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Minimal surfaces in symmetric spaces with parallel second fundamental form

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Abstract. In this paper, we study geometry of isometric minimal immersions of Riemannian surfaces in a symmetric space by moving frames and prove that the Gaussian curvature must be constant if the immersion is of parallel second fundamental form. In particular, when the surface is S^2, we discuss the special case and obtain a necessary and sufficient condition such that its second fundamental form is parallel. We also consider isometric minimal two-spheres immersed in complex two-dimensional Kähler symmetric spaces with parallel second fundamental form, and prove that the immersion is totally geodesic with constant Kähler angle if it is neither holomorphic nor anti-holomorphic with Kähler angle $\alpha \neq 0$ (resp. $\alpha \neq \pi$) everywhere on S^2.

Keywords. Isometric minimal immersion; Gaussian curvature; Kähler angle; second fundamental form; symmetric space.

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1. Introduction

Let M and N be a Riemannian surface and an n-dimensional Riemannian manifold respectively, and $f : M \to N$ is an isometric minimal immersion with parallel second fundamental form. In this paper we will develop, by moving frames, the basic formulae of minimal surfaces, both in a symmetric space and a complex two-dimensional Kähler symmetric space. Here we use a general moving frame, not a Darboux frame.

It is an interesting problem to study parallel submanifolds in a special Riemannian symmetric space. In fact, these submanifolds have been discussed by several authors when ambient spaces are real space forms and complex space forms (see [7,10]). In 1976, Nakagawa and Takagi [8] studied some properties about Kähler imbeddings of compact Hermitian symmetric spaces in $\mathbb{C}P^n$ and gave a classification of Kähler submanifolds in the complex projective space $\mathbb{C}P^n$ with parallel second fundamental form. Minimal surfaces immersed in various Riemannian symmetric spaces with parallel second fundamental form have also been a well studied subject. Recently, a number of authors [3–6] studied conformal minimal immersions of 2-spheres in $G(k, n; \mathbb{C})$, Q_n and HP^n with parallel second fundamental form respectively, and gave some geometric properties of them; Tsukada [9] gave a classification of parallel submanifolds in a quaternion projective space and its non-compact dual by totally real, totally complex, and invariant immersions. In this paper, we discuss some properties of isometric minimal immersions of Riemannian surfaces in symmetric spaces with second fundamental form σ.

This paper is organized as follows. In § 2, let f be an isometric minimal immersion of a Riemannian surface M in an n-dimensional Riemannian symmetric space N with second fundamental form σ, we give some basic formulae of this immersion and compute the Laplacian of $\|\sigma\|^2$. In § 3, by using these formulae, we prove that the Gaussian curvature K must be constant if σ is parallel (here the condition of minimal is not necessary) (Theorem 3.1), and obtain a necessary and sufficient condition such that σ is parallel in the special case $M = S^2$ (Proposition 3.3). In § 4, under the assumption that σ is parallel, we consider isometric minimal 2-spheres immersed in complex two-dimensional Kähler symmetric spaces, and prove that the immersion is totally geodesic with constant Kähler angle α if it is neither holomorphic nor anti-holomorphic with $\alpha \neq 0$ (resp. $\alpha \neq \pi$) everywhere on S^2 (Theorem 4.1 and Corollary 4.3).

2. Some basic formulae on minimal surfaces immersed in symmetric spaces

In this section, for a minimal surface immersed in a symmetric space, we will develop some basic formulae by moving frames. Here we use general moving frame, not the Darboux frame. Using these formulae, the Laplacian of the square length of its second fundamental form can be computed.

Let M be a Riemannian surface and N be an n-dimensional Riemannian manifold, $T(M)$ and $T(N)$ be their tangent bundles respectively. Choose $e_1,\ e_2$ and $\varepsilon_1, \dots, \varepsilon_n$ be the unit orthogonal tangent frames of M and N respectively. With respect to these two frame fields, let θ^1, θ^2 and $\omega^1, \dots, \omega^n$ be their dual frames.

Throughout this paper, we will agree on the following ranges of indices:

$$1 \le i, j, k, \dots \le 2,\ 1 \le A, B, C, \dots \le n,\ 1 \le a, b, c, \dots \le 2.$$

Here the summation convention is used.

Let $f : M \to N$ be an isometric immersion with the second fundamental form σ, $\bar{\nabla}$ be the Riemannian connection of N, and ∇ be the induced Riemannian connection of M by f. Then $\forall X, Y \in T(M)$ we have

$$\sigma(X, Y) = \nabla df(X, Y) = \tilde{\nabla}_X f_* Y - f_*(\nabla_X Y), \tag{2.1}$$

where $\tilde{\nabla}$ is the induced connection of the pull-back bundle $f^{-1}T(N)$, which is defined by $\tilde{\nabla}_X W = \bar{\nabla}_{f_* X} W$ for $W \in f^{-1}T(N)$ and $X \in T(M)$.

Let $f_*(e_i) = f_i^A \varepsilon_A, \nabla e_i = \theta_i^j e_j$ and $\bar{\nabla}\varepsilon_A = \omega_A^B \varepsilon_B$, where θ_i^j and ω_A^B are the connection forms of M and N respectively. Then the Cartan structure equations of M and N are as follows:

$$\begin{cases} d\theta^i = -\theta^i_j \wedge \theta^j,\ \theta^i_j + \theta^j_i = 0, \\ d\theta^i_j = -\theta^i_k \wedge \theta^k_j + \Omega^i_j,\ \Omega^i_j = \frac{1}{2} R^i_{jkl} \theta^k \wedge \theta^l \end{cases} \tag{2.2}$$

and

$$\begin{cases} d\omega^A = -\omega^A_B \wedge \omega^B, \ \omega^A_B + \omega^B_A = 0, \\ d\omega^A_B = -\omega^A_C \wedge \omega^C_B + \Phi^A_B, \ \Phi^A_B = \frac{1}{2}\bar{K}^A_{BCD}\omega^C \wedge \omega^D, \end{cases} \tag{2.3}$$

where R^i_{jkl} and $\bar{K}^A_{BCD}$ are the Riemannian curvature tensors of M and N respectively. Here $R^i_{jkl} = R_{ijkl}$ and $\bar{K}^A_{BCD} = \bar{K}_{ABCD}$ because e_1, e_2 and $\varepsilon_1, \ldots, \varepsilon_n$ are both unit orthogonal tangent frames.

Since f is an isometric immersion, our basic fact is

$$\sum_A f^A_i f^A_j = \delta_{ij} \tag{2.4}$$

for any $i, j = 1, 2$.

Let $\sigma = f^A_{ij}\theta^i \otimes \theta^j \otimes \varepsilon_A$. By (2.1), it is easy to see that

$$f^A_{ij}\theta^j = df^A_i - f^A_j\theta^j_i + f^B_i\omega^A_B, \tag{2.5}$$

where $f^A_{ij} = f^A_{ji}$.

It follows from (2.4) that $\sum_A f^A_1 f^A_1 = 1$, $\sum_A f^A_2 f^A_2 = 1$ and $\sum_A f^A_1 f^A_2 = 0$. Taking the exterior derivative of these three equations and apply the structure equations, we get

$$\sum_A f^A_1 f^A_{11} = 0, \sum_A f^A_1 f^A_{12} = 0, \ \sum_A f^A_2 f^A_{21} = 0,$$
$$\sum_A f^A_2 f^A_{22} = 0, \ \sum_A f^A_2 f^A_{11} = 0, \ \sum_A f^A_1 f^A_{22} = 0,$$

from which we conclude that

$$\sum_A f^A_i f^A_{jk} = 0 \tag{2.6}$$

for any $1 \leq i, j, k \leq 2$.

Let $\nabla^{\perp}$ be the induced normal connection of the normal bundle $T^{\perp}(M)$ of M by f. Then the *first order convariant derivative* of σ induced by $\nabla^{\perp}$ is defined as follows:

$$\nabla\sigma(X, Y, Z) = \nabla^{\perp}_Z\sigma(X, Y) - \sigma(\nabla_Z X, Y) - \sigma(X, \nabla_Z Y). \tag{2.7}$$

Similarly, the *second order convariant derivative* of σ induced by $\nabla^{\perp}$ is given by

$$\begin{aligned} \nabla^2\sigma(X, Y, Z, W) = \nabla^{\perp}_W\nabla\sigma(X, Y, Z) - \nabla\sigma(\nabla_W X, Y, Z) \\ -\nabla\sigma(X, \nabla_W Y, Z) - \nabla\sigma(X, Y, \nabla_W Z), \end{aligned} \tag{2.8}$$

for $X, Y, Z, W \in T(M)$.

Let $\nabla\sigma = f^A_{ijk}\theta^i \otimes \theta^j \otimes \theta^k \otimes \varepsilon_A$ and $\nabla^2\sigma = f^A_{ijkl}\theta^i \otimes \theta^j \otimes \theta^k \otimes \theta^l \otimes \varepsilon_A$. From (2.7) and (2.8), direct calculations give

$$f^A_{ijk}\theta^k = df^A_{ij} + f^B_{ij}\omega^A_B - f^A_{kj}\theta^k_i - f^A_{ik}\theta^k_j - \sum_{k,B} f^A_k f^B_k(df^B_{ij} + f^C_{ij}\omega^B_C), \tag{2.9}$$

and

$$f^A_{ijkl}\theta^l = df^A_{ijk} + f^B_{ijk}\omega^A_B - f^A_{ljk}\theta^l_i - f^A_{ilk}\theta^l_j - f^A_{ijl}\theta^l_k - \sum_{l,B} f^A_l f^B_l (df^B_{ijk} + f^C_{ijk}\omega^B_C). \tag{2.10}$$

By exterior differentiating both sides of (2.5) and using (2.2) and (2.3), the *Codazzi equation* of f is given as follows:

$$f^A_{ijk} - f^A_{ikj} = -f^B_i f^C_j f^D_k \bar{K}^A_{BCD} + \sum_l f^A_l f^B_l f^C_i f^D_j f^E_k \bar{K}_{BCDE}. \tag{2.11}$$

Let

$$\tilde{f}^A_{ijk}\theta^k = df^A_{ij} - f^A_{kj}\theta^k_i - f^A_{ik}\theta^k_j + f^B_{ij}\omega^A_B. \tag{2.12}$$

Using (2.9) and (2.12) together, we get

$$f^A_{ijk} = \tilde{f}^A_{ijk} - \sum_{l,B} f^A_l f^B_l \tilde{f}^B_{ijk}. \tag{2.13}$$

Taking the exterior derivative of $\sum_A f^A_i f^A_{jk} = 0$, we get

$$\sum_A f^A_i \tilde{f}^A_{jkl} = -\sum_A f^A_{il} f^A_{jk}. \tag{2.14}$$

Exterior differentiation of the equation in (2.9) yields

$$f^A_{ijkl} - f^A_{ijlk} = -f^B_{ij} f^C_k f^D_l \bar{K}^A_{BCD} + \sum_h f^A_h f^B_h f^C_{ij} f^D_k f^E_l \bar{K}_{BCDE} + f^A_{ih} R^h_{jkl} + f^A_{hj} R^h_{ikl} + \sum_{B,h} (f^A_{lh} f^B_{ij} f^B_{kh} - f^A_{kh} f^B_{ij} f^B_{lh}). \tag{2.15}$$

From (2.3) we get $f^*\Phi^A_B = \frac{1}{2} f^C_i f^D_j \bar{K}^A_{BCD}\theta^i \wedge \theta^j$, the derivative $(f^C_i f^D_j \bar{K}_{ABCD})_{,k}$ is defined by

$$(f^C_i f^D_j \bar{K}_{ABCD})_{,k}\theta^k = d(f^C_i f^D_j \bar{K}_{ABCD}) - f^C_k f^D_j \bar{K}_{ABCD}\theta^k_i - f^C_i f^D_k \bar{K}_{ABCD}\theta^k_j - f^C_i f^D_j \bar{K}_{EBCD}\omega^E_A - f^C_i f^D_j \bar{K}_{AECD}\omega^E_B. \tag{2.16}$$

This covariant derivative $(f^C_i f^D_j \bar{K}_{ABCD})_{,k}$ must be distinguished from the covariant derivative of $\bar{K}_{ABCD}$ as a curvature tensor of N, which will be denoted by $\bar{K}_{ABCD;E}$ (see [1]).

In this section, we shall assume that N is *symmetric*, i.e., $\bar{K}_{ABCD;E} = 0$. It follows from (2.16) that

$$(f^C_i f^D_j \bar{K}_{ABCD})_{,k} = (f^C_{ik} f^D_j + f^C_i f^D_{jk})\bar{K}_{ABCD}. \tag{2.17}$$

Denote $\bar{K} = f_1^A f_2^B f_1^C f_2^D \bar{K}_{ABCD}$, then we prove as follows.

PROPOSITION 2.1

Let M be a Riemannian surface and N be a Riemannian symmetric space. Suppose that $f : M \to N$ is an isometric immersion with the second fundamental form σ. Then the Laplacian of the square length of σ satisfies the following equation:

$$\begin{aligned}\frac{1}{2}\triangle\|\sigma\|^2 = {} & \|\nabla\sigma\|^2 + \sum_{A,i,j,k} f_{ij}^A f_{kkij}^A + \sum_{A,B,i,j,k,l} f_{ij}^A f_{jl}^A f_{il}^B f_{kk}^B \\ & +2\left[2\|\sigma\|^2 - \sum_A (f_{11}^A + f_{22}^A)^2\right]\bar{K} \\ & -\sum_{i,j,k}\left(4 f_{ij}^A f_{kj}^B f_i^C f_k^D + f_{ij}^A f_k^B f_{ij}^C f_k^D + f_{ij}^A f_i^B f_j^C f_{kk}^D\right) \\ & \bar{K}_{ABCD} - \sum_{A,B,i,j}\left[\sum_k (f_{ik}^A f_{jk}^B - f_{jk}^A f_{ik}^B)\right]^2 \\ & -\sum_{A,B}\left(\sum_{i,j} f_{ij}^A f_{ij}^B\right)^2 . \end{aligned} \tag{2.18}$$

Proof. Using (2.11) and (2.15), routine computations show that

$$\begin{aligned} f_{ijkk}^A = {} & f_{kkij}^A + \sum_l (f_l^A f_l^B f_k^C f_i^D f_k^E \bar{K}_{BCDE})_{,j} \\ & -(f_k^B f_i^C f_k^D \bar{K}_{BCD}^A)_{,j} + f_{il}^A R_{kjk}^l + f_{kl}^A R_{ijk}^l \\ & +\sum_l f_l^A f_l^B f_{ik}^C f_j^D f_k^E \bar{K}_{BCDE} \\ & -f_{ik}^B f_j^C f_k^D \bar{K}_{BCD}^A + \sum_{B,l} (f_{kl}^A f_{ik}^B f_{jl}^B - f_{jl}^A f_{ik}^B f_{kl}^B) \\ & +\sum_l (f_l^A f_l^B f_i^C f_j^D f_k^E \bar{K}_{BCDE})_{,k} \\ & -(f_i^B f_j^C f_k^D \bar{K}_{BCD}^A)_{,k}. \end{aligned} \tag{2.19}$$

Substituting (2.17) into (2.19), by (2.6), we get

$$\begin{aligned}\sum_{A,i,j} f_{ij}^A \triangle f_{ij}^A = {} & \sum_{A,i,j,k} f_{ij}^A f_{kkij}^A - \sum_{i,j,k}\Big(4 f_{ij}^A f_{kj}^B f_i^C f_k^D + f_{ij}^A f_k^B f_{ij}^C f_k^D \\ & + f_{ij}^A f_i^B f_j^C f_{kk}^D\Big) \bar{K}_{ABCD} \\ & +2 \sum_{E,i,j,k,l} \Big(f_{ij}^E f_{jl}^E f_l^A f_k^B f_i^C f_k^D \bar{K}_{ABCD} \end{aligned}$$

$$+f_{ij}^E f_{kl}^E f_l^A f_i^B f_j^C f_k^D \bar{K}_{ABCD}\Big) + \sum_{A,i,j,k} f_{ij}^A f_{jl}^A f_{il}^B f_{kk}^B$$

$$- \sum_{A,B,i,j} \left[\sum_k (f_{ik}^A f_{jk}^B - f_{jk}^A f_{ik}^B) \right]^2$$

$$- \sum_{A,B} \left(\sum_{i,j} f_{ij}^A f_{ij}^B \right)^2. \tag{2.20}$$

Since

$$\frac{1}{2}\triangle \|\sigma\|^2 = \|\nabla\sigma\|^2 + \sum_{A,i,j} f_{ij}^A \triangle f_{ij}^A \tag{2.21}$$

and

$$\begin{cases} \displaystyle\sum_{E,i,j,k} f_{ij}^E f_{jl}^E f_l^A f_k^B f_i^C f_k^D \bar{K}_{ABCD} = \|\sigma\|^2 \bar{K}, \\ \displaystyle\sum_{A,i,j,k} f_{ij}^E f_{kl}^E f_l^A f_i^B f_j^C f_k^D \bar{K}_{ABCD} = \left[\|\sigma\|^2 - \sum_A (f_{11}^A + f_{22}^A)^2 \right] \bar{K}, \end{cases} \tag{2.22}$$

(2.18) can be obtained by (2.20) (2.21) and (2.22). □

We say that f is *minimal* if $\operatorname{tr}\sigma = 0$, i.e., $\sum_i f_{ii}^A = 0$ for any A. This is an immediate consequence of Proposition 2.1.

COROLLARY 2.2

Let $f : M \to N$ be an isometric minimal immersion of a Riemannian surface M in a symmetric space N with the second fundamental form σ. Then the Laplacian of the square length of σ satisfies the following equation:

$$\frac{1}{2}\triangle \|\sigma\|^2 = \|\nabla\sigma\|^2 - \sum_{i,j,k} \left(4 f_{ij}^A f_{kj}^B f_i^C f_k^D + f_{ij}^A f_k^B f_{ij}^C f_k^D\right) \bar{K}_{ABCD}$$

$$+4\|\sigma\|^2 \bar{K} - \sum_{A,B,i,j} \left[\sum_k \left(f_{ik}^A f_{jk}^B - f_{jk}^A f_{ik}^B\right) \right]^2$$

$$- \sum_{A,B} \left(\sum_{i,j} f_{ij}^A f_{ij}^B \right)^2. \tag{2.23}$$

Remark 2.3. In Corollary 2.2, if we assume that the ambient space N is a space of constant curvature c, i.e., $\bar{K}_{ABCD} = c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC})$, then (2.23) reduces to

$$\frac{1}{2}\triangle\|\sigma\|^2 = \|\nabla\sigma\|^2 + 2c\|\sigma\|^2 - \sum_{A,B,i,j}\left[\sum_k (f_{ik}^A f_{jk}^B - f_{jk}^A f_{ik}^B)\right]^2 - \sum_{A,B}\left(\sum_{i,j} f_{ij}^A f_{ij}^B\right)^2. \tag{2.24}$$

For each A, let F_A denote the matrix (f_{ij}^A), and set $S_{AB} = \sum_{i,j} f_{ij}^A f_{ij}^B$. Then the $(n \times n)$-matrix (S_{AB}) is symmetric. Now (2.24) can be rewritten as follows:

$$\frac{1}{2}\triangle\|\sigma\|^2 = \|\nabla\sigma\|^2 + 2c\|\sigma\|^2 - \sum_{A,B} N(F_A F_B - F_B F_A) - \sum_{A,B} S_{AB}^2, \tag{2.25}$$

where $N(F_A F_B - F_B F_A)$ denote the square of the normal of $F_A F_B - F_B F_A$, i.e.,

$$N(F_A F_B - F_B F_A) = \mathrm{tr}(F_A F_B - F_B F_A) \cdot^t (F_A F_B - F_B F_A).$$

The formulae we obtained in this section are consistent with the formulae in [1] when the frame is a Darboux frame.

3. Minimal surfaces immersed in symmetric spaces with parallel second fundamental form

Let $f : M \to N$ be an isometric immersion of a Riemannian surface M in a symmetric space N. In this section, we concentrate primarily on geometric properties of f with parallel second fundamental form. The main purpose of this part is the relation between Gaussian curvature K and second fundamental form σ of f.

We say that σ is *parallel* if $\nabla\sigma = 0$, i.e., $f_{ijk}^A = 0$ for any $1 \le A \le n$ and $1 \le i, j, k \le 2$.

Since f has parallel second fundamental form, (2.11) can be rewritten as

$$\sum_l f_l^A f_l^B f_i^C f_j^D f_k^E \bar{K}_{BCDE} = f_i^B f_j^C f_k^D \bar{K}_{BCD}^A. \tag{3.1}$$

Then we have

$$f_2^B f_1^C f_2^D \bar{K}_{ABCD} = f_1^A \bar{K},\ \ f_1^B f_2^C f_1^D \bar{K}_{ABCD} = f_2^A \bar{K}.$$

Similarly, (2.13) can be rewritten as

$$\tilde{f}_{ijk}^A = \sum_{l,B} f_l^A f_l^B \tilde{f}_{ijk}^B. \tag{3.2}$$

It follows from (2.6) and (3.2) that

$$\sum_A f^A_{mn} \tilde{f}^A_{ijk} = 0. \tag{3.3}$$

Therefore we have as follows.

Theorem 3.1. *Let $f : M \to N$ be an isometric immersion of a Riemannian surface M in a symmetric space N. If its second fundamental form σ is parallel, then the Gaussian curvature K must be constant.*

Proof. Exterior differentiating both sides of the equation $f_2^B f_1^C f_2^D \bar{K}_{ABCD} = f_1^A \bar{K}$, we have

$$\begin{aligned} &d(f_2^B f_1^C f_2^D \bar{K}_{ABCD}) \\ &\quad = (f_{2i}^B f_1^C f_2^D + f_2^B f_{1i}^C f_2^D + f_2^B f_1^C f_{2i}^D)\bar{K}_{ABCD}\theta^i \\ &\qquad + f_1^B f_1^C f_2^D \bar{K}_{ABCD}\theta_2^1 + f_2^B f_1^C f_2^D \bar{K}_{EBCD}\omega_A^E \end{aligned}$$

and

$$d(f_1^A \bar{K}) = f_{1i}^A \bar{K}\theta^i + f_2^A \bar{K}\theta_1^2 - f_1^E \bar{K}\omega_E^A + f_1^A d\bar{K}.$$

These two equations and (3.1) imply that

$$f_1^A d\bar{K} = [(f_{2i}^B f_1^C f_2^D + f_2^B f_{1i}^C f_2^D + f_2^B f_1^C f_{2i}^D)\bar{K}_{ABCD} - f_{1i}^A \bar{K}]\theta^i. \tag{3.4}$$

Multiplying (3.4) by f_1^A, summing on A and using (2.4) and (2.6), we obtain

$$d\bar{K} = (f_1^A f_{2i}^B f_1^C f_2^D + f_1^A f_2^B f_{1i}^C f_2^D + f_1^A f_2^B f_1^C f_{2i}^D)\bar{K}_{ABCD}\theta^i. \tag{3.5}$$

By (2.6) and (3.1) one verifies that $d\bar{K} = 0$, i.e., $\bar{K}$ is a constant. Moreover, from (2.12) and (3.3), we get

$$d\left[\sum_A (f_{11}^A f_{22}^A - f_{12}^A f_{12}^A)\right] = 0.$$

The result follows from the Gaussian equation $K = \bar{K} + \sum_A (f_{11}^A f_{22}^A - f_{12}^A f_{12}^A)$. □

Remark 3.2. For an isometric immersion $f : M \to N$ of a Riemannian surface M in a symmetric space N with parallel second fundamental form, eq. (3.4) implies that

$$f_{1i}^A \bar{K} = \left(f_{2i}^B f_1^C f_2^D + f_2^B f_{1i}^C f_2^D + f_2^B f_1^C f_{2i}^D\right)\bar{K}_{ABCD}. \tag{3.6}$$

For the equation $f_1^B f_2^C f_1^D \bar{K}_{ABCD} = f_2^A \bar{K}$, a similar calculation gives

$$f_{2i}^A \bar{K} = (f_{1i}^B f_2^C f_1^D + f_1^B f_{2i}^C f_1^D + f_1^B f_2^C f_{1i}^D)\bar{K}_{ABCD}. \tag{3.7}$$

In the following, we discuss the special case $M = S^2$. This case exhibits many special features not present in the general case. These arise from the fact that S^2 has no holomorphic differential forms of positive degree except zero. From our analytical data we are able to construct such forms and obtain strong conclusions.

For an isometric immersion $f : S^2 \to N$, it follows from Proposition 2.1 that σ is parallel if and only if the following equation holds:

$$\begin{aligned} 0 = &\sum_{A,i,j,k} f_{ij}^A f_{kkij}^A + \sum_{A,B,i,j,k,l} f_{ij}^A f_{jl}^A f_{il}^B f_{kk}^B \\ &+2\left[2\|\sigma\|^2 - \sum_A (f_{11}^A + f_{22}^A)^2\right]\bar{K} \\ &- \sum_{i,j,k}\left(4 f_{ij}^A f_{kj}^B f_i^C f_k^D + f_{ij}^A f_k^B f_{ij}^C f_k^D + f_{ij}^A f_i^B f_j^C f_{kk}^D\right)\bar{K}_{ABCD} \\ &- \sum_{A,B,i,j}\left[\sum_k (f_{ik}^A f_{jk}^B - f_{jk}^A f_{ik}^B)\right]^2 - \sum_{A,B}\left(\sum_{i,j} f_{ij}^A f_{ij}^B\right)^2. \end{aligned} \tag{3.8}$$

If $f : S^2 \to N$ is a minimal immersion, we prove the following result.

PROPOSITION 3.3

Let $f : S^2 \to N$ be an isometric minimal immersion of two-spheres S^2 in a symmetric space N with the second fundamental form σ. Then σ is parallel if and only if the following equation holds:

$$\begin{aligned} \|\sigma\|^2 \bar{K} = &\, 2 f_{11}^A f_{12}^B f_1^C f_2^D \bar{K}_{ABCD} + 4\sum_{A,B} (f_{11}^A f_{12}^B - f_{12}^A f_{11}^B)^2 \\ &+2\sum_{A,B}(f_{11}^A f_{11}^B + f_{12}^A f_{12}^B)^2. \end{aligned} \tag{3.9}$$

Proof. Since f is minimal, by (3.8) we see that, σ is parallel if and only if f satisfies the following equation:

$$\begin{aligned} 4\|\sigma\|^2 \bar{K} = &\sum_{i,j,k}\left(4 f_{ij}^A f_{kj}^B f_i^C f_k^D + f_{ij}^A f_k^B f_{ij}^C f_k^D\right)\bar{K}_{ABCD} \\ &+ \sum_{A,B,i,j}\left[\sum_k (f_{ik}^A f_{jk}^B - f_{jk}^A f_{ik}^B)\right]^2 \\ &+ \sum_{A,B}\left(\sum_{i,j} f_{ij}^A f_{ij}^B\right)^2. \end{aligned} \tag{3.10}$$

Here we note that $1 \le i, j, k \le 2$. Hence

$$f_{ij}^A f_{kj}^B f_i^C f_k^D \bar{K}_{ABCD} = 4 f_{11}^A f_{12}^B f_1^C f_2^D \bar{K}_{ABCD} \tag{3.11}$$

and

$$f_{ij}^A f_k^B f_{ij}^C f_k^D \bar{K}_{ABCD} = 2(f_{11}^A f_1^B f_{11}^C f_1^D + f_{11}^A f_2^B f_{11}^C f_2^D + f_{12}^A f_1^B f_{12}^C f_1^D + f_{12}^A f_2^B f_{12}^C f_2^D)\bar{K}_{ABCD}.$$

Using (3.6) and (3.7), a direct computation shows that

$$f_{ij}^A f_k^B f_{ij}^C f_k^D \bar{K}_{ABCD} = 2(\|\sigma\|^2 \bar{K} - 6 f_{11}^A f_{12}^B f_1^C f_2^D \bar{K}_{ABCD}) \tag{3.12}$$

On the other hand, we have

$$\sum_{A,B,i,j} \left[\sum_k (f_{ik}^A f_{jk}^B - f_{jk}^A f_{ik}^B) \right]^2 = 8 \sum_{A,B} (f_{11}^A f_{12}^B - f_{12}^A f_{11}^B)^2, \tag{3.13}$$

and

$$\sum_{A,B} \left(\sum_{i,j} f_{ij}^A f_{ij}^B \right)^2 = 4 \sum_{A,B} (f_{11}^A f_{11}^B + f_{12}^A f_{12}^B)^2. \tag{3.14}$$

Substituting (3.11)–(3.14) into (3.10), we get the result. □

To simplify (3.9), we have as follows.

Lemma 3.4. *Let $f : S^2 \to N$ be an isometric minimal immersion from S^2 to a symmetric space N. If f has parallel second fundamental form, then the following equations hold*:

$$\sum_A (f_{11}^A)^2 = \sum_A (f_{12}^A)^2, \ \sum_A f_{11}^A f_{12}^A = 0 \tag{3.15}$$

Proof. Let $\varphi = \theta^1 + \sqrt{-1}\theta^2$, which is a complex-valued one-form. Then we have $d\varphi = \sqrt{-1}\rho \wedge \varphi$, where ρ is the connection form with respect to φ. Here $\varphi = \lambda dz$, where z is a local holomorphic coordinate in S^2 and λ is a complex-valued smooth local function on S^2.

Consider the holomorphic 4-form

$$Q = \sum_A (f_{11}^A - \sqrt{-1} f_{12}^A)^2 \varphi^4$$

of type (4, 0). Let $g = \sum_A \big(f_{11}^A - \sqrt{-1} f_{12}^A\big)^2 \lambda^4$. We wish to show that its differential is zero mod φ, since

$$\begin{aligned} dg = & \, 2\sum_A (f_{11}^A - \sqrt{-1} f_{12}^A)[(\tilde{f}_{11i}^A - \sqrt{-1}\tilde{f}_{12i}^A)\theta^i \\ & + 2\sqrt{-1}(f_{11}^A - \sqrt{-1} f_{12}^A)\theta_1^2]\lambda^4 \\ & - 2\sum_A \big(f_{11}^A - \sqrt{-1} f_{12}^A\big)\big(f_{11}^B - \sqrt{-1} f_{12}^B\big)\omega_B^A \lambda^4 \\ & + 4\sum_A \big(f_{11}^A - \sqrt{-1} f_{12}^A\big)^2 \lambda^3 d\lambda. \end{aligned}$$

It is a routine computation to check that

$$dg \equiv 2\sum_A \big(f_{11}^A - \sqrt{-1} f_{12}^A\big)\big(\tilde{f}_{11i}^A - \sqrt{-1}\tilde{f}_{12i}^A\big)\lambda^4 \theta^i, \quad \text{mod } \varphi.$$

This formula and (3.3) yield

$$dg \equiv 0, \quad \text{mod } \varphi,$$

which implies that Q is a holomorphic (4, 0) form on S^2. It is easy to verify that Q is independent of the choice of frames and it is globally defined on S^2. Thus we have

$$\sum_A \big(f_{11}^A - \sqrt{-1} f_{12}^A\big)^2 = 0.$$

This completes the proof. □

Remark 3.5. Let $f : S^2 \to N$ be an isometric minimal immersion from S^2 to a symmetric space N. Suppose the second fundamental form of f is parallel, it follows from (2.14), (3.2) and (3.15) that

$$\begin{aligned} \tilde{f}_{111}^A &= -\frac{\|\sigma\|^2}{4} f_1^A, \quad \tilde{f}_{112}^A = \frac{\|\sigma\|^2}{4} f_2^A, \\ \tilde{f}_{121}^A &= -\frac{\|\sigma\|^2}{4} f_2^A, \quad \tilde{f}_{122}^A = -\frac{\|\sigma\|^2}{4} f_1^A, \\ \tilde{f}_{211}^A &= -\frac{\|\sigma\|^2}{4} f_2^A, \quad \tilde{f}_{212}^A = -\frac{\|\sigma\|^2}{4} f_1^A, \\ \tilde{f}_{221}^A &= \frac{\|\sigma\|^2}{4} f_1^A, \quad \tilde{f}_{222}^A = -\frac{\|\sigma\|^2}{4} f_2^A. \end{aligned}$$

By Proposition 3.3 and Lemma 3.4, a direct computation shows as follows

PROPOSITION 3.6

Let $f : S^2 \to N$ be an isometric minimal immersion of two-spheres S^2 in a symmetric space N. Suppose that f has parallel second fundamental form. Then the following equation holds.

$$\|\sigma\|^2\left(K-\frac{\|\sigma\|^2}{4}\right)-2f_{11}^A f_{12}^B f_1^C f_2^D \bar{K}_{ABCD}=0. \tag{3.16}$$

COROLLARY 3.7

Let $f : S^2 \to N$ be an isometric minimal immersion from S^2 to a symmetric space N. If its second fundamental form is parallel, then f satisfies that

$$\begin{cases} f_{12}^B f_1^C f_2^D \bar{K}_{ABCD}-2f_{11}^A\left(K-\frac{\|\sigma\|^2}{4}\right)=0,\\ f_{11}^B f_1^C f_2^D \bar{K}_{ABCD}+2f_{12}^A\left(K-\frac{\|\sigma\|^2}{4}\right)=0. \end{cases} \tag{3.17}$$

Proof. Since f has parallel second fundamental form, formula (2.9) becomes

$$df_{ij}^A+f_{ij}^B\omega_B^A-\sum_{k,B} f_k^A f_k^B(df_{ij}^B+f_{ij}^C\omega_C^B)-f_{kj}^A\theta_i^k-f_{ik}^A\theta_j^k=0.$$

Taking its exterior derivative and using the following four equations

$$\left[df_{ij}^B+f_{ij}^C\omega_C^B-\sum_{k,C} f_k^B f_k^C(df_{ij}^C+f_{ij}^D\omega_D^C)-f_{kj}^B\theta_i^k-f_{ik}^B\theta_j^k\right]\wedge\omega_B^A=0,$$

$$\left[df_{kj}^A+f_{kj}^B\omega_B^A-\sum_{l,B} f_l^A f_l^B(df_{kj}^B+f_{kj}^C\omega_C^B)-f_{lj}^A\theta_k^l-f_{kl}^A\theta_j^l\right]\wedge\theta_i^k=0,$$

$$\left[df_{ik}^A+f_{ik}^B\omega_B^A-\sum_{l,B} f_l^A f_l^B(df_{ik}^B+f_{ik}^C\omega_C^B)-f_{lk}^A\theta_i^l-f_{il}^A\theta_k^l\right]\wedge\theta_j^k=0,$$

$$\sum_{h,B} f_h^A f_{lh}^B\theta^l\wedge(f_{kj}^B\theta_i^k+f_{ik}^B\theta_j^k-df_{ij}^B-f_{ij}^C\omega_C^B)=0,$$

we get

$$\sum_{l,B} f_{lh}^A f_{ij}^B f_{lk}^B\theta^h\wedge\theta^k+f_{ij}^B\Phi_B^A-f_{kj}^A\Omega_i^k-f_{ik}^A\Omega_j^k$$
$$-\sum_{l,B} f_l^A f_l^B f_{ij}^C\Phi_C^B=0.$$

By (3.1), we have $f_l^A f_l^B f_{ij}^C\Phi_C^B=0$, therefore

$$\sum_{h,B} f_{hk}^A f_{ij}^B f_{lh}^B-\sum_{h,B} f_{hl}^A f_{ij}^B f_{hk}^B+f_{ij}^B f_k^C f_l^D \bar{K}_{BCD}^A$$
$$-f_{hj}^A R_{ikl}^h-f_{ih}^A R_{jkl}^h=0.$$

It is then easy to see that $f_{11}^B f_1^C f_2^D\bar{K}_{ABCD}+2f_{12}^A(K-\frac{\|\sigma\|^2}{4})=0$ and $f_{12}^B f_1^C f_2^D\bar{K}_{ABCD}-2f_{11}^A(K-\frac{\|\sigma\|^2}{4})=0$ by taking $i=j=k=1, l=2$ and $i=k=1, j=l=2$ respectively, which ends the proof. □

Remark 3.8. When N is a specific symmetric space, for example, the complex projective space $\mathbb{C}P^n$, the complex hyperquadric Q_n and the complex Grassmann manifold $G(k,n;\mathbb{C})$, its curvature form Φ_B^A may be expressed under a unitary frame, and some good geometric properties should be obtained if f has parallel second fundamental form.

4. Minimal two-spheres immersed in Kähler symmetric spaces with parallel second fundamental form

In this section, we wish to apply the results obtained in § 3 to the special case that N is a Kähler symmetric space of complex dimension 2. In this case our results in § 3 have further geometrical interpretation.

Let N be a complex two-dimensional Kähler symmetric space, whose metric we write as

$$ds^2 = \sum_a \phi^a \bar{\phi}^a, \tag{4.1}$$

where here, as well as throughout this paper, we use the index range $1 \le a, b, c, \ldots \le 2$. The forms ϕ^a are of type $(1,0)$ and are defined up to a unitary transformation. They constitute a unitary coframe.

To get hold of the underlying Riemannian structure we set

$$\phi^a = \omega^a + \sqrt{-1}\omega^{2+a}. \tag{4.2}$$

Since $\omega = (\omega_B^A)$ has values in the real representation of $\mathfrak{u}(n)$, we have

$$\omega_b^a = \omega_{2+b}^{2+a},\ \omega_{2+b}^a = -\omega_b^{2+a},\ \omega_b^a = -\omega_a^b,\ \omega_{2+b}^a = \omega_{2+a}^b. \tag{4.3}$$

The notation of a minimal surface in N is defined in terms of its underlying Riemannian structure.

Consider now an immersed 2-sphere $f : S^2 \to N$. In fact, the complex structure on S^2 can be defined by the complex-valued one-form $\varphi = \theta^1 + \sqrt{-1}\theta^2$. We have restricted to M,

$$\phi^a = s_a\varphi + t_a\bar{\varphi}. \tag{4.4}$$

Since $ds_{S^2}^2 = \varphi\bar{\varphi}$, it follows from (4.1) and (4.4) that

$$\sum_a (|s_a|^2 + |t_a|^2) = 1,\ \sum_a s_a\bar{t}_a = 0. \tag{4.5}$$

Using (4.2) and (4.4) we also have

$$\begin{cases} f_1^a = \frac{1}{2}(s_a + \bar{s}_a + t_a + \bar{t}_a),\ f_1^{2+a} = \frac{1}{2\sqrt{-1}}(s_a - \bar{s}_a + t_a - \bar{t}_a), \\ f_2^a = \frac{\sqrt{-1}}{2}(s_a - \bar{s}_a - t_a + \bar{t}_a),\ f_2^{2+a} = \frac{1}{2}(s_a + \bar{s}_a - t_a - \bar{t}_a). \end{cases} \tag{4.6}$$

By (4.6), with a simple test we can check that (4.5) is equal to (2.4).

It is then well known [2] that there exists a real-valued continuous function $\alpha \in [0, \pi]$ satisfying

$$f^*\left(\frac{\sqrt{-1}}{2}\sum_a \phi^a \wedge \bar{\phi}^a\right) = \cos\alpha\theta^1 \wedge \theta^2, \tag{4.7}$$

which is called the *Kähler angle* of f. If f is neither holomorphic nor anti-holomorphic, then, except at possible isolated points, the Kähler angle α is smooth.

Recall that the immersion f is *holomorphic* or *anti-holomorphic* if and only if $\sin\alpha = 0$ on S^2.

Substituting (4.4) into (4.7) we get

$$\cos\alpha = \sum_a (|s_a|^2 - |t_a|^2). \tag{4.8}$$

By (4.5) (4.8) and a unitary transformation at each point of S^2, it is possible to set

$$s_1 = \cos\frac{\alpha}{2},\ s_2 = 0,\ t_1 = 0,\ t_2 = \sin\frac{\alpha}{2}. \tag{4.9}$$

This together with (4.6) implies that

$$\begin{cases} f_1^1 = \cos\frac{\alpha}{2},\ f_1^2 = \sin\frac{\alpha}{2},\ f_1^3 = 0,\ f_1^4 = 0, \\ f_2^1 = 0,\ f_2^2 = 0,\ f_2^3 = \cos\frac{\alpha}{2},\ f_2^4 = -\sin\frac{\alpha}{2}. \end{cases} \tag{4.10}$$

In this section we mainly concern local behavior of the Kähler angle of isometric minimal 2-spheres immersed in complex two-dimensional Kähler symmetric space under the assumption that the second fundamental form is parallel. Since we know that, for such an isometric minimal immersion, if it is holomorphic or anti-holomorphic, its Kähler angle is 0 or π and it is not always totally geodesic under the condition that its second fundamental form is parallel (see [4]). From now on we may assume that it is neither holomorphic nor anti-holomorphic.

By taking the exterior derivatives of (4.10) and making using of (4.3), we have the following relations:

$$\begin{aligned} f_{11}^2 &= -\cot\frac{\alpha}{2} f_{11}^1,\ f_{11}^3 = f_{12}^1 + \sin\frac{\alpha}{2}\alpha_2, \\ f_{11}^4 &= \cot\frac{\alpha}{2}\left(f_{12}^1 + \sin\frac{\alpha}{2}\alpha_2\right), \end{aligned} \tag{4.11}$$

$$\begin{aligned} f_{12}^2 &= -\cot\frac{\alpha}{2} f_{12}^1,\ f_{12}^3 = -\left(f_{11}^1 + \sin\frac{\alpha}{2}\alpha_1\right), \\ f_{12}^4 &= -\cot\frac{\alpha}{2}\left(f_{11}^1 + \sin\frac{\alpha}{2}\alpha_1\right), \end{aligned} \tag{4.12}$$

$$\sin\frac{\alpha}{2}\omega_4^1 = -\frac{1}{2}\left[\left(2f_{12}^1 + \sin\frac{\alpha}{2}\alpha_2\right)\theta^1 - \left(2f_{11}^1 + \sin\frac{\alpha}{2}\alpha_1\right)\theta^2\right], \tag{4.13}$$

$$\sin\frac{\alpha}{2}\omega_2^1 = \frac{1}{2}\left[\left(2f_{11}^1 + \sin\frac{\alpha}{2}\alpha_1\right)\theta^1 + \left(2f_{12}^1 + \sin\frac{\alpha}{2}\alpha_2\right)\theta^2\right], \tag{4.14}$$

$$\theta_2^1 - \omega_3^1 = \frac{1}{2}\tan\frac{\alpha}{2}(\alpha_2\theta^1 - \alpha_1\theta^2), \tag{4.15}$$

$$\theta_2^1 + \omega_4^2 = -\frac{1}{2}\cot\frac{\alpha}{2}(\alpha_2\theta^1 - \alpha_1\theta^2), \tag{4.16}$$

where $d\alpha = \alpha_1\theta^1 + \alpha_2\theta^2$.

Equations (4.11) (4.12) and Lemma 3.4 yield

$$\begin{cases} (f_{11}^1)^2 + \left(f_{12}^1 + \sin\frac{\alpha}{2}\alpha_2\right)^2 = \frac{\|\sigma\|^2}{4}\sin^2\frac{\alpha}{2}, \\ \left(f_{11}^1 + \sin\frac{\alpha}{2}\alpha_1\right)^2 + (f_{12}^1)^2 = \frac{\|\sigma\|^2}{4}\sin^2\frac{\alpha}{2}, \\ f_{11}^1 f_{12}^1 - \left(f_{11}^1 + \sin\frac{\alpha}{2}\alpha_1\right)\left(f_{12}^1 + \sin\frac{\alpha}{2}\alpha_2\right) = 0, \end{cases}$$

which implies the following two cases:

(I) Kähler angle α is constant and $(f_{11}^1)^2 + (f_{12}^1)^2 = \frac{\|\sigma\|^2}{4}\sin^2\frac{\alpha}{2}$;

(II) $2f_{11}^1 + \sin\frac{\alpha}{2}\alpha_1 = 0,\ 2f_{12}^1 + \sin\frac{\alpha}{2}\alpha_2 = 0$ and $(f_{11}^1)^2 + (f_{12}^1)^2 = \frac{\|\sigma\|^2}{4}\sin^2\frac{\alpha}{2}$.

Then we have as follows.

Theorem 4.1. *Let $f : S^2 \to N$ be an isometric minimal immersion of two-spheres S^2 in a complex two-dimensional Kähler symmetric space N, which is neither holomorphic nor anti-holomorphic. Suppose the Kähler angle $\alpha \neq 0$ (resp. $\alpha \neq \pi$) everywhere on S^2, and f has parallel second fundamental form, then α must be constant.*

Proof. We prove only case (II). Since f has parallel second fundamental form, exterior differentiating both sides of equations $f_{11}^1 + f_{12}^3 = -\sin\frac{\alpha}{2}\alpha_1$ and $f_{11}^3 - f_{12}^1 = \sin\frac{\alpha}{2}\alpha_2$, and using Remark 3.5 and (4.11)–(4.16), a direct computation shows that

$$\begin{aligned} \triangle\alpha = \cot\frac{\alpha}{2}\|\sigma\|^2 - \frac{\cos\frac{\alpha}{2}}{\sin^3\frac{\alpha}{2}}\Bigg[\left(2f_{11}^1 + \sin\frac{\alpha}{2}\alpha_1\right)^2 \\ + \left(2f_{12}^1 + \sin\frac{\alpha}{2}\alpha_2\right)^2\Bigg] - \frac{1}{\sin\alpha}(\alpha_1^2 + \alpha_2^2). \end{aligned} \tag{4.17}$$

In case (II), $2f_{11}^1 + \sin\frac{\alpha}{2}\alpha_1 = 0,\ 2f_{12}^1 + \sin\frac{\alpha}{2}\alpha_2 = 0$ and $(f_{11}^1)^2 + (f_{12}^1)^2 = \frac{\|\sigma\|^2}{4}\sin^2\frac{\alpha}{2}$, (4.17) becomes

$$\triangle\alpha = \cot\alpha\|\sigma\|^2. \tag{4.18}$$

Using equations $\triangle\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{2\cos^3\frac{\alpha}{2}}(\alpha_1^2 + \alpha_2^2) + \frac{1}{2\cos^2\frac{\alpha}{2}}\triangle\alpha$ and $\triangle\cot\frac{\alpha}{2} = \frac{\cos\frac{\alpha}{2}}{2\sin^3\frac{\alpha}{2}}(\alpha_1^2 + \alpha_2^2) - \frac{1}{2\sin^2\frac{\alpha}{2}}\triangle\alpha$, we obtain

$$\triangle\tan\frac{\alpha}{2} = \frac{1}{2\sin\alpha\cos^2\frac{\alpha}{2}}\|\sigma\|^2, \quad \triangle\cot\frac{\alpha}{2} = \frac{1}{2\sin\alpha\sin^2\frac{\alpha}{2}}\|\sigma\|^2.$$

Since $\alpha \neq 0$ (resp. $\alpha \neq \pi$) everywhere on S^2, then $\triangle\cot\frac{\alpha}{2} \geq 0$ (resp. $\triangle\tan\frac{\alpha}{2} \geq 0$), and the Hopf maximum principle implies that the Kähler angle α of f must be constant. □

It follows from (4.3) that $\Phi_3^2 = \Phi_4^1$, which can be rewritten as

$$f_1^C f_2^D \bar{K}_{23CD} = f_1^C f_2^D \bar{K}_{14CD}. \tag{4.19}$$

Coupling this with (4.11) and (4.12), we get

$$\begin{aligned} f_{11}^A f_{12}^B f_1^C f_2^D \bar{K}_{ABCD} = \Big[& f_{11}^1 \left(f_{11}^1 + \sin\frac{\alpha}{2}\alpha_1 \right) \\ & + f_{12}^1 \left(f_{12}^1 + \sin\frac{\alpha}{2}\alpha_2 \right) \Big] f_1^C f_2^D \\ & \left(\cot^2\frac{\alpha}{2} \bar{K}_{24CD} - \bar{K}_{13CD} \right). \end{aligned} \tag{4.20}$$

Theorem 4.2. *Let $f : S^2 \to N$ be an isometric minimal immersion of two-spheres S^2 in a complex two-dimensional Kähler symmetric space N, which is neither holomorphic nor anti-holomorphic. If the second fundamental form of f is parallel, then f is totally geodesic or $K = \|\sigma\|^2$.*

Proof. In case (I), Kähler angle α is constant and $(f_{11}^1)^2 + (f_{12}^1)^2 = \frac{\|\sigma\|^2}{4} \sin^2\frac{\alpha}{2}$. Equation (4.20) becomes

$$\begin{aligned} & f_{11}^A f_{12}^B f_1^C f_2^D \bar{K}_{ABCD} \\ & = \frac{\|\sigma\|^2}{4} \left(\cos^2\frac{\alpha}{2} f_1^C f_2^D \bar{K}_{24CD} - \sin^2\frac{\alpha}{2} f_1^C f_2^D \bar{K}_{13CD} \right). \end{aligned} \tag{4.21}$$

It follows from (4.15) and (4.16) that

$$\theta_2^1 = \omega_3^1, \ \theta_2^1 = -\omega_4^2.$$

Exterior differentiating both sides of these two equations, using (4.13) and (4.14) we get

$$K + \frac{\|\sigma\|^2}{2} = f_1^C f_2^D \bar{K}_{13CD}, \ K + \frac{\|\sigma\|^2}{2} = -f_1^C f_2^D \bar{K}_{24CD}. \tag{4.22}$$

Substituting (4.22) into (4.21) and comparing with (3.16) we obtain

$$\|\sigma\|^2 = 0,$$

i.e., f is totally geodesic.

In case (II), $2f_{11}^1 + \sin\frac{\alpha}{2}\alpha_1 = 0, \ 2f_{12}^1 + \sin\frac{\alpha}{2}\alpha_2 = 0$ and $(f_{11}^1)^2 + (f_{12}^1)^2 = \frac{\|\sigma\|^2}{4}\sin^2\frac{\alpha}{2}$, equation (4.20) becomes

$$\begin{aligned} f_{11}^A f_{12}^B f_1^C f_2^D \bar{K}_{ABCD} = -\frac{\|\sigma\|^2}{4} \Big(& \cos^2\frac{\alpha}{2} f_1^C f_2^D \bar{K}_{24CD} \\ & - \sin^2\frac{\alpha}{2} f_1^C f_2^D \bar{K}_{13CD} \Big). \end{aligned} \tag{4.23}$$

By a similar calculation to case (I), eq. (4.22) will be also derived. Substituting (4.22) into (4.23) and comparing with (3.16) we obtain

$$\|\sigma\|^2(K - \|\sigma\|^2) = 0.$$

□

In Theorem 4.2, moreover, suppose $\alpha \neq 0$ (resp. $\alpha \neq \pi$) everywhere on S^2. Then we have as follows.

COROLLARY 4.3

Let $f : S^2 \to N$ be an isometric minimal immersion of two-spheres S^2 in a complex two-dimensional Kähler symmetric space N, which is neither holomorphic nor anti-holomorphic. Suppose $\alpha \neq 0$ (resp. $\alpha \neq \pi$) everywhere on S^2, and f has parallel second fundamental form, then f must be totally geodesic.

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References

[1] Chern S S and Wolfson J G, Minimal surfaces by moving frames, *Am. J. Math.* **105** (1983) 59–83

[2] Chern S S, do Carmo M and Kobayshi S, Minimal submanifolds of a sphere with second fundamental form of constant length, in: Functional Analysis and Related Fields (Proc. Conf. for M. Stone, Univ. Chicago, Chicago, III. 1968) (1970) (Springer) pp. 59–75

[3] He L and Jiao X X, Classification of all conformal minimal immersions of S^2 with parallel second fundamental form, *Annali di Matematica*. **194** (2015) 1301–1317

[4] Jiao X X, On minimal two-spheres immersed in complex Grassmann manifolds with parallel second fundamental form, *Monatsh. Math.* **168** (2012) 381–401

[5] Jiao X X and Wang J, Minimal surfaces in a complex hyperquadric Q_2, *Manuscripta Math.* **140** (2013) 597–611

[6] Jiao X X and Li M Y, On conformal minimal immersions of two-spheres in a complex hyperquadric with parallel second fundamental form, *J. Geom. Anal.* **26** (2016) 185–205

[7] Naitoh H, Isotropic submanifolds with parallel second fundamental form in $P^m(C)$, *Osaka J. Math.* **18** (1981) 427–464

[8] Nakagawa H and Takagi R, On locally symmetric Kaehler submanifolds in a complex projective space, *J. Math. Soc. Japan* **28(4)** (1976) 638–667

[9] Tsukada K, Parallel submanifolds in a quaternion projective space, *Osaka J. Math.* **22** (1985) 187–241

[10] Vilms J, Submanifolds of Euclidean space with parallel second fundamental form, *Proc. Am. Math. Soc.* **32(1)** (1972) 263–267

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