. This completes the proof of the Lemma 4.2.3. □

Remark 1. Let T be a proper tree in K. Let $G := \{e \in EG(K) \mid V(e) \cap V(T) \neq \emptyset\}$ be a graph. Let $G_1 := G \backslash T$. By the argument of Lemma 4.2.3, G_1 is a proper graph of type-II. Therefore, the graph G can be decompose into proper tree and proper graph of type-II. Thus, $G = G_1 \cup T$.

4.3 *Proper graph of type-III*

Let K be a polyhedral map on n vertices and M denote its dual map. Let $C(u_1, \ldots, u_n)$ be a non-contractible separating Hamiltonian cycle in K and $G := (V, E)$ be its corresponding dual graph in M. Then, we have:

Lemma 4.3.1. *The graph $G_1 := G(V_M, E_M \backslash E)$ has two components none of which is a proper tree.*

Proof. Since C is a separating cycle so it is a cut cycle. Let S_1 and S_2 be two disjoint subsets of the set of faces $F(K)$ of K, such that $S_1 \cup S_2 = F(K)$ and $\partial|S_1| \cap \partial|S_2| = C$. Let X_i denote the dual graph corresponding to S_i for $i = 1, 2$. Thus, $EG(M) = E(X_1) \cup E(X_2) \cup G$ where G is a cut graph. Therefore, $G_1 = E(X_1) \cup E(X_2)$ has two components $EG(X_1)$ and $EG(X_2)$. Now, we show that none of $EG(M_1)$ and $EG(M_2)$ is proper tree. Assume without loss of generality that $E(M_1)$ is proper tree. Then, by Remark 1, the graph G is proper graph of type-II. This gives that the cycle C is contractible. A contradiction. Therefore, the graph G_1 has two components where none of them is proper tree. This completes the proof of the Lemma 4.3.1. □

Lemma 4.3.2. *Let H_1 be a proper graph of type-III in M and C_1 be its dual in K. Let $G := G(V_M, E_M \backslash E(H_1))$ consists of two components G_1 and G_2 none of which is a proper tree. Then C_1 is a non-contractible separating cycle in K.*

Proof. Since G_1 and G_2 are two components of G, so $G = G_1 \cup G_2$ and $G_1 \cap G_2 = \emptyset$. We show that C_1 is non-contractible. Suppose C_1 is contractible. Then C_1 bounds a 2-disk

D. Let $\partial|D| = C_1$. If G_1 is the dual of D (this is possible as G_1 is one component of G). By Remark 1, the graph G_1 is a proper tree. This is a contradiction as none of G_1 and G_2 is a proper tree. So, the cycle C_1 is non-contractible. The graph G is disconnected. So, by Lemma 4.1.4, the cycle C_1 separating. Therefore, the cycle C_1 is non-contractible separating cycle. This completes the proof of the Lemma 4.3.2. □

5. Proofs of Theorems 1, 2 and 3

Proof of Theorem 1. As before, let K be a polyhedral map on n vertices and M denote its dual map. Let $C(u_1, \ldots, u_n)$ be a non-separating Hamiltonian cycle in K. Let $G := G(V, E)$ denote the dual graph corresponding to C in M. By Lemma 4.1.1, the graph G consists of n edges, *i.e.*, $|E| = n$. By Lemma 4.1.2, the number of elements in $E(G) \cap F = 2$ for all faces F of M. By Lemma 4.1.3, there exists a face sequence, say, $F_1, F_2, \ldots, F_n$ of faces of M such that $F_1 \cap F_n \in E$ and $F_i \cap F_{i+1} \in E$ for $1 \le i \le n-1$. By Lemma 4.1.4, $G(V_M, E_M \backslash E)$ is connected. Therefore, the graph G is a proper graph of type-I.

Conversely, let $G := G(V, E)$ be a proper graph of type-I. By the third property of G, let $F_1, F_2, \ldots, F_n$ be a face sequence in M where $F_1 \cap F_n \in E$ and $F_i \cap F_{i+1} \in E$ for $1 \le i \le n - 1$. Let $u_i \in V(K)$ denote the dual vertex corresponding to face F_i. By definition, $F_i \cap F_{i+1}$ is an edge in G which is a common edge between the faces F_i and F_{i+1}. So, the edge $F_i \cap F_{i+1}$ is dual edge corresponding to $u_i u_{i+1}$. Let a set E_1 consist of edges $u_i u_{i+1}$. Since $F_i \cap F_{i+1}$ consists of exactly one edge for $i \in \{1, 2, \ldots, n-1\}$. So, the set E_1 contains exactly n elements. Thus, $E_1 = \{u_1u_2, u_2u_3, \ldots, u_iu_{i+1}, \ldots, u_{n-1}u_n, u_nu_1\}$. Define a cycle of length n by $C := (u_1, u_2, \ldots, u_i, u_{i+1}, \ldots, u_n)$. The cycle C contains all the vertices of K. So, C is Hamiltonian cycle in K. Further, since $G(V_M, E_M \backslash E)$ is connected, so by Lemma 4.1.5, C is non-separating. Thus we conclude that the edge graph $EG(K)$ contains a non-separating Hamiltonian cycle. This completes the proof. □

Proof of Theorem 2. Let K be a polyhedral map on n vertices and M denote its dual. Let $C := (u_1, \ldots, u_n)$ be a contractible Hamiltonian cycle. Let $G := G(V, E)$ denote the dual graph corresponding to C in M. By Lemmas 4.1.1, 4.1.2 and 4.1.3, G is an admissible graph. By Lemma 4.2.1, the graph $G(V_M, E_M \backslash E)$ has two components and one of them is proper tree. Therefore, the graph G is a proper graph of type-II.

Conversely, let $G := G(V, E)$ be a proper graph of type-II. By the third property of G, let $F_1, F_2, \ldots, F_n$ be a face sequence in M where $F_1 \cap F_n \in E$ and $F_i \cap F_{i+1} \in E$ for $1 \le i \le n - 1$. Let u_i denote the dual vertex corresponding to the face F_i. Then we get a set of edges $E_1 := \{u_1u_2, u_2u_3, \ldots, u_iu_{i+1}, \ldots, u_{n-1}u_n, u_nu_1\}$ in K. Define a cycle of length n as $C := (u_1, u_2, \ldots, u_i, u_{i+1}, \ldots, u_n)$. The cycle C contains all the vertices of K. So, C is Hamiltonian. The graph $G(V_M, E_M \backslash E)$ is of type-II. So, it has two components and one of them is proper tree. By Lemma 4.2.2, C is contractible. Therefore, $EG(K)$ contains contractible Hamiltonian cycles. This completes the proof. □

Proof of Theorem 3. Let K be a polyhedral map on n vertices and M denote its dual. Let $C := (u_1, \ldots, u_n)$ be a non-contractible separating Hamiltonian cycle. Let $G := G(V, E)$ denote the dual graph corresponding to C in M. By Lemmas 4.1.1, 4.1.2 and 4.1.3, graph G is admissible. By Lemma 4.3.1, the graph $G(V_M, E_M \backslash E)$ has two components and none of them is proper tree. Therefore, the graph G is a proper graph of type-III.

Conversely, let $G := G(V, E)$ be a proper graph of type-III. By the third property of G, let $F_1, F_2, \ldots, F_n$ be a face sequence in M where $F_1 \cap F_n \in E$ and $F_i \cap F_{i+1} \in E$ for $1 \leq i \leq n-1$. Let u_i be dual vertex corresponding to face F_i. So as before we get a set of edges $\{u_1u_2, u_2u_3, \ldots, u_iu_{i+1}, \ldots, u_{n-1}u_n, u_nu_1\}$ in K. Define a cycle of length n by $C := (u_1, u_2, \ldots, u_i, u_{i+1}, \ldots, u_n)$. The cycle C contains all the vertices of K. So, C is Hamiltonian. The graph $G(V_M, E_M \backslash E)$ has two components and none of them is proper tree. So by Lemma 4.3.2, C is non-contractible and separating. Therefore, the edge graph $EG(K)$ contains non-contractible separating Hamiltonian cycles. This completes the proof. □

6. Algorithm

Algorithm 6.1. *The steps for searching Hamiltonian cycle in equivelar maps*

The following steps may be implemented as a computer program to search contractible, non-separating and non-contractible separating Hamiltonian cycles. The terms and notations are same as those used in Lemma 4.1.4. Let $V(K)$, $E(K)$ and $F(K)$ denote set of vertices, edges and faces respectively and $d(K)$ denote dual of K. $FS(F_1, F_{k_1})$ denote a face sequence between F_1, F_{k_1} and $CEFS(F_1, F_{k_1})$ denote a set of common edges between consecutive faces in the sequence $FS(F_1, F_{k_1})$. We denote C_{NS}, C_C and C_{NCS} to be a non separating, contractible and non-contractible separating Hamiltonian cycle respectively. We also denote number of connected components in a graph G by $C(G)$ and the cardinality of a set S by $|S|$. The $\mathbb{N}$ denote the set of natural numbers. Let K_1 be a map of type $\{p, q\}$ where $|V(K_1)| = n$, $|E(K_1)| = m$, $|F(K_1)| = k$. Denote $V(K_1) = \{v_1, v_2, \ldots, v_n\}$ and $F(K_1) = \{F_1, F_2, \ldots, F_k\}$. Construct a face adjacent matrix $(F(i, j))$ which is a $k \times k$ matrix. The entry $F(i, j) = 1$ if $F_i \cap F_j$ is an edge, otherwise, $F(i, j) = 0$. The face set of K_1 is vertex set of its dual. We also construct a vertex-face incident matrix $(V(i, j))$. The entry $V(i, j) = 1$ if $v_i \in V(F_j)$, otherwise, $V(i, j) = 0$. We consider non zero entries of a row in $(V(i, j))$ which gives boundary edges of a dual face in the dual and use matrix $(F(i, j))$ to define face which is bounded by these edges. Thus, we get face, edge and vertex set of the dual map. It takes $O(k^2) + O(nk) + O(nq^2)$ times. We choose n edges from E_M and put in a set S_1. The possible number of subsets of E_M of order n is $\binom{m}{n}$. Consider a set of n edges. Now, each edge belongs to exactly two faces. We consider pair of faces for each edge. If every face belongs to exactly two pairs then we consider the set S_1, otherwise, we repeat this process. In this case, we get a face sequence $FS(F_{t_1}, F_{t_n})$ in $F(M)$ such that $CEFS(F_{t_1}, F_{t_n}) = S_1$. Thus, consider dual edges of S_1 in S_2. The set S_2 is a Hamiltonian cycle. We classify the cycle S_2. By [12], if $\frac{n-2}{p-2}$ is not an integer then S_2 is non-contractible Hamiltonian cycle. Consider the graph $G' = G(V_M, E_M \backslash S_1)$. Search spanning tree in G'. If it does not contain spanning tree then search its forest. By Kruskal's algorithm, it takes $O((m-n)\log(m-n))$ times. If G' is connected i.e. contains spanning tree then the cycle S_2 is separating Hamiltonian cycle. If forrest contains two trees and none of them is proper tree, then, the cycle S_2 is non-contractible separating Hamiltonian cycle. Again, if $\frac{n-2}{p-2}$ is an integer then similarly, consider the graph $G'' = G(V_M, E_M \backslash S_1)$. If G'' is connected, then, by Theorem 1, the cycle S_2 is separating Hamiltonian cycle. If G'' has two components and one of them is proper tree, then, by Theorem 2, the cycle S_2 is contractible Hamiltonian cycle. If G'' has two components and none of them is proper tree, then, by Theorem 3, the cycle S_2 is non-contractible separating Hamiltonian cycle. In this process, consider components of G''. Now, proper tree is a tree that touches all faces of the map. So, we consider a component which is a tree and touches all the faces (if exists). Now, for a face say F, if $S_1 \bigcap \partial F$ either does not form a path or has number of edges $>$ length$(\partial F) - 2$, then, the tree is not a proper tree (by the definition of proper tree). Otherwise, the tree is a proper tree. It takes $O(n)$ times as we check each face exactly once. We stop unless we get contractible, non-separating and non-contractible separating Hamiltonian cycle (if they exists) after finitely many steps.

Algorithm 6.1. Searching Hamiltonian cycle in equivelar maps.

1: Let K denote an equivelar map of type $\{p, q\}$ on n.
2: SET $M' = d(K)$, $M = F(M') = \{F_1, \dots, F_n\}$, $E_M = E(M)$ & $S_1 = \emptyset$
3: SET $S_1 \subset E_M$ & $|S_1| = n$
4: FOR $F \in M$ DO
5:     IF $|E(F) \bigcap S_1| \neq 2$ THEN
6:         GOTO STEP 3
7:     END IF
8: END FOR
9: SET $S_2 = d(S_1)$
10: FOR $FS(F_{t_1}, F_{t_n})$ in M DO
11:     IF $CEFS(F_{t_1}, F_{t_n}) = S_1$ THEN
12:         GOTO STEP 17
13:     END IF
14: END FOR
15: GOTO STEP 3
16: Note: Set S_2 is a Hamiltonian cycle and we classify S_2 in STEP 4.
17: IF $\frac{n-2}{p-2} \notin \mathbb{N}$ THEN
18:     IF $C(G(V_M, E_M \backslash S_1)) = 1$ THEN
19:         SET $C_{NS} = S_2$
20:     ELSE IF $C(G(V_M, E_M \backslash S_1)) = 2$ THEN
21:         SET $C_{NCS} = S_2$
22:     END IF
23: END IF
24: IF $\frac{n-2}{p-2} \in \mathbb{N}$ THEN
25:     IF $C(G(V_M, E_M \backslash S_1)) = 1$ THEN
26:         SET $C_{NS} = S_2$
27:     ELSE IF $C(G(V_M, E_M \backslash S_1)) = 2$ THEN
28:             DENOTE $G(V_M, E_M \backslash S_1) = H_1 \bigcup H_2$
29:             IF $H_1 =$ proper tree or $H_1 =$ proper tree THEN
30:                 SET $C_C = S_2$
31:             ELSE
32:                 SET $C_{NCS} = S_2$
33:             END IF
34:     END IF
35: END IF

So, the runtime of the algorithm is $O(k^2) + O(nk) + O(nq^2) + O(\binom{m}{n}(m-n)\log(m-n)) + O(n)$, i.e., $O(k^2) + O(nk) + O(nq^2) + O(\binom{m}{n}(m-n)\log(m-n))$. We have a relation $m \leq 3n - 3\chi$ which is among number of vertices (denoted by n), number of edges (denoted by m) and Euler characteristic value (denoted by χ). So, $m \leq 3n - 3\chi \implies m \leq 3n - 3(n - m + k) \implies 3k \leq 2m$. Now, $m = \frac{nq}{2}$ and $q \leq n - 1$. So, $k \leq \frac{1}{3}(n^2 - n) \implies nk \leq \frac{1}{3}(n^3 - n^2)$, $nq^2 \leq (n^3 - 2n^2 + n)$, $k^2 \leq \frac{1}{9}(n^4 - 2n^3 + n^2)$, $(m - n) = \frac{nq}{2} - n \leq \frac{1}{2}(n^2 - 3n)$ and $\log(m - n) \leq \log(n^2 - 3n) - \log(2)$. Hence, $O(k^2) + O(nk) + O(nq) + O(\binom{m}{n}(m - n)\log(m - n)) \approx O(n^4) + O(n^3) + O(n^3) + O(\binom{\frac{n^2}{2}}{n} n^2 \log(n^2 - 3n)) \approx O(n^4) + O(\binom{\frac{n^2}{2}}{n} n^2 \log(n^2 - 3n))$. Now, $\binom{\frac{n^2}{2}}{n} = \frac{(\frac{n^2}{2})!}{n!(\frac{n^2}{2} - n)!} \approx \frac{\sqrt{2\pi \frac{n^2}{2}}(\frac{n^2}{2e})^{\frac{n^2}{2}}}{\sqrt{2\pi n}(\frac{n}{e})^n \sqrt{2\pi(\frac{n^2}{2} - n)}[\frac{\frac{n^2}{2} - n}{e}]^{\frac{n^2}{2} - n}}$ (by Stirling's approximation) $= \frac{1}{2^n\sqrt{2\pi}}(1 + \frac{2}{n-2})^{\frac{n^2}{2}}(n - 2)^{n - \frac{1}{2}}$. So, $O(n^4) + O(\binom{\frac{n^2}{2}}{n} n^2 \log(n^2 - 3n)) \approx O(\frac{n^2}{2^n}(1 + \frac{2}{n-2})^{\frac{n^2}{2}}(n - 2)^{n - \frac{1}{2}}\log(n^2 - 3n))$. Therefore, the Algorithm 6.1 is an exponential time algorithm.

Algorithm 6.2. *The steps for searching a non-contractible Hamiltonian cycle in equivelar maps*

We use terms and notations are same as those used in Algorithm 6.1. Let $EG(K)$ be the edge graph of a $\{p, q\}$ equivelar map K and $|V(K)| = n$. Let C_{NC} denote non-contractible Hamiltonian cycle. Let $EG(K)$ be the edge graph of a $\{p, q\}$ equivelar map K and $\#V(EG(K)) = n$. Let M be the set of all p-gonal faces and i denote the number of steps. We construct two sets D and V as follows. Choose an element $P_0 \in M$. Define $D := \{P_0\}$, $V := \{V(P_0)\}$ and $i = 1$. We have either $\#V = \#V(EG(K))$ or $\#V < \#V(EG(K))$. We go to the following steps. If $\#V < \#V(EG(K))$, then, we observe the set D at i-th and $(i + 1)$-th steps. Let $v \in V(K)\setminus V$. There exists a face P (see Algorithm 1 of [12]) such that $v \in V(P)$, $V(P) \cap V = \{v_1, v_2\}$ and $E(P) \cap E(P_1) = \{v_1v_2\}$ for some $P_1 \in D$. Put $D = D \cup \{P\}$, $V = V \cup V(P)$ and $i = i + 1$. If $D \cup \{P\}$ is a 2-disk then, we put $D = D \cup \{P\}$ and $V = V \cup V(P)$ and $i = i + 1$. Go to the next step and continue. Do this until we get either $V = V(EG(K))$ or D is a 2-disk and $D \cup \{P\}$ is not a 2-disk. In this process, we consider at most n number of faces and thus, it takes $O(n)$ times. Then, we go to next step. If $V = V(EG(K))$, then, we add a face P_2 of $M\setminus D$ in D. The geometric carrier $\partial|D \cup P_2|$ contains more than one cycle as $V(P_2) \subset V$. Denote it by C_1. Otherwise, at the end of processes in step (23), we get a face F which has at least one vertex which is already in V. In this case, we consider the geometric carrier $\partial|D \cup F|$ which contains at least one cycle and denote it by C_1. Here, the cycle C_1 does not bound any 2-disk in both the cases. So, the cycle is non-contractible. Define $V_1 := V(C_1)$. We go to next step. If $\#V_1 = \#V(EG(K))$. We stop here. We get a non-contractible Hamiltonian cycle C_1. If $\#V_1 < \#V(EG(K))$, then, there is a vertex in $V(K)\setminus V_1$. We get a new face P_3 at j-th step such that cycle C_1 and ∂P_3 have common path (in step (30)). We concatenate C_1 and ∂P_3 if the concatenated cycle has length bigger than the length of C_1. Hence, we get new cycle and denote it by C_1. Put, $V = V(C_1)$ and $j = j + 1$. Go to next step and continue. Do this until we get $V_1 = V(EG(K))$. This gives a non-contractible Hamiltonian cycle C_1. It takes $O(n)$ times to extend the cycle C_1 to a Hamiltonian cycle. Thus, it is an linear (in vertex) time algorithm. We stop unless we get a non-contractible Hamiltonian cycle (if it exists) after finite steps.

Algorithm 6.2. Searching non-contractible Hamiltonian cycle in equivelar maps.

SET $M = F(K)$ & $i = 1$
$D(i)$ = face set at i^{th} step & $V(i)$ = vertex set at i^{th} step
PICK $P_0 \in M$
SET $i = 1$, $D(1) = \{P_0\}$ & $V(1) = \{V(P_0)\}$
IF $|V(i)| < n$ THEN
    PICK $v \in V(K) \backslash V(i)$
    FOR $P \in M$ DO
        IF $v \in V(P)$ & $V(P) \bigcap V(i) = \{v_1, v_2\}$ THEN
            FOR $P_1 \in D$ DO
                IF $E(P) \bigcap E(P_1) = \{v_1 v_2\}$ THEN
                    SET $i = i + 1$,
                    $D(i + 1) = D(i) \cup \{P\}$ &
                    $V(i + 1) = V(i) \cup V(P)$
                END IF
            END FOR
        END IF
    END FOR
    IF $D(i + 1)$ is a 2-disk THEN
        GOTO step 5
    ELSE
        GOTO step 24
    END IF
END IF
SET $D = D(i + 1) \bigcup \{P_2\}$
SET C_1 = cycle in $\partial |D \cup P_2|$
SET $V_1 := V(C_1)$
IF $|V_1| = n$ THEN
    SET $C_{NC} = C_1$
ELSE
    FOR P_3 in M DO
        IF $C_1 \bigcap \partial P_3$ = a path THEN
            C_2 = Concatenate$(C_1, \partial P_3)$
            IF length$(C_2) >$ length(C_1) THEN
                SET $C_1 = C_2$
            END IF
        END IF
    END FOR
END IF

Remark 2. We have given an algorithm in [12] to construct contractible Hamiltonian cycle in equivelar maps. It is also is a linear (in vertex) time algorithm by the similar argument of Algorithm 6.2.

Acknowledgements

The authors are grateful to the anonymous referee whose comments led to a substantial improvement in the paper. The work of the second author is partially supported by SERB, DST Grant No. SR/S4/MS:717/10.

References

[1] Altshuler A, Construction and enumeration of regular maps on the torus, *Discrete Math.* **4** (1973) 201–217

[2] Archdeacon D, Topological graph theory: a survey (surveys in graph theory) (1995) (San Francisco, CA)

[3] Bondy J A and Murthy U S R, Graph theory with applications (1982) (North Holland, Amsterdam)

[4] Brehm U and Schulte E, Polyhedral maps, Handbook of Discrete and Computational Geometry, edited by J E Goodman and J O Rourke (1997) (CRC Press) pp. 345–358

[5] Brunet R, Nakamoto A and Negami S, Every 5-connected triangulations of the Klein bottle is Hamiltonian, *Yokohama Math. J.* **47** (1999) 239–244

[6] Brunet R and Richter R B, Hamiltonicity of 5-connected toroidal triangulations. *J. Graph Theory* **20** (1995) 267–286

[7] Cabello S, Verdière E C D, Lazarus F, Finding cycles with topological properties in embedded graphs. *SIAM J. Discrete Math.* **25** (2011) 1600–1614

[8] Fujisawa J, Nakamoto A and Ozeki K, Hamilton cycles in bipartite toroidal graphs with a partite set of degree four vertices (preprint) http://www.fbc.keio.ac.jp/~fujisawa/papers.html

[9] Grünbaum B, Polytopes, graphs and complexes, *Bull. Amer. Math. Soc.* **76** (1970) 1131–1201

[10] Kawarabayashi Ken-ichi and Ozeki K, Hamilton cycles in 4-connected troidal triangulations. *Electronic Notes in Discrete Math.* **38** (2011) 493–498

[11] Jennings D L G, Separating cycles in triangulations of the double torus, Ph.D. thesis (2003) (Vanderbilt University)

[12] Maity D and Upadhyay A K, Contractible Hamiltonian Cycles in Polyhedral Maps, *Discrete Math. Algorithms Appl.* **4** (2012)

[13] McMullen P and Schulte E, Abstract Regular Polytopes (2002) (CUP)

[14] Mohar B and Thomassen C, Graphs on Surfaces (2001) (The John Hopkins Univ. Press)

[15] Thomas R, and Yu X, Five-connected toroidal graphs are Hamiltonian. *J. Combin. Theory Ser. B* **69** (1997) 79–96

[16] Tutte W T, A theorem on planar graphs, *Trans. Amer. Math. Soc.* **82** (1956) 99–116

[17] Upadhyay A K, Contractible Hamiltonian Cycles in Triangulated Surfaces, *Elemente der Mathematik* **69** (1) 23–29

[18] Whitney H, A theorem on graphs, *Ann. Math.* **32** (1931) 378–390

COMMUNICATING EDITOR: Sharad S Sane