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Hypersurfaces in nearly Kaehler manifold $\mathbb{S}^3 \times \mathbb{S}^3$

BIAOGUI YANG[1,*] and QINGQING ZHU[2]

[1]School of Mathematics and Computer Science, Fujian Normal University, Qishan Campus, Fuzhou 350117, People's Republic of China
[2]Math Teaching and Research Group, Zhangzhou Branch Campus of Xiamen Shuangshi Middle School, Zhangzhou 363107, People's Republic of China
*Corresponding author.
E-mail: bgyang@163.com

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Abstract. In this paper, we initiate the study of contact and minimal hypersurfaces in nearly Kaehler manifold $\mathbb{S}^3 \times \mathbb{S}^3$ with a conformal vector field. There are three almost contact metric structures on a hypersurface of $\mathbb{S}^3 \times \mathbb{S}^3$, and we will give some important properties of them. Besides, we study the influence of the conformal vector field on the almost contact metric structures and use it to characterize the hypersurfaces in $\mathbb{S}^3 \times \mathbb{S}^3$.

Keywords. Nearly Kaehler manifold; contact hypersurface; contact metric structure; minimal hypersurface; shape operator; conformal vector field.

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1. Introduction

It is known that the six-dimensional unit sphere $\mathbb{S}^6$ has a nearly Kaehler structure (J, g), the almost complex structure J can be defined in terms of the vector cross product on $\mathbb{R}^7$. Recently it has been shown by Butruille [4] that the only homogeneous 6-dimensional nearly Kaehler manifolds are the nearly Kaehler 6-sphere $\mathbb{S}^3 \times \mathbb{S}^3$, the projective space $\mathbb{C}P^3$ and the flag manifold $SU(3)/U(1) \times U(1)$. Besides, Bolton [3] introduced some important properties of almost complex surfaces in nearly Kaehler manifold $\mathbb{S}^3 \times \mathbb{S}^3$. In this paper, we will study hypersurfaces in nearly Kaehler manifold $\mathbb{S}^3 \times \mathbb{S}^3$.

Okumura [10] studied the differential geometric properties of contact hypersurfaces of Kaehlerian manifold of constant holomorphic sectional curvature. Deshmukh [5,8] studied hypersurfaces of the nearly Kaehler manifold $\mathbb{S}^6$. However, hypersurfaces of the nearly Kaehler manifold $\mathbb{S}^3 \times \mathbb{S}^3$ have not been studied so extensively. As we all know, a contact manifold is a smooth $(2n-1)$-dimensional manifold M together with a one-form η satisfying $\eta \wedge (d\eta)^{n-1} \neq 0$, $n \geq 2$ [1,2]. In this paper, first, we show the existence of two more almost contact metric structures on the hypersurfaces in nearly Kaehler manifold $\mathbb{S}^3 \times \mathbb{S}^3$ and study their properties. In the second part of this paper, we will study the contact hypersurfaces in nearly Kaehler manifold $\mathbb{S}^3 \times \mathbb{S}^3$ by use of these almost contact metric structures. We know the conformal vector fields are very useful tools in studying the geometry of Riemannian manifolds (see [6,7]). In the third part of this paper, we will use a conformal vector field on $\mathbb{S}^3 \times \mathbb{S}^3$ and the three almost contact metric structures to find the properties of minimal hypersurfaces in nearly Kaehler manifold $\mathbb{S}^3 \times \mathbb{S}^3$.

2. Preliminaries

An almost Hermitian manifold $(\bar{M}, g, J)$ is a manifold endowed with an almost complex structure J that it is compatible with the metric g, i.e. an endomorphism $J : T\bar{M} \to T\bar{M}$ such that $J_P^2 = -Id$ for every $p \in \bar{M}$ and $g(JX, JY) = g(X, Y)$. A nearly Kaehler manifold is an almost Hermitian manifold with the extra condition that the (1, 2)-tensor field $G = \bar{\nabla} J$ is skew-symmetric:

$$(\bar{\nabla}_X J)Y + (\bar{\nabla}_Y J)X = 0,$$

for every $X, Y \in T\bar{M}$. Here $\bar{\nabla}$ stands for the Levi–Civita connection of the metric g. The tensor field G of type (2, 1) defined on $\bar{M}$ has the properties described as follows (see [2,3,9]):

$$G(X, Y) = -G(Y, X), \tag{2.1}$$

$$G(X, JY) = -JG(X, Y), \tag{2.2}$$

$$g(G(X, Y), Z) = -g(G(X, Z), Y), \tag{2.3}$$

$$g(G(X, Y), Z) = g(G(Y, Z), X) = g(G(Z, X), Y), \tag{2.4}$$

$$(\bar{\nabla} G)(X, Y, Z) = \frac{1}{3}(g(Y, JZ)X + g(X, Z)JY - g(X, Y)JZ), \tag{2.5}$$

$$\begin{aligned} G(G(X, Y), Z) = \frac{1}{3}(&g(X, Z)Y - g(Y, Z)X + g(X, JZ)JY \\ &- g(Y, JZ)JX), \end{aligned} \tag{2.6}$$

$$\begin{aligned} g(G(X, Y), G(Z, W)) = \frac{1}{3}(&g(X, Z)g(Y, W) - g(X, W)g(Y, Z) \\ &+ g(JX, Z)g(JW, Y) - g(JX, W)g(JZ, Y)), \end{aligned} \tag{2.7}$$

where X, Y, Z, W are arbitrary vector fields on $\mathbb{S}^3 \times \mathbb{S}^3$.

Let M be an orientable hypersurface of the nearly Kaehler manifold $\mathbb{S}^3 \times \mathbb{S}^3$. If ∇ denote the Riemannian connection induced on M, then the Gauss and Weingarten formulas are

$$\bar{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \bar{\nabla}_X N = -AX, \tag{2.8}$$

where $X, Y \in TM$ and $N \in T^{\perp}M$, and A is the shape operator of the hypersurface M. The second fundamental form σ and the shape operator A are related to each other by

$$g(\sigma(X, Y), N) = g(AX, Y). \tag{2.9}$$

Now let us review almost contact metric structures (see [6,7]). A $(2n-1)$-dimensional smooth manifold M is said to be an almost contact metric manifold if carries a global 1-form η, a vector field ξ, a (1, 1)-tensor field φ, and a Riemannian metric g satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \ \ \eta(\xi) = 1, \ \ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.10}$$

We know that $\varphi\xi = 0$, $\eta \circ \varphi = 0$, $g(X, \xi) = \eta(X)$, and ξ is a unit vector field. The almost contact metric structure (η, ξ, g) on M is called a contact metric structure if $d\eta(X, Y) = g(X, \varphi Y)$.

On the hypersurface M of the nearly Kaehler manifold $\mathbb{S}^3 \times \mathbb{S}^3$, we define three operators $\varphi_1, \varphi_2, \varphi_3$ by

$$\varphi_1 X = JX - \eta(X)N, \quad \varphi_2 X = \sqrt{3}G(X, N), \quad \varphi_3 X = \sqrt{3}G(X, \xi), \tag{2.11}$$

where $X \in TM$, $\xi = -JN$, N is the unit vector field normal to the hypersurface M and η is the smooth 1-form dual to ξ. Then, we can prove the following result.

Lemma 2.1. *Let M be an orientable hypersurface of the nearly Kaehler $\mathbb{S}^3 \times \mathbb{S}^3$, then* $(\varphi_i, \xi, \eta, g), i = 1, 2, 3$ *are almost contact metric structures on M and satisfies* $\varphi_1 \circ \varphi_2 = -\varphi_2 \circ \varphi_1 = \varphi_3, \varphi_2 \circ \varphi_3 = -\varphi_3 \circ \varphi_2 = \varphi_1,\ \varphi_3 \circ \varphi_1 = \varphi_1 \circ \varphi_3 = \varphi_2$.

Proof. It is trivial to see that $(\varphi_1, \xi, \eta, g)$ satisfies

$$\varphi_1^2 = -I + \eta \otimes \xi,\ \varphi_1(\xi) = 0,\ \eta \circ \varphi_1 = 0,\ \eta(\xi) = 1,$$
$$g(\varphi_1 X, \varphi_1 Y) = g(X, Y) - \eta(X)\eta(Y),\ X, Y \in TM,$$

that it is an almost contact metric structure on M. Now, note that $\sqrt{3}G(X, N) \in TM$ for $X \in TM$, which follows by (2.6), we have

$$\begin{aligned} \varphi_2^2 X &= \sqrt{3}\varphi_2 G(X, N) = 3G(G(X, N), N) \\ &= 3 \times \frac{1}{3}(-X + g(X, JN)JN) = -X + g(X, \xi)\xi \end{aligned}$$

and using (2.2), (2.3) and (2.7),

$$\begin{aligned} \varphi_2\xi &= \sqrt{3}G(\xi, N) = \sqrt{3}G(\xi, J\xi) = -\sqrt{3}JG(\xi, \xi) = 0, \\ \eta(\varphi_2 X) &= \sqrt{3}g(G(X, N), \xi) = -\sqrt{3}g(JX, G(\xi, \xi)) = 0, \\ g(\varphi_2 X, \varphi_2 X) &= 3g(G(X, N), G(Y, N)) = g(X, Y) - \eta(X)\eta(Y). \end{aligned}$$

Hence, $(\varphi_2, \xi, \eta, g)$ is also an almost contact metric structure on M. Similarly, it is easy to show that $(\varphi_3, \xi, \eta, g)$ is an almost contact metric structure on M

By using (2.1), (2.2) and (2.6), we can obtain

$$\begin{aligned} \varphi_1 \circ \varphi_2(X) &= \sqrt{3}\varphi_1 G(X, N) = \sqrt{3}JG(X, N) \\ &= \sqrt{3}G(X, \xi) = \varphi_3 X, \\ \varphi_2 \circ \varphi_3(X) &= \sqrt{3}\varphi_1 G(X, \xi) \\ &= 3G(G(X, \xi), N) = JX - \eta(X)N = \varphi_1 X. \end{aligned}$$

Similarly, we can prove the other equalities. □

In the following lemma, we find the covariant derivatives of the structure tensor fields of the almost contact metric structures on the hypersurface M. We shall denote by $G(X, Y)^T$, the tangential component of $G(X, Y)$ for $X, Y \in TM$.

Lemma 2.2. *Let* $(\varphi_i, \xi, \eta, g)$, $i = 1, 2, 3$ *be almost contact metric structures on the hypersurface* M *of the nearly Kaehler* $\bar{M}$. *Then the following hold for* $X, Y \in TM$:

$$(a)\ \nabla_X \xi = \varphi_1 AX - \frac{\sqrt{3}}{3}\varphi_2 X;$$

$$(b)\ (\nabla\varphi_1)(X, Y) = \eta(Y)AX - g(AX, Y)\xi + G(X, Y)^T;$$

$$(c)\ (\nabla\varphi_2)(X, Y) = \frac{\sqrt{3}}{3}(g(X, Y)\xi - \eta(Y)X) + \sqrt{3}G(AX, Y)^T;$$

$$(d)\ (\nabla\varphi_3)(X, Y) = \frac{\sqrt{3}}{3}(\eta(X)\varphi_1 Y - \eta(Y)\varphi_1 X - g(X, \varphi_1 Y)\xi + g(AX, \xi)\varphi_2 Y) - \sqrt{3}G(\varphi_1 AX, Y)^T.$$

Proof. Firstly, we see Lemma 2.2(a), by using $\xi = -JN$ and (2.8). We get

$$\begin{aligned}\bar{\nabla}_X \xi = \bar{\nabla}_X(-JN) &= -G(X, N) + JAX \\ &= -G(X, N) + \varphi_1 AX + \eta(AX)N.\end{aligned}$$

Taking the tangential parts, we get the first equation. Next, using (2.8) and (2.11),

$$\begin{aligned}G(X, Y) = \bar{\nabla}_X JY - J\bar{\nabla}_X Y &= \bar{\nabla}_X(\varphi_1 Y + \eta(Y)N) - J(\nabla_X Y + g(X, AY)N) \\ &= \bar{\nabla}_X \varphi_1 Y + X\eta(Y)N - \eta(Y)AX - \varphi_1(\nabla_X Y) - \eta(\nabla_X Y)N \\ &\quad + g(X, AY)\xi \\ &= \nabla_X \varphi_1 Y + g(X, A\varphi_1 Y)N + X\eta(Y)N - \eta(Y)AX - \varphi_1(\nabla_X Y) \\ &\quad - \eta(\nabla_X Y)N + g(X, AY)\xi.\end{aligned}$$

By comparing the tangential parts $(\nabla\varphi_1)(X, Y) = \eta(Y)AX - g(AX, Y)\xi + G(X, Y)^T$ holds. Similarly, by (2.5), we get

$$\begin{aligned}(\bar{\nabla}\varphi_2)(X, Y) &= \sqrt{3}\bar{\nabla}_X G(Y, N) - \sqrt{3}G(\bar{\nabla}_X Y, N) = \sqrt{3}(\bar{\nabla}_X G)(Y, N) \\ &\quad - \sqrt{3}G(\bar{\nabla}_X N, Y) \\ &= \frac{\sqrt{3}}{3}(g(X, Y)\xi - \eta(Y)X) + \sqrt{3}G(AX, Y).\end{aligned}$$

By comparing the tangential parts, $(\nabla\varphi_2)(X, Y) = \frac{\sqrt{3}}{3}(g(X, Y)\xi - \eta(Y)X) + \sqrt{3}G(AX, Y)^T$ holds.

Finally, by (2.5) and Lemma 2.2(a),

$$\begin{aligned}(\bar{\nabla}\varphi_3)(X, Y) &= \sqrt{3}\bar{\nabla}_X G(Y, \xi) - \sqrt{3}G(\bar{\nabla}_X Y, \xi) = \sqrt{3}(\bar{\nabla}_X G)(Y, \xi) \\ &\quad - \sqrt{3}G(\bar{\nabla}_X \xi, Y) \\ &= \frac{\sqrt{3}}{3}(\eta(X)(\varphi_1 Y + \eta(Y)N) - g(X, Y)N) - \sqrt{3}G(\varphi_1 AX, Y) \\ &\quad + G(\varphi_2 X, Y) + \frac{\sqrt{3}}{3}\eta(AX)\varphi_2 Y\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sqrt{3}}{3}\eta(X)\varphi_1 Y + \sqrt{3}G(G(X,N),Y) + \frac{\sqrt{3}}{3}g(AX,\xi)\varphi_2 Y \\
&\quad - \sqrt{3}G(\varphi_1 AX, Y) \\
&= \frac{\sqrt{3}}{3}(\eta(X)\varphi_1 Y - \eta(Y)\varphi_1 X - g(X,\varphi_1 Y)\xi + g(AX,\xi)\varphi_2 Y) \\
&\quad - \sqrt{3}G(\varphi_1 AX, Y).
\end{aligned}
$$

By comparing the tangential parts, Lemma 2.2(d) is proved. □

As we know, there exists a conformal vector field v on the sphere $\mathbb{S}^n(c)$ of constant curvature c in the Euclidean space $\mathbb{R}^{n+1}$, In fact, if we denote by ξ the tangential component of a nonzero constant vector field Z on $\mathbb{R}^{n+1}$ and by N the unit normal vector field on $\mathbb{S}^n(c)$, then we have

$$\nabla_X \xi = -\sqrt{c}\rho X, \quad \nabla\rho = \sqrt{c}\xi,$$

where ρ is the smooth function that is the normal component of Z and $\nabla\rho$ is the gradient of ρ. It follows that ξ is the conformal vector field on the sphere $\mathbb{S}^n(c)$ with potential function $f = -\sqrt{c}\rho$.

If we suppose v_i $(i = 1, 2)$ are conformal vector fields on Riemannian manifolds (M_i, g_i) $(i = 1, 2)$ with potential function ρ_i $(i = 1, 2)$ respectively, i.e.

$$\mathscr{L}_{v_i} g_i = \rho_i g_i, \quad i = 1, 2.$$

Then, we consider Riemannian product manifold $(M_1 \times M_2, g = g_1 + g_2)$ and let $v = (\rho_2 v_1, \rho_1 v_2) \in TM_1 \times TM_2$, i.e. $v_{(x,y)} = (\rho_2(y)(v_1)_x, \rho_1(x)(v_2)_y)$, for $(x, y) \in M_1 \times M_2$. It is easy to see that

$$\mathscr{L}_v g = \mathscr{L}_{\rho_2 v_1} g_1 + \mathscr{L}_{\rho_1 v_2} g_2 = \rho_2 \mathscr{L}_{v_1} g_1 + \rho_1 \mathscr{L}_{v_2} g_2 = \rho g,$$

where $\rho(x, y) = \rho_1(x)\rho_2(y)$. Therefore, $v = (\rho_2 v_1, \rho_1 v_2)$ is a conformal vector field on $(M_1 \times M_2, g = g_1 + g_2)$. In particular, there exists a conformal vector field on the nearly Kähler manifold $\mathbb{S}^3 \times \mathbb{S}^3$.

Now, we suppose v is a conformal vector field on $\mathbb{S}^3 \times \mathbb{S}^3$. We restrict it to the hypersurface M and express it as

$$\bar{\nabla}_X v = -\rho X, \quad X(\rho) = g(X, v), \quad v = w + sN, \tag{2.12}$$

where $w \in \chi(M)$ and $s = g(v, N)$. Then using (2.8) and (2.12), we have

$$\nabla_X w = -\rho X + sAX, \quad \nabla\rho = w, \quad \nabla s = -Aw, \tag{2.13}$$

where $\nabla\rho$, ∇s are the gradients of the functions ρ and s on the hypersurface M. Also, it follows that

$$\|w\|^2 = 1 - \rho^2 - s^2. \tag{2.14}$$

We use the vector field $w \in \chi(M)$ to define $w_i = \varphi_i w$, and they satisfy

$$\varphi_1 w_1 = -w + h\xi,\ \varphi_1 w_2 = w_3,\ \varphi_1 w_3 = -w_2,$$
$$\varphi_2 w_1 = -w_3,\ \varphi_2 w_2 = -w + h\xi,\ \varphi_2 w_3 = w_1,$$
$$\varphi_3 w_1 = w_2,\ \varphi_3 w_2 = -w_1,\ \varphi_3 w_3 = -w + h\xi,$$

where $h = g(w, \xi)$. It is easy to see that w, w_1, w_2, w_3 are mutually orthogonal and that

$$\|w_i\|^2 = \|w\|^2 - h^2, \quad i = 1, 2, 3. \tag{2.15}$$

Next, we will prove the following Lemma.

Lemma 2.3. *The covariant derivatives of the vector fields w_i are given by*

(*a*) $\nabla_X w_1 = hAX - g(AX, w)\xi + G(X, w)^T - \rho\varphi_1 X + s\varphi_1 AX;$

(*b*) $\nabla_X w_2 = \frac{\sqrt{3}}{3}(g(X, w)\xi - hX) + \sqrt{3}G(AX, w)^T - \rho\varphi_2 X + s\varphi_2 AX;$

(*c*) $\nabla_X w_3 = \frac{\sqrt{3}}{3}(\eta(X)w_1 - h\varphi_1 X - g(X, w_1)\xi) + g(AX, \xi)w_2$
$\qquad - \sqrt{3}G(\varphi_1 AX, W)^T - \rho\varphi_3 X + s\varphi_3 AX.$

Proof. Using (2.8) and Lemma 2.2, we can get,

$$\begin{aligned}
\bar{\nabla}_X w_1 &= \bar{\nabla}_X(Jw - \eta(w)N) = (\bar{\nabla}_X J)w + J\bar{\nabla}_X w - X\eta(w)N + \eta(w)AX \\
&= G(X, w) + J(\nabla_X w + g(AX, w)N) - X\eta(w)N + \eta(w)AX \\
&= G(X, w) + \varphi_1\nabla_X w + \eta(\nabla_X w)N - g(AX, w)\xi - X\eta(w)N \\
&\quad + \eta(w)AX \\
&= hAX - g(AX, w)\xi + G(X, w) - \rho\varphi_1 X + s\varphi_1 AX.
\end{aligned}$$

The above equation explains that Lemma 2.3(a) holds. Besides, (2.5) and Lemma 2.2 indicate Lemma 2.3(b) as follows:

$$\begin{aligned}
\bar{\nabla}_X w_2 &= \sqrt{3}(\bar{\nabla}_X G)(w, N) + \sqrt{3}G(\bar{\nabla}_X w, N) + \sqrt{3}G(w, \bar{\nabla}_X N) \\
&= \frac{\sqrt{3}}{3}(g(w, JN)X + g(X, N)Jw - g(X, w)JN) \\
&\quad + \sqrt{3}G(\nabla_X w + g(AX, w)N, N) - \sqrt{3}G(w, AX) \\
&= \frac{\sqrt{3}}{3}(-hX + g(X, w)\xi) + \sqrt{3}G(AX, w) + \varphi_2\nabla_X w \\
&= \frac{\sqrt{3}}{3}(g(X, w)\xi - hX) + \sqrt{3}G(AX, w) - \rho\varphi_2 X + s\varphi_2 AX.
\end{aligned}$$

Similarly, we can prove Lemma 2.3(c). The lemma is proved. □

3. Contact hypersurface in nearly Kaehler manifold $\mathbb{S}^3 \times \mathbb{S}^3$

Let A be the shape operator of hypersurface M in nearly Kaehler manifold $\mathbb{S}^3 \times \mathbb{S}^3$ defined by $AX = -\bar{\nabla}_X N$, where $\bar{\nabla}$ denotes the Levi–Civita covariant derivative on $\mathbb{S}^3 \times \mathbb{S}^3$.

Denote by Φ the fundamental 2-form on M given by $\Phi(X, Y) = g(X, \varphi_1 Y)$. First of all, we shall prove the following result.

PROPOSITION 3.1

The fundamental 2-form Φ *on a hypersurface in a nearly Kaehler* $\mathbb{S}^3 \times \mathbb{S}^3$ *satisfies* $d\Phi(X, Y, Z) = -3g(G(X, Y), Z)$.

Proof. We have from Lemma 2.2,

$$\begin{aligned}(\nabla_X \varphi_1)Y = G(X, Y) - g(X, A\varphi_1 Y)N - X\eta(Y)N \\ + \eta(\nabla_X Y)N + \eta(Y)AX - g(AY, X)\xi.\end{aligned} \tag{3.1}$$

By definition, we have

$$\begin{aligned}d\Phi(X, Y, Z) &= d(\Phi(Y, Z))(X) + d(\Phi(Z, X))(Y) + d(\Phi(X, Y))(Z) \\ &\quad -\Phi([X, Y], Z) - \Phi([Y, Z], X) - \Phi([Z, X], Y) \\ &= Xg(Y, \varphi_1 Z) + Yg(Z, \varphi_1 X) + Zg(X, \varphi_1 Y) \\ &\quad -g(\nabla_X Y - \nabla_Y X, \varphi_1 Z) \\ &\quad -g(\nabla_Y Z - \nabla_Z Y, \varphi_1 X) - g(\nabla_Z X - \nabla_X Z \varphi_1 Y) \\ &= g(Y, \nabla_X \varphi_1 Z) - g(Y, \varphi_1 \nabla_X Z) + g(Z, \nabla_Y \varphi_1 X) \\ &\quad -g(Z, \varphi_1 \nabla_Y X) + g(X, \nabla_Z \varphi_1 Y) - g(X, \varphi_1 \nabla_Z Y) \\ &= g(\nabla_Z \varphi_1 Y - \varphi_1 \nabla_Z Y, X) + g(\nabla_X \varphi_1 Z - \varphi_1 \nabla_X Z, Y) \\ &\quad +g(\nabla_Y \varphi_1 X - \varphi_1 \nabla_Y X, Z) \\ &= g((\nabla_Z \varphi_1)Y, X) + g((\nabla_X \varphi_1)Z, Y) + g((\nabla_Y \varphi_1)X, Z).\end{aligned}$$

Inserting the expression for $\nabla\varphi_1$ as in (3.1) into the previous equation gives

$$d\Phi(X, Y, Z) = -3g(G(X, Y), Z).$$

The proposition is proved. □

We introduced that the contact hypersurface satisfies $d\eta(X, Y) = \Phi(X, Y)$ for all tangent vector fields X, Y on M. Next, we state and prove the following proposition.

PROPOSITION 3.2

Let M *be a connected orientable real hypersurface of a nearly Kaehler* $\mathbb{S}^3 \times \mathbb{S}^3$. *Then* M *is a contact hypersurface if and only if the shape operator* A *and* φ_1, φ_2 *satisfy*

$$A\varphi_1 + \varphi_1 A = \frac{2\sqrt{3}}{3}\varphi_2 - \varphi_1.$$

Proof. Using the definition for the exterior derivative, we obtain

$$\begin{aligned} d\eta(X,Y) &= d(\eta(Y))(X) - d(\eta(X))(Y) - \eta([X,Y]) \\ &= Xg(Y,\xi) - Yg(X,\xi) - g(\xi,[X,Y]) \\ &= g(\nabla_X\xi, Y) + g(\xi, \nabla_X Y) - g(\nabla_Y X, \xi) \\ &\quad - g(X, \nabla_Y \xi) - g(\xi, [X,Y]) \\ &= g(\nabla_X\xi, Y) - g(X, \nabla_Y\xi). \end{aligned}$$

Taking Lemma 2.2(a) into the above equation, we get $d\eta(X,Y) = g(X, (-A\varphi_1 - \varphi_1 A + \frac{2\sqrt{3}}{3}\varphi_2)Y)$. Comparing it with $d\eta(X,Y) = \Phi(X,Y)$, we have $A\varphi_1 + \varphi_1 A = \frac{2\sqrt{3}}{3}\varphi_2 - \varphi_1$. □

The almost contact metric structure is called nearly cosymplectic, if

$$\begin{aligned} &(\nabla_X\varphi_1)Y + (\nabla_Y\varphi_1)X = 0, \\ &(\nabla_X\eta)Y + (\nabla_Y\eta)X = 0, \;\; X, Y \in TM \end{aligned} \tag{3.2}$$

and a cosymplectic manifold is a normal almost contact metric manifold such that η and Φ are both closed. This definition is equivalent to $\nabla\varphi_1 = 0$ on an almost contact metric manifold (see [2]).

COROLLARY 3.1

The contact hypersurface in nearly Kaehler $\mathbb{S}^3 \times \mathbb{S}^3$ is nearly cosymplectic if and only if $\varphi_1 A = A\varphi_1$.

Proof. From (3.1), we see that

$$\begin{aligned} (\nabla_Y\varphi_1)X &= G(Y,X) - g(Y, A\varphi_1 X)N - Y\eta(X)N - \eta(\nabla_Y X) \\ &\quad + \eta(X)AY - g(AX,Y)\xi. \end{aligned} \tag{3.3}$$

Combining with (3.2) and taking normal component, we have

$$g((\varphi_1 A - A\varphi_1)X, Y)N - ((\nabla_X\eta)Y + (\nabla_Y\eta)X)N = 0. \tag{3.4}$$

Because of the arbitrariness of X and Y, we have $\varphi_1 A = A\varphi_1$. □

COROLLARY 3.2

The contact hypersurface in nearly Kaehler $\mathbb{S}^3 \times \mathbb{S}^3$ is cosymplectic if and only if $\varphi_1 A = A\varphi_1 = 0$.

Proof. Using Proposition 3.2, it is straightforward to see that

$$-A\varphi_1 - \varphi_1 A + \frac{2\sqrt{3}}{3}\varphi_2 = 0. \tag{3.5}$$

Combining with Corollary 3.1 and Proposition 3.1, $\varphi_1 A = A\varphi_1 = 0$ holds. □

4. Minimal hypersurface in nearly Kaehler manifold $\mathbb{S}^3 \times \mathbb{S}^3$

In this section, we explore the influence of the conformal vector field on the geometry of the hypersurface M with different restrictions on v. It is interesting to note that the conformal vector field v not only dictates the geometry of the hypersurface, but also influences the eigenvalues of the Laplacian operator on the hypersurface. First, we prove the following.

Lemma 4.1. *Let M be a compact orientable hypersurface of the nearly Kaehler manifold $\mathbb{S}^3 \times \mathbb{S}^3$. Then the smooth function $h = g(w, \xi)$ on M satisfies*

$$\int_M h = 0.$$

Proof. Choose a local orthonormal frame $\{e_1, \ldots, e_5\}$ on M that diagonalizes A with $Ae_i = \lambda_i e_i$. Then

$$\sum_i g(G(Ae_i, w)^T, e_i) = \sum_i -\lambda_i g(G(w, e_i), e_i) = \sum_i \lambda_i g(w, G(e_i, e_i)) = 0,$$
$$\sum_i g(\varphi_2 Ae_i, e_i) = \sum_i \lambda_i g(\varphi_2 e_i, e_i) = 0.$$

Using Lemma 2.3(b) and the above equations, we compute the divergence of the vector field w_2 to get

$$\operatorname{div} w_2 = -\frac{4\sqrt{3}}{3} h,$$

which on integration proves the lemma. □

Lemma 4.2. *Let M be a compact orientable hypersurface of the nearly Kaehler manifold $\mathbb{S}^3 \times \mathbb{S}^3$. Then the unit vector $\xi \in TM$ defined by $\xi = -JN$ satisfies*

$$\operatorname{div} \xi = 0. \tag{4.1}$$

Proof. Using Lemma 2.2(a), it is easy to see that

$$\operatorname{div} \xi = \sum_i g(\nabla_{e_i} \xi, e_i) = 0 \tag{4.2}$$

which proves the lemma. □

Theorem 4.1. *Let M be a compact minimal hypersurface of constant scalar curvature in $\mathbb{S}^3 \times \mathbb{S}^3$. Then there exists an eigenvalue of the Laplace operator on M satisfying $\|A\|^2 = \lambda - \frac{7}{3}$.*

Proof. By calculating, we find that the gradient ∇h of the smooth function $h = g(w, \xi)$ is

$$\nabla h = \frac{\sqrt{3}}{3} w_2 - \rho \xi - A w_1 + s A \xi. \tag{4.3}$$

Choosing a local orthonormal frame $\{e_1, \dots, e_5\}$ on M and using Lemma 4.1, we compute

$$\begin{aligned}
\operatorname{div} w_2 &= -\frac{4\sqrt{3}}{3}h,\\
\operatorname{div}(Aw_1) &= \sum g(\nabla_{e_i} Aw_1, e_i) = \sum g(\nabla_{e_i} w_1, Ae_i)\\
&= \sum g(hAe_i - g(Ae_i, w)\xi + G(e_i, w) - \rho\varphi_1 e_i + s\varphi_1 Ae_i, Ae_i)\\
&= h\ \|A\|^2 - g(Aw, A\xi),
\end{aligned}$$

where we used skew symmetry of φ_1, $\sum_i g(\varphi_1 e_i, Ae_i) = 0$ and $\sum_i g(G(e_i, w), Ae_i) = 0$ and Lemma 2.3(a). Similarly we compute

$$\begin{aligned}
\operatorname{div}(\rho\xi) &= \sum g(\nabla_{e_i} \rho\xi, e_i) = \sum g(\nabla_{e_i}\rho)\xi + \rho\nabla_{e_i}\xi, e_i) = h,\\
\operatorname{div}(sA\xi) &= \sum g((\nabla_{e_i} s)A\xi + s\nabla_{e_i} A\xi, e_i)\\
&= \sum g((\nabla_{e_i} s)A\xi, e_i) + s\sum g(\nabla_{e_i}\xi, Ae_i) = -g(Aw, A\xi).
\end{aligned}$$

Using the above equationss, by computing the Laplacian Δh, we arrive at

$$\Delta h = -\left(\frac{7}{3} + \|A\|^2\right)h. \tag{4.4}$$

Equation (4.3) implies that h is an eigenfunction of the Laplace operator Δ corresponding to the eigenvalue $\lambda = \frac{7}{3} + \|A\|^2$ and this proves that $\|A\|^2 = \lambda - \frac{7}{3}$. □

COROLLARY 4.1

Let M be an orientable compact minimal hypersurface of the nearly Kaehler manifold $\mathbb{S}^3 \times \mathbb{S}^3$ tangential to the conformal vector field v and the smooth function $h \neq 0$. Then the first nonzero eigenvalue λ_1 of the Laplacian operator Δ of M, satisfies $\lambda_1 \leq \frac{7}{3}$

Proof. Since the conformal vector field v is tangent to the hypersurface M, we have $s = 0$ and consequently, using (2.13), we get $Aw = 0$. By calculating, we find the gradient ∇h of the smooth function $h = g(w, \xi)$ as

$$\nabla h = \frac{\sqrt{3}}{3}w_2 - \rho\xi - Aw_1. \tag{4.5}$$

By using Theorem 4.1, we get

$$\Delta h = -\frac{7}{3}h. \tag{4.6}$$

If the function h is a constant, then Lemma 4.1 implies that $h = 0$ which is contradictory to the hypothesis. Hence, the function h is not a constant, and therefore by (4.6), we get that the first nonzero eigenvalue λ_1 of the Laplacian operator Δ satisfies $\lambda_1 \leq \frac{7}{3}$. □

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