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Homological algebra in n-abelian categories

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Abstract. In this paper, we study the homological theory in n-abelian categories. First, we prove some useful properties of n-abelian categories, such as $(n+2)\times(n+2)$-lemma, 5-lemma and n-Horseshoes lemma. Secondly, we introduce the notions of right(left) n-derived functors of left(right) n-exact functors, n-(co)resolutions, and n-homological dimensions of n-abelian categories. For an n-exact sequence, we show that the long n-exact sequence theorem holds as a generalization of the classical long exact sequence theorem. As a generalization of $\mathsf{Ext}^*(-,-)$, we study the n-derived functor $\mathsf{nExt}^*(-,-)$ of hom-functor $\mathrm{Hom}(-,-)$. We give an isomorphism between the abelian group of equivalent classes of m-fold n-extensions $\mathsf{nE}^m(A,B)$ of A, B and $\mathsf{nExt}^m_{\mathcal{A}}(A,B)$ using n-Baer sum for $m,n\geq 1$.

Keywords. n-Abelian category; n-derived functor; nExt^*-correspondence; n-Baer sum; n-cluster tilting.

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1. Introduction

Recently, a new class of categories called 2-cluster tilting subcategories that appeared in representation theory were introduced by Buan *et al.* [4], and the class of n-cluster tilting subcategories was developed by Iyama and Yoshino [12]. And then, from the viewpoint of higher Auslander-Reiten theory, Iyama [8–10] investigated and introduced the notion of n-almost-split sequences which are n-exact sequences in the sense of Jasso [13]. He developed the classical abelian category and exact category theory to higher-dimensional n-abelian category and n-exact category theory [13]. He also proved that n-cluster tilting subcategories are n-abelian categories. These new discoveries have broken new ground in category theory.

Homological algebra, as a connected system of notions and results, was first developed for categories of modules by Cartan and Eilenberg [6] and was immediately generalized by Buchsbaum [5], Mac Lane [14] and Heller [7] to exact categories and abelian categories. Homological algebra can also construct on various nonabelian categories, such as pre-abelian category [20], all of their derived functors are defined on right(left) exact functors of certain short exact sequences via (co)homology of (co)resolutions under the right(left) exact functors.

In this paper, we study the homological theory of n-abelian categories as a generalization of homological theory of abelian categories compared to higher homological theory of abelian categories via higher (co)homology of n-(co)resolutions under right(left) n-exact functors for short n-exact sequences.

This paper is organized as follows. In § 2, we recall some notions and notations of n-abelian categories, and study some properties. In section 3, we study the relationship between n-exact sequences and exact sequences, and introduce the $(n+2) \times (n+2)$-lemma and 5-lemma of n-abelian categories as generalizations of classical 3×3-lemma and 5-lemma of abelian categories. In § 4, we introduce the notions of right(left) n-derived functors of left(right) n-exact functors, n-(co)resolution, and n-homological dimensions, specially, we introduce the functor $\mathsf{nExt}^*(-,-)$ as a generalization of $\mathsf{Ext}^*(-,-)$. We study some basic properties of n-derived functors, and prove the long n-exact sequence theorem and n-Horseshoes lemma. In § 5, we study the n-extension and m-fold n-extension groups, and we prove that there is an isomorphism $\mathsf{nE}^m(A,B) \simeq \mathsf{nExt}^m_{\mathcal{A}}(A,B)$ of the group of equivalence classes of m-fold n-extension group $\mathsf{nE}^m(A,B)$ (this is an abelian group under n-Baer sum) and $\mathsf{nExt}^m_{\mathcal{A}}(A,B)$ in n-abelian categories. This proves that we can define $\mathsf{nExt}^m_{\mathcal{A}}(A,B)$ without mentioning projectives or injectives.

2. *n*-Abelian categories

Let n be a positive integer and $\mathcal{C}$ an additive category. We denote the category of cochain complexes of $\mathcal{C}$ by $\mathsf{Ch}(\mathcal{C})$ and the homotopy category of $\mathcal{C}$ by $\mathsf{H}(\mathcal{C})$. Also, we denote by $\mathsf{Ch}^n(\mathcal{C})$ the full subcategory of $\mathsf{Ch}(\mathcal{C})$ given by all complexes

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}$$

which are concentrated in degrees $0, 1, \ldots, n+1$. We write $\mathcal{C}(X,Y)$ for the morphisms in $\mathcal{C}$ from X to Y, if $X, Y \in \mathsf{ob}\mathcal{C}$.

2.1 *n-Kernels, n-cokernels and n-exact sequences*

Let $\mathcal{C}$ be an additive category and $d^0 : X^0 \to X^1$ a morphism in $\mathcal{C}$. An *n-cokernel of* d^0 is a sequence of morphisms

$$(d^1, \ldots, d^n) : X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} X^3 \to \cdots \xrightarrow{d^n} X^{n+1}$$

such that for all $Y \in \mathcal{C}$ the induced sequence of abelian groups

$$0 \to \mathcal{C}(X^{n+1}, Y) \to \mathcal{C}(X^n, Y) \to \cdots \to \mathcal{C}(X^1, Y) \to \mathcal{C}(X^0, Y)$$

is exact. Equivalently, the sequence $(d^1, \ldots, d^n)$ is an n-cokernel of d^0 if for all $1 \leq k \leq n-1$ the morphism d^k is a weak cokernel of d^{k-1}, and d^n is moreover a cokernel of d^{n-1}. The concept of *n-kernel* of a morphism is defined dually. If $n \geq 2$, the n-cokernels and n-kernels are not unique in general, but they are unique up to isomorphism in $\mathsf{H}(\mathcal{C})$ [13].

An *n-exact sequence in* $\mathcal{C}$ is an n-kernel-n-cokernel pair, i.e., a complex $X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}$ in $\mathsf{Ch}^n(\mathcal{C})$ such that $(d^0, \ldots, d^{n-1})$ is an n-kernel of d^n, and $(d^1, \ldots, d^n)$ is an n-cokernel of d^0.

We recall the Comparison lemma, together with its dual, plays a central role in the sequel.

Lemma 2.1 [13, *Comparison lemma* 2.1]. *Let $\mathcal{C}$ be an additive category and $X \in \mathsf{Ch}^{\geq 0}(\mathcal{C})$ a complex such that for all $k \geq 0$ the morphism d_X^{k+1} is a weak cokernel of d_X^k. If $f: X \to Y$ and $g: X \to Y$ are morphisms in $\mathsf{Ch}^{\geq 0}(\mathcal{C})$ such that $f^0 = g^0$, then there exists a homotopy $h: f \to g$ such that $h^1 = 0$.*

Lemma 2.2. *In additive category, any n-exact sequence, whose number of nonzero terms less than $n+2$, are contractible.*

Proof. For any n-exact sequence X, the number of nonzero terms is less than $n+2$. We can split X into direct sum of three classes of n-exact sequences as follows:

(a) $Y^0 \to Y^1 \to \cdots \to Y^i \to 0 \to \cdots \to 0$ for some $i \leq n$, $Y_s \neq 0$ for any s.
(b) $0 \to \cdots \to 0 \to Y^i \to \cdots \to Y^j \to 0 \to \cdots \to 0$ for some $1 \leq i < j \leq n$, $Y_s \neq 0$ for any s.
(c) $0 \to \cdots \to 0 \to Y^i \to \cdots \to Y^{n+1}$ for some $i \geq 1$, $Y_s \neq 0$ for any s.

Since $M \to 0$ and $0 \to M$ are split epimorphism and split monomorphism respectively, by [13, Proposition 2.6], (a), (b) and (c) are contractible n-exact sequences so is X. □

2.2 *n-Pushout, n-pullback and n-bicartesian diagrams*

Let $f : X \to Y$ be a morphism of complexes in $\mathsf{Ch}^{n-1}(\mathcal{C})$

$$\begin{array}{ccccccccccc} X & & X^0 & \longrightarrow & X^1 & \longrightarrow & \cdots & \longrightarrow & X^{n-1} & \longrightarrow & X^n \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} \\ Y & & Y^0 & \longrightarrow & Y^1 & \longrightarrow & \cdots & \longrightarrow & Y^{n-1} & \longrightarrow & Y^n \end{array} .$$

The *mapping cone* $C = C(f) \in \mathsf{Ch}^n(\mathcal{C})$ is

$$X^0 \overset{d_C^{-1}}{\to} X^1 \oplus Y^0 \overset{d_C^0}{\to} \cdots \overset{d_C^{n-2}}{\to} X^n \oplus Y^{n-1} \overset{d_C^{n-1}}{\to} Y^n,$$

where

$$d_C^k := \begin{pmatrix} -d_X^{k+1} & 0 \\ f^{k+1} & d_Y^k \end{pmatrix} : X^{k+1} \oplus Y^k \to X^{k+2} \oplus Y^{k+1}$$

for each $k \in \{-1, 0, \ldots, n-1\}$. In particular, $d_C^{-1} = \begin{pmatrix} -d_X^0 \\ f^0 \end{pmatrix}$ and $d_C^{n-1} = (f^n \; d_Y^{n-1})$.

(1) The diagram $f : X \to Y$ is called an *n-pullback diagram of Y along f^n* if the sequence $(d_C^{-1}, \ldots, d_C^{n-2})$ is an n-kernel of d_C^{n-1};
(2) The diagram $f : X \to Y$ is called an *n-pushout diagram of X along f^0* if the sequence $(d_C^0, \ldots, d_C^{n-1})$ is an n-cokernel of d_C^{-1};
(3) The diagram $f : X \to Y$ is called an *n-bicartesian* (*or, n-exact diagram*) if the sequence $C(f) = (d_C^{-1}, d_C^0, \ldots, d_C^{n-1})$ is an n-exact sequence.

Lemma 2.3 [13, *Proposition* 2.13]. *Let $\mathcal{C}$ be an additive category, $g\colon X \to Z$ a morphism of complexes in $\mathsf{Ch}^{n-1}(\mathcal{C})$ and suppose there exists an n-pushout diagram of X along g^0,*

$$\begin{array}{ccccccccccc} X & & X^0 & \longrightarrow & X^1 & \longrightarrow & \cdots & \longrightarrow & X^{n-1} & \longrightarrow & X^n \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g^0} & & \downarrow & & & & \downarrow & & \downarrow \\ Y & & Y^0 = Z^0 & \longrightarrow & Y^1 & \longrightarrow & \cdots & \longrightarrow & Y^{n-1} & \longrightarrow & Y^n \end{array}.$$

Then, there exists a morphism of complexes $p\colon Y \to Z$ such that $p^0 = 1_{Z^0}$ and a homotopy $h\colon fp \to g$ with $h^1 = 0$. Moreover, these properties determine p uniquely up to homotopy.

Lemma 2.3 shows that n-pushout is unique up to homotopy equivalence. If $h = 0$ in Lemma 2.3, we say that the morphism $f : X \to Y$ is a *good n-pushout diagram of X along f^0*. Dually, we can define the good n-pullback diagram. Definition-Proposition 2.14 of [13] (resp., its dual) has proved that if there exists n-pushout (resp., n-pullback), then there exists a good n-pushout (resp., good n-pullback).

We consider a commutative diagram in an abelian category

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & \text{(I)} & \downarrow & \text{(II)} & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}.$$

There are two well-known pullback lemmas [14]:

(a) If squares (I) and (II) are pullbacks, then (I + II) is a pullback (i.e., diagrams compose).
(b) if (I + II) and (II) are pullbacks, then (I) is a pullback.

The third possibility: ‘if (I + II) and (I) are pullbacks, then (II) is a pullback’. Generally, this does not hold (see [17]).

We now show that (a) and (b) hold for any positive integer n.

PROPOSITION 2.4

Let $\mathcal{C}$ be an additive category, $Z \xrightarrow{g} Y$ and $Y \xrightarrow{f} X$ are morphisms in $\mathsf{Ch}^{n-1}(\mathcal{C})$. Then we have the following statements:

(*i*) *If $Y \xrightarrow{f} X$ is an n-pullback diagram of X along f^n and $Z \xrightarrow{g} Y$ is an n-pullback diagram of Y along g^n, then $Z \xrightarrow{fg} X$ is an n-pullback diagram of X along $f^n g^n$.*
(*ii*) *If $Y \xrightarrow{f} X$ is an n-pullback diagram of X along f^n and $Z \xrightarrow{fg} X$ is an n-pullback diagram of X along $f^n g^n$, then $Z \xrightarrow{g} Y$ is an n-pullback diagram of Y along g^n.*

Proof. We have a commutative diagram

$$\begin{array}{ccccccccccccc} C(g) & & Z^0 & \xrightarrow{d^{-1}_{C(g)}} & Z^1 \oplus Y^0 & \xrightarrow{d^{0}_{C(g)}} & Z^2 \oplus Y^1 & \xrightarrow{d^{1}_{C(g)}} & \cdots & \longrightarrow & Z^n \oplus Y^{n-1} & \xrightarrow{d^{n-1}_{C(g)}} & Y^n \\ \downarrow{\scriptstyle \varphi} & & \|{\scriptstyle \varphi^{-1}=1_{Z^0}} & & \downarrow{\scriptstyle \varphi^0} & & \downarrow{\scriptstyle \varphi^1} & & & & \downarrow{\scriptstyle \varphi^{n-1}} & & \downarrow{\scriptstyle \varphi^n = f^n} \\ C(fg) & & Z^0 & \xrightarrow{d^{-1}_{C(fg)}} & Z^1 \oplus X^0 & \xrightarrow{d^{0}_{C(fg)}} & Z^2 \oplus X^1 & \xrightarrow{d^{1}_{C(fg)}} & \cdots & \longrightarrow & Z^n \oplus X^{n-1} & \xrightarrow{d^{n-1}_{C(fg)}} & X^n \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi^{-1}=g^0} & & \downarrow{\scriptstyle \psi^0} & & \downarrow{\scriptstyle \psi^1} & & & & \downarrow{\scriptstyle \psi^{n-1}} & & \|{\scriptstyle \psi^n = 1_{X^n}} \\ C(f) & & Y^0 & \xrightarrow{d^{-1}_{C(f)}} & Y^1 \oplus X^0 & \xrightarrow{d^{0}_{C(f)}} & Y^2 \oplus X^1 & \xrightarrow{d^{1}_{C(f)}} & \cdots & \longrightarrow & Y^n \oplus X^{n-1} & \xrightarrow{d^{n-1}_{C(f)}} & X^n \end{array}, \tag{1}$$

where

$$\varphi^k = \begin{pmatrix} 1_{Z^{k+1}} & 0 \\ 0 & f^k \end{pmatrix}, \quad \psi^k = \begin{pmatrix} g^{k+1} & 0 \\ 0 & 1_{X^k} \end{pmatrix}$$

for $k \in \{1, 2, \ldots, n-1\}$.

(i) $Y \xrightarrow{f} X$ is an n-pullback diagram of X along f^n and $Z \xrightarrow{g} Y$ is an n-pullback diagram of Y along g^n, then $(d_{C(f)}^{-1}, \ldots, d_{C(f)}^{n-2})$ and $(d_{C(g)}^{-1}, \ldots, d_{C(g)}^{n-2})$ are n-kernels of $d_{C(f)}^{n-1}$ and $d_{C(g)}^{n-1}$ respectively, $d_{C(f)}^{-1}$ and $d_{C(g)}^{-1}$ are monomorphisms. Hence $d_{C(fg)}^{-1}$ is a monomorphism. Indeed, let $u : M \to Z^0$ be a morphism such that $d_{C(fg)}^{-1}u = 0$. Then $d_Z^0 u = 0$. But $0 = \psi^0 d_{C(fg)}^{-1} u = d_{C(f)}^{-1} g^0 u$, this implies $g^0 u = 0$ since $d_{C(f)}^{-1}$ is a monomorphism. Thus $d_{C(g)}^{-1} u = 0$ so is u since $d_{C(g)}^{-1}$ is a monomorphism. So, $d_{C(fg)}^{-1}$ is a monomorphism. Thus $d_{C(fg)}^{i-1}$ is a weak kernel of $d_{C(fg)}^{i}$ for $i = 0, \ldots, n-1$ (we consider X^{-1}, Y^{-1}, Y^{-1}, X^{n+1}, Y^{n+1}, Y^{n+1} as 0 objects). Indeed, let $\begin{pmatrix} u^{i+1} \\ v^i \end{pmatrix} : M \to Z^{i+1} \oplus X^i$ be a morphism such that $d_{C(fg)}^{i} \begin{pmatrix} u^{i+1} \\ v^i \end{pmatrix} = 0$, hence we have $d_{C(f)}^{i} \psi^i \begin{pmatrix} u^{i+1} \\ v^i \end{pmatrix} = 0$. Then $d_Z^{i+1} u^{i+1} = 0$, $f^{i+1} g^{i+1} u^{i+1} + d_X^i v^i = 0$, and there exists a morphism $\begin{pmatrix} w^i \\ t^{i-1} \end{pmatrix} : M \to Y^i \oplus X^{i-1}$ such that $d_{C(f)}^{i-1} \begin{pmatrix} w^i \\ t^{i-1} \end{pmatrix} = \psi_i \begin{pmatrix} u^{i+1} \\ v^i \end{pmatrix}$ since $d_{C(f)}^{i-1}$ is a weak kernel of $d_{C(f)}^{i}$. Then

$$g^{i+1} u^{i+1} + d_Y^i w^i = 0, \quad f^i w^i + d_X^{i-1} t^{i-1} = v^i.$$

Then, we have

$$d_{C(g)}^{i} \begin{pmatrix} u^{i+1} \\ w^i \end{pmatrix} = \begin{pmatrix} -d_Z^{i+1} & 0 \\ g^{i+1} & d_Y^i \end{pmatrix} \begin{pmatrix} u^{i+1} \\ w^i \end{pmatrix} = \begin{pmatrix} -d_Z^{i+1} u^{i+1} \\ g^{i+1} u^{i+1} + d_Y^i w^i \end{pmatrix} = 0.$$

Therefore, since $d_{C(g)}^{i-1}$ is a weak kernel of $d_{C(g)}^{i}$, there exists a morhpism $\begin{pmatrix} s^i \\ h^{i-1} \end{pmatrix} : M \to Z^i \oplus Y^{i-1}$ such that $d_{C(g)}^{i-1} \begin{pmatrix} s^i \\ h^{i-1} \end{pmatrix} = \begin{pmatrix} u^{i+1} \\ w^i \end{pmatrix}$. We have that

$$u^{i+1} + d_Z^i s^i = 0, \quad g^i s^i + d_Y^{i-1} h^{i-1} = w^i.$$

Set $\begin{pmatrix} s^i \\ f^{i-1} h^{i-1} + t^{i-1} \end{pmatrix} : M \to Z^i \oplus X^{i-1}$, we have

$$\begin{aligned} d_{C(fg)}^{i-1} \begin{pmatrix} s^i \\ f^{i-1} h^{i-1} + t^{i-1} \end{pmatrix} &= \begin{pmatrix} -d_Z^i & 0 \\ f^i g^i & d_X^{i-1} \end{pmatrix} \begin{pmatrix} s^i \\ f^{i-1} h^{i-1} + t^{i-1} \end{pmatrix} \\ &= \begin{pmatrix} u^{i+1} \\ v^i \end{pmatrix}. \end{aligned}$$

This proves that $d^{i-1}_{C(fg)}$ is a weak kernel of $d^{i}_{C(fg)}$ for $i=0,\dots,n-1$, thus $Z \overset{fg}{\to} X$ is an n-pullback diagram of Y along $f^n g^n$.

(ii) Because $Y \overset{f}{\to} X$ is an n-pullback diagram of X along f^n and $Z \overset{fg}{\to} X$ is an n-pullback diagram of Z along $f^n g^n$, $(d^{-1}_{C(f)},\dots,d^{n-2}_{C(f)})$ and $(d^{-1}_{C(fg)},\dots,d^{n-2}_{C(fg)})$ are n-kernels of $d^{n-1}_{C(f)}$ and $d^{n-1}_{C(fg)}$ respectively, $d^{-1}_{C(f)}$ and $d^{-1}_{C(fg)}$ are monomorphisms, so is $d^{-1}_{C(g)}$. Hence $d^{i-1}_{C(g)}$ is a weak kernel of $d^{i}_{C(g)}$ for $i=0,\dots,n-1$. Indeed, let $\begin{pmatrix} u^{i+1} \\ v^i \end{pmatrix} : M \to Z^{i+1} \oplus Y^i$ be a morphism such that $d^i_{C(g)} \begin{pmatrix} u^{i+1} \\ v^i \end{pmatrix} = 0$, we have

$$d^{i+1}_Z u^{i+1} = 0, \quad g^{i+1}u^{i+1} + d^i_Y v^i = 0, \quad d^i_{C(fg)}\varphi^i \begin{pmatrix} u^{i+1} \\ v^i \end{pmatrix} = 0.$$

Therefore, since $d^{i-1}_{C(fg)}$ is a weak kernel of $d^{i}_{C(fg)}$, there exists a morhpism $\begin{pmatrix} s^{i} \\ h^{i-1} \end{pmatrix}$: $M \to Z^i \oplus X^{i-1}$ such that $d^{i-1}_{C(fg)} \begin{pmatrix} s^{i} \\ h^{i-1} \end{pmatrix} = \varphi^i \begin{pmatrix} u^{i+1} \\ v^i \end{pmatrix}$. Thus we have

$$d^i_Z s^i + u^{i+1} = 0, \quad f^i g^i s^i + d^{i-1}_X h^{i-1} = f^i v^i$$

and

$$d^{i-1}_{C(f)} \begin{pmatrix} v^i - g^i s^i \\ -h^{i-1} \end{pmatrix} = \begin{pmatrix} -d^i_Y v^i + d^i_Y g^i s^i \\ f^i v^i - f^i g^i s^i - d^{i-1}_X h^{i-1} \end{pmatrix} = 0.$$

Therefore, since $d^{i-2}_{C(f)}$ is a weak kernel of $d^{i-1}_{C(f)}$, there exists a morhpism $\begin{pmatrix} w^{i-1} \\ t^{i-2} \end{pmatrix}$: $M \to Y^{i-1} \oplus X^{i-2}$ such that $d^{i-2}_{C(f)} \begin{pmatrix} w^{i-1} \\ t^{i-2} \end{pmatrix} = \begin{pmatrix} v^i - g^i s^i \\ -h^{i-1} \end{pmatrix}$. Thus, we have $-d^{i-1}_Y w^{i-1} = v^i - g^i s^i$. Then, $d^{i-1}_{C(g)} \begin{pmatrix} s^i \\ -w^{i-1} \end{pmatrix} = \begin{pmatrix} -d^i_Z s^i \\ g^i s^i - d^{i-1}_Y w^{i-1} \end{pmatrix} = \begin{pmatrix} u^{i+1} \\ v^i \end{pmatrix}$. This proves that $d^{i-1}_{C(g)}$ is a weak kernel of $d^{i}_{C(g)}$ for $i=0,\cdots,n-1$, thus $Z \overset{g}{\to} Y$ is an n-pullback diagram of Z along g^n. □

2.3 *n-Abelian categories*

As a generalization of the notion of classical abelian categories, Jasso introduced the n-abelian categories in [13] as follows.

DEFINITION 2.5 (n-abelian category, [13, Definition 3.1])

An *n-abelian category* is an additive category $\mathcal{A}$ which satisfies the following axioms:

(A0) The category $\mathcal{A}$ is idempotent complete.
(A1) Every morphism in $\mathcal{A}$ has an n-kernel and an n-cokernel.

(A2) For every monomorphism $f^0\colon X^0 \to X^1$ in $\mathcal{A}$ there exists an n-exact sequence:

$$X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} \cdots \xrightarrow{f^{n-1}} X^n \xrightarrow{f^n} X^{n+1}.$$

(A2$^{\mathrm{op}}$) For every epimorphism $f^n\colon X^n \to X^{n+1}$ in $\mathcal{A}$ there exists an n-exact sequence:

$$X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} \cdots \xrightarrow{f^{n-1}} X^n \xrightarrow{f^n} X^{n+1}.$$

Note that 1-abelian categories are precisely abelian categories in the usual sense. For $m \neq n$, any category are both m-abelian category and n-abelian category if and only if it is a semisimple category [13, Corollary 3.10].

Lemma 2.6 [13, *Proposition* 3.7]. *Let $\mathcal{A}$ be an additive category which satisfies axioms* (A1) *and* (A2), *and let X a complex in $\mathsf{Ch}^{n-1}(\mathcal{A})$. If for all $0 \le k \le n-1$ the morphism d^k is a weak cokernel of d^{k-1}, then d^{n-1} admits a cokernel in $\mathcal{A}$.*

Consider a pushout diagram (in abelian category)

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ A' & \xrightarrow{f'} & B' \end{array}.$$

It is well known that coker $f \simeq$ coker f', and pushout of monomorphism yields monomorphism. We can generalize this property to the n case.

PROPOSITION 2.7

Let $\mathcal{C}$ be an additive category which satisfies axioms (A0) *and* (A1), *let X be a complex in $\mathsf{Ch}^{n-1}(\mathcal{C})$ such that the morphism d_X^k is a weak cokernel of d_X^{k-1} for $1 \le k \le n-1$. Let $f^0 : X^0 \to Y^0$ be a morphism. Then there exists a cokernel $d_X^n : X^n \to X^{n+1}$ of d_X^{n-1} such that $(d_X^1, \ldots, d_X^n)$ is an n-cokernel of d_X^0, and for any n-pushout diagram*

$$\begin{array}{ccccccccccc} X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & \cdots & \longrightarrow & X^{n-1} & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & X^{n+1} \\ \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} & & \| \\ Y^0 & \xrightarrow{d_Y^0} & Y^1 & \xrightarrow{d_Y^1} & \cdots & \longrightarrow & Y^{n-1} & \xrightarrow{d_Y^{n-1}} & Y^n & \xrightarrow{d_Y^n} & X^{n+1} \end{array}$$

of $(d_X^0, \ldots, d_X^{n-1})$ along f^0, there exists a cokernel $d_Y^n : Y^n \to X^{n+1}$ of d_Y^{n-1} such that $(d_Y^1, \ldots, d_Y^n)$ is an n-cokernel of d_Y^0 and $d_X^n = d_Y^n f^n$. Moreover, if d_X^0 is a monomorphism, both rows are n-exact sequences.

Proof. The existence of $d_X^n : X^n \to X^{n+1}$ of d_X^{n-1} is immediately by Lemma 2.6. Since d_C^{n-1} is a cokernel of d_C^{n-2}, there exists an unique morphism $d_Y^n : Y^n \to X^{n+1}$ such that $d_X^n = d_Y^n f^n$ and $d_Y^n d_Y^{n-1} = 0$. Since d_X^n is an epimorphism so is d_Y^n. It remains to show that d_Y^n is a cokernel of d_Y^{n-1}. Let $u : Y^n \to M$ be a morphism such that $u d_Y^{n-1} = 0$. Then

$$(uf^n)d_X^{n-1} = u(d_Y^{n-1} f^{n-1}) = 0.$$

Since d_X^n is a cokernel of d_X^{n-1}, there exists a morphism $v : X^{n+1} \to M$ such that $uf^n = vd_X^n$. It follows that

$$uf^n = vd_X^n = (vd_Y^n)f^n \quad \text{and} \quad ud_Y^{n-1} = 0 = (vd_Y^n)d_Y^{n-1}.$$

Since d_C^{n-1} is a cokernel of d_C^{n-2}, $u = vd_Y^n$. This shows that the epimorphism d_Y^n is a cokernel of d_Y^{n-1}.

We show that the morphism d_Y^{k+1} is a weak cokernel of d_Y^k for $2 \leq k \leq n$, this shows that $(d_Y^1, \cdots, d_Y^n)$ is an n-cokernel of d_Y^0. Indeed, let $u : Y^{k+1} \to M$ be a morphism such that $ud_Y^k = 0$. Then $uf^{k+1}d_X^k = ud_Y^k f^k = 0$, there exists $v^{k+2} : X^{k+2} \to M$ such that $uf^{k+1} = v^{k+2}d_X^{k+1}$ since d_X^{k+1} is a weak cokernel of d_X^k. Then there exists morphisms $v^{k+3} : X^{k+2} \to M$ and $u^{k+2} : Y^{k+2} \to M$(we set $X^{n+1} = 0$) such that $u^{k+2}d_Y^{k+1} = u$ since d_C^{k+1} is a weak cokernel of d_C^k. This shows that d_Y^{k+1} is a weak cokernel of d_Y^k.

Moreover, if d_X^0 is a monomorphism, by [13, Theorem 3.8(ii)], n-pushout preserve monic, then d_Y^0 is a monomorphism. By axiom (A2), both the two rows are n-exact sequences. □

Remark 2.8. In Proposition 2.7, we call $Y^0 \to Y^1 \to \cdots \to Y^n \to X^{n+1}$ *an n-exact sequence induced by n-pushout of* $X^0 \to \cdots \to X^{n+1}$ *along* f^0, $(f^0, \ldots, f^n, 1)$ is *a morphism induced by n-pushout of* $X^0 \to \cdots \to X^{n+1}$ *along* f^0. Then $Y^0 \to Y^1 \to \cdots \to Y^n \to X^{n+1}$ is uniquely determined up to equivalence of n-exact sequences by Proposition 5.3.

Proposition 2.7 has also been generalized [13, Proposition 2.12] to be a necessary and sufficient condition. The following proposition is a characterization of composition of n-exact diagrams as an application of Proposition 2.4.

PROPOSITION 2.9

Let $\mathcal{A}$ be an n-abelian category, $Z \xrightarrow{g} Y$ and $Y \xrightarrow{f} X$ are morphisms in $\mathsf{Ch}^{n-1}(\mathcal{A})$. If $Y \xrightarrow{f} X$ is an n-pullback diagram of X along f^n and $Z \xrightarrow{g} Y$ is an n-pullback diagram of Y along g^n, then $Z \xrightarrow{g} Y$ and $Y \xrightarrow{f} X$ are n-exact diagrams if and only if $Z \xrightarrow{fg} X$ is an n-exact diagram.

Proof.

(⇒) This is obvious by the dual of Proposition 2.4.
(⇐) Because $Y \xrightarrow{f} X$ is a n-pullback diagram of X along f^n, $(d_{C(f)}^{-1}, \ldots, d_{C(f)}^{n-2})$ is a n-kernel of $d_{C(f)}^{n-1}$ in the bottom row of diagram (1). The middle row of diagram (1) is an n-exact sequence by the definition of n-exact diagram and $d_{C(f)}^{n-1}$ is an epimorphism since $d_{C(fg)}^{n-1}$ is an epimorphism. By axiom ($\mathrm{A2^{op}}$) of n-abelian category, the bottom row of diagram (1) is an n-exact sequence, thus $Y \xrightarrow{g} X$ is an n-exact diagram.

We show that $Z \xrightarrow{g} Y$ is an n-exact diagram. It is enough to show that $d_{C(g)}^{n-1}$ is an epimorphism by axiom ($\mathrm{A2^{op}}$) of n-abelian category since $(d_{C(g)}^{-1}, \ldots, d_{C(g)}^{n-2})$ is an n-kernel of $d^{n-1}_{C(g)}$. Indeed, let $u : y^n \to M$ be a morphism such that $ud^{n-1}_{C(g)} = (ug^n \; ud^{n-1}_Y) = 0$. Then, $(u \; 0) : Y^n \oplus X^{n-1} \to M$ is a morphism such that $(u \; 0)d^{n-2}_{C(f)} = 0$. Therefore, since $d^{n-1}_{C(f)}$ is the cokernel of $d^{n-2}_{C(f)}$, there exists a morphism $h : X^n \to M$ such that $h(f^n \; d^{n-1}_X) = (u \; 0)$. Thus $hf^n = u$, $hd^{n-1}_X = 0$. Hence $h = 0$ since $hd^{n-1}_{C(fg)} = (hd^{n-1}_X \; hf^n g^n) = (0 \; ug^n) = 0$ and $d^{n-1}_{C(fg)}$ is an epimorphism. Thus, $u = 0$. This proves our assertion. □

2.4 *n-Cluster tilting subcategories*

Recall that a full subcategory $\mathcal{D}$ of an abelian category $\mathcal{A}$ is *cogenerating* if for every object $X \in \mathcal{A}$ there exists an object $Y \in \mathcal{D}$ and a monomorphism $X \to Y$. The concept of *generating* subcategory is defined dually.

Let $\mathcal{A}$ be an abelian category and $\mathcal{D}$ a generating–cogenerating full subcategory of $\mathcal{A}$. $\mathcal{D}$ is called an *n-cluster-tilting subcategory of* $\mathcal{A}$ if $\mathcal{D}$ is functorially finite in $\mathcal{A}$ and

$$\begin{aligned}\mathcal{D} =&\{X \in \mathcal{A} | \forall i \in \{1, \ldots, n-1\} \; \mathsf{Ext}^i_{\mathcal{A}}(X, \mathcal{D}) = 0\}\\ =&\{X \in \mathcal{A} | \forall i \in \{1, \ldots, n-1\} \; \mathsf{Ext}^i_{\mathcal{A}}(\mathcal{D}, X) = 0\}.\end{aligned}$$

Note that $\mathcal{A}$ itself is the unique 1-cluster-tilting subcategory of $\mathcal{A}$.

Lemma 2.10 (*Theorem* 3.16 *of* [13]). *Let $\mathcal{A}$ be an abelian category and $\mathcal{D}$ an n-cluster tilting subcategory of $\mathcal{A}$. Then, $\mathcal{D}$ is an n-abelian category.*

Let $\mathcal{C}$ be a small additive category. A *$\mathcal{C}$-module* is a contravariant functor $G : \mathcal{C} \to \operatorname{Mod} \mathbb{Z}$. The category $\operatorname{Mod} \mathcal{C}$ of $\mathcal{C}$-modules is an abelian category. Morphisms in $\operatorname{Mod} \mathcal{C}$ are natural transformations of contravariant functors. As a consequence of Yoneda's lemma, functors $\mathcal{C}(-, N)$ are projective objects in $\operatorname{Mod} \mathcal{C}$. The *category of coherent $\mathcal{C}$-modules*, denoted by $\operatorname{mod} \mathcal{C}$, is the full subcategory of $\operatorname{Mod} \mathcal{C}$ whose objects are the $\mathcal{C}$-modules G such that there exists a morphism $f : X \to Y$ in $\mathcal{C}$ and an exact sequence of functors

$$\mathcal{C}(-, X) \xrightarrow{f^*} \mathcal{C}(-, Y) \to G \to 0.$$

Note that $\operatorname{mod} \mathcal{C}$ is closed under cokernels and extensions in $\operatorname{Mod} \mathcal{C}$. Moreover, $\operatorname{mod} \mathcal{C}$ is closed under kernels in $\operatorname{Mod} \mathcal{C}$ if and only if $\mathcal{C}$ has weak kernels, in which case $\operatorname{mod} \mathcal{C}$ is an abelian category [1].

Let $\mathcal{C}$ be a small projectively generated n-abelian category. Let $\mathcal{P}$ be the category of projective objects in $\mathcal{C}$. $\mathcal{P}$ has weak kernels since $\mathcal{C}$ is a projectively generated n-abelian category, thus $\operatorname{mod} \mathcal{P}$ is an abelian category.

Lemma 2.10 says that any n-cluster tilting subcategory of abelian category is an n-abelian category. The following lemma says that certain n-abelian categories can be realized as n-cluster tilting subcategories of abelian categories.

Lemma 2.11 (*Lemma* 3.22 *of* [13] *and Theorem* 1.3 *of* [18]). *Let $\mathcal{C}$ be a small projectively generated n-abelian category. Let $\mathcal{P}$ be the category of projective objects in $\mathcal{C}$ and $F : \mathcal{C} \to$* $\operatorname{mod} \mathcal{P}$ *the functor defined by $FX := \mathcal{C}(-, X)|_{\mathcal{P}}$. Also, let $F\mathcal{C} := \{M \in \operatorname{mod} \mathcal{P} | \exists X \in$*

$\mathcal{C}$ such that $M \simeq FX\}$ be the essential image of F. Then, the following statements hold:

(*i*) *F is a fully faithful functor.*
(*ii*) *$F\mathcal{C}$ is an n-cluster tilting subcategory of* mod $\mathcal{P}$.

In the rest of this paper, all functors 'F' are the functor defined in Lemma 2.11 unless otherwise specified.

3. $(n+2) \times (n+2)$-Lemma and 5-lemma

In this section, we study the relationship between exact sequences and n-exact sequences in n-cluster tilting subcategories of abelian categories and prove the $(n+2) \times (n+2)$-lemma and 5-lemma in n-abelian categories as generalizations of the classical 3×3-lemma and 5-lemma in abelian categories.

PROPOSITION 3.1

Let $\mathcal{A}$ be an abelian category, $\mathcal{D}$ a full additive subcategory of $\mathcal{A}$. Let $X:\ X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}$ be a complex in $\mathsf{Ch}^n(\mathcal{D})$. Then we have

(*i*) *If $\mathcal{D}$ is a contravariantly finite generating subcategory of $\mathcal{A}$ and $(d^0, \ldots, d^{n-1})$ is an n-kernel of d^n, then $0 \to X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}$ is an exact sequence of $\mathcal{A}$*;
(*ii*) *If $\mathcal{D}$ is a covariantly finite cogenerating subcategory of $\mathcal{A}$ and $(d^1, \ldots, d^n)$ is an n-cokernel of d^0, then $X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \to 0$ is an exact sequence of $\mathcal{A}$*;
(*iii*) *If $\mathcal{D}$ is a functorially finite generating–cogenerating subcategory of $\mathcal{A}$ and X is an n-exact sequence of $\mathcal{D}$, then $0 \to X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \to 0$ is an exact sequence of $\mathcal{A}$.*

Proof. We only prove (i); (ii) follows from the dual of (i) and (iii) follows from (i), (ii). We need to show that d^0 is a monomorphism and $\mathsf{Ker}\, d^k = \mathsf{Im}\, d^{k-1}$ for $k = 1, 2, \ldots, n$ in $\mathcal{A}$.

First, d^0 is a monomorphism in $\mathcal{A}$. Indeed, let $u : M \to X^0$ be a morphism in $\mathcal{A}$ such that $d^0 u = 0$. Because $\mathcal{D}$ is a contravariantly finite generating subcategory of $\mathcal{A}$, there exists a right $\mathcal{D}$-approximation $\pi : D \to M$ in $\mathcal{A}$ (note that, π is an epimorphism in $\mathcal{A}$ since $\mathcal{D}$ is a generating subcategory of $\mathcal{A}$). Then, $d^0(u\pi) = 0$. We have that $u\pi = 0$ since $u\pi$ is a morphism in $\mathcal{D}$ and d^0 is a monomorphism in $\mathcal{D}$. Also, $u = 0$ since π is an epimorphism in $\mathcal{A}$. So d^0 is a monomorphism in $\mathcal{A}$.

Second, $\mathsf{Ker}\, d^k = \mathsf{Im}\, d^{k-1}$ for $k = 1, 2, \ldots, n$ in $\mathcal{A}$, this is an adaptation of the proof of [19, Yoneda lemma 1.6.11]. It is trivial that $d^k d^{k-1} = 0$. We have an inclusion $\iota : \mathsf{Ker}\, d^k \to X^k$ and a right $\mathcal{D}$-approximation $\pi : E \to \mathsf{Ker}\, d^k$ in $\mathcal{A}$. Therefore, since $d^k \iota\pi = 0$ and $E \in \mathcal{D}$ and d^{k-1} is a weak kernel of d^k, there exists a morphism $\sigma : E \to X^{k-1}$ such that $\iota\pi = d^{k-1}\sigma$. It follows that $\mathsf{Ker}\, d^k = \mathsf{Im}\, \iota\pi \subset \mathsf{Im}\, d^{k-1}$. □

PROPOSITION 3.2

Let $\mathcal{D}$ be an n-cluster tilting subcategory of an abelian category $\mathcal{A}$ and $X : X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}$ a complex in $\mathsf{Ch}^n(\mathcal{D})$. Then we have

(*i*) $0 \to X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}$ *is an exact sequence of $\mathcal{A}$ if and only if $(d^0, \ldots, d^{n-1})$ is an n-kernel of d^n in $\mathcal{D}$.*

(*ii*) $X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \to 0$ *is an exact sequence of $\mathcal{A}$ if and only if $(d^1, \ldots, d^n)$ is an n-cokernel of d^0 in $\mathcal{D}$.*

(*iii*) $0 \to X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \to 0$ *is an exact sequence of $\mathcal{A}$ if and only if X is an n-exact sequence of $\mathcal{D}$.*

Proof. We only prove (i); (ii) follows from the dual of (i) and (iii) follows from (i), (ii). We only need to prove the 'only if' part, the 'if' part is a corollary of Proposition 3.1.

Put $L^i := \operatorname{Im} d^{i-1}$ in $\mathcal{A}$. Then we have exact sequences

$$0 \to L^i \xrightarrow{f^i} X^i \xrightarrow{g^i} L^{i+1} \to 0 \quad (1 \le i \le n-1) \text{ and } 0 \to L^n \xrightarrow{f^n} X^n \xrightarrow{g^n} L^{n+1},$$

where $L^1 = X^0$, $L^{n+1} = X^{n+1}$. Let $M \in \mathcal{D}$. Applying $\mathcal{A}(M, -)$, we have

$$\operatorname{Ext}^j_{\mathcal{A}}(M, L^1) = 0 \quad (1 \le j \le n-1) \text{ and } \operatorname{Ext}^j_{\mathcal{A}}(M, L^2) = 0 \quad (1 \le j \le n-2)$$

and

$$\begin{aligned}\operatorname{Ext}^1_{\mathcal{A}}(M, L^{i-1}) &\simeq \operatorname{Ext}^2_{\mathcal{A}}(M, L^{i-2}) \\ &\simeq \cdots \simeq \operatorname{Ext}^{i-2}_{\mathcal{A}}(M, L^2) = 0 \quad (3 \le i \le n).\end{aligned}$$

Now, we show that d^{i-1} is a weak kernel of d^i for $i = 1, \ldots, n$. Indeed, let $u : M \to X^i$ in $\mathcal{D}$ be any morphism such that $d^i u = 0$, then $g^i u = 0$ since f^{i+1} is a monomorphism($f^{n+1} = 1_{X^{n+1}}$). Then, there exists a morphism $v : M \to L^i$ such that $f^i v = u$. If $i = 1$, this proves d^0 is a kernel of d^1 since d^0 is a monomorphism. For $i > 1$, we have an exact sequence

$$0 \to \mathcal{A}(M, L^{i-1}) \to \mathcal{A}(M, X^{i-1}) \to \mathcal{A}(M, L^i) \to 0$$

since $\operatorname{Ext}^1_{\mathcal{A}}(M, L^{i-1}) = 0$ for $i = 2$ and

$$\begin{aligned}\operatorname{Ext}^1_{\mathcal{A}}(M, L^{i-1}) &\simeq \operatorname{Ext}^2_{\mathcal{A}}(M, L^{i-2}) \\ &\simeq \cdots \simeq \operatorname{Ext}^{i-2}_{\mathcal{A}}(M, L^2) = 0 \quad (3 \le i \le n).\end{aligned}$$

There exists a morphism $t : M \to X^{i-1}$ such that $g^{i-1} t = v$. Then $d^{i-1} t = f^i g^{i-1} t = f^i v = u$. This proves that d^{i-1} is a weak kernel of d^i for $i = 1, \cdots, n$ and d^0 is a kernel of d^1, so that $(d^0, \ldots, d^{n-1})$ is an n-kernel of d^n. □

As an application, since every small projectively generated n-abelian category can be embedded in an abelian category by Lemma 2.11, we have the following Corollary.

COROLLARY 3.3

Let $\mathcal{A}$ be a small projectively generated n-abelian category. Let $\mathcal{P}$ be the category of projective objects in $\mathcal{A}$ and X a complex in $\mathsf{Ch}^n(\mathcal{A})$. Then we have

(*i*) $0 \to FX^0 \xrightarrow{Fd^0} FX^1 \xrightarrow{Fd^1} \cdots \xrightarrow{Fd^{n-1}} FX^n \xrightarrow{Fd^n} FX^{n+1}$ *is an exact sequence of* $\operatorname{mod}\mathcal{P}$ *if and only if* $(d^0, \dots, d^{n-1})$ *is an n-kernel of* d^n *in* $\mathcal{A}$.

(*ii*) $FX^0 \xrightarrow{Fd^0} FX^1 \xrightarrow{Fd^1} \cdots \xrightarrow{Fd^{n-1}} FX^n \xrightarrow{Fd^n} FX^{n+1} \to 0$ *is an exact sequence of* $\operatorname{mod}\mathcal{P}$ *if and only if* $(d^1, \dots, d^n)$ *is an n-cokernel of* d^0 *in* $\mathcal{A}$.

(*iii*) $0 \to FX^0 \xrightarrow{Fd^0} FX^1 \xrightarrow{Fd^1} \cdots \xrightarrow{Fd^{n-1}} FX^n \xrightarrow{Fd^n} FX^{n+1} \to 0$ *is an exact sequence of* $\operatorname{mod}\mathcal{P}$ *if and only if X is an n-exact sequence of* $\mathcal{A}$.

Proof. We only prove (i); (ii) follows from the dual of (i) and (iii) follows from (i), (ii).

$\Leftarrow$: The exact sequence at FX^i for any $k \in \{1, 2, \dots, n\}$. Indeed, it is trivial that $Fd^kFd^{k-1} = 0$. We have an inclusion $\iota : \mathsf{Ker}\, Fd^k \to FX^k$ and an epic right $F\mathcal{A}$-approximation $\pi : FE \to \mathsf{Ker}\, Fd^k$. Then there exists a morphism $\delta : E \to X^k$ such that $F\delta = \iota\pi$ since F is fully faithful. Therefore, since $F(d^k\delta) = 0$ and d^{k-1} is a weak kernel of d^k, there exists a morphism $\sigma : E \to X^{k-1}$ such that $\delta = d^{k-1}\sigma$. It follows that

$$\mathsf{Ker}\, Fd^k = \mathsf{Im}\, \iota\pi = \mathsf{Im}\, F\delta = \mathsf{Im}\, F(d^{k-1}\sigma) \subset \mathsf{Im}\, Fd^{k-1}.$$

Fd^0 is a monomorphism. Indeed, let $u : G \to FX^0$ be a morphism in $\operatorname{mod}\mathcal{P}$ such that $(Fd^0)u = 0$. There exists an epic right $F\mathcal{A}$-approximation $v : FM \to G$. Therefore, there exists a morphism $s : M \to X^0$ such that $Fs = uv$. We have that $Fd^0Fs = 0$, so $sd^0 = 0$. $s = 0$ since d^0 is a monomorphism, so is Fs. This implies $u = 0$ since v is an epimorphism. It follows that Fd^0 is a monomorphism.

$\Rightarrow$: For $k \in \{1, 2, \dots, n\}$, let $u : M \to X^k$ be a morphism such that $d^ku = 0$. It follows that $Fd^kFu = 0$, therefore, since F is fully faithful and Fd^{k-1} is a weak kernel of Fd^k (Proposition 3.2(i)), there exists a morphism $v : M \to X^{k-1}$ such that $Fd^{k-1}Fv = Fu$, so $d^{k-1}v = u$. This proves that d^{k-1} is a weak kernel of d^k. Hence d^0 is a monomorphism and follows from Fd^0 which is a monomorphism and F is fully faithful. □

Lemma 3.4. *Let $\mathcal{A}$ be an abelian category. Then*

(*i*) $\mathsf{Ch}(\mathcal{A})$ *is an abelian category*;

(*ii*) *If* $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ *is a short exact sequence of cochain complexes*, *then*, *whenever two of the three complexes* $A^\bullet, B^\bullet, C^\bullet$ *are exact complexes*, *so is the third*;

(*iii*) *If* $0 \to A_1^\bullet \to A_2^\bullet \to \cdots \to A_r^\bullet \to 0$ *is an exact sequence of cochain complexes for* $r \geq 3$, *then*, *whenever* $r - 1$ *of the r complexes* $A_1^\bullet, A_2^\bullet, \dots, A_r^\bullet$ *are exact complexes*, *so is the remainder.*

Proof. The proof follows immediately from [19, Theorem 1.2.3, Exercise 1.3.1]. □

The following theorem is a generalization of 3×3-lemma of the abelian category.

Theorem 3.5 **($(n+2)\times(n+2)$-lemma).** *Let $\mathcal{A}$ be a small projectively generated n-abelian category, $\mathcal{P}$ the category of projective objects in $\mathcal{A}$. Let*

$$\begin{array}{ccccccc}
A^{1,1} & \longrightarrow & A^{1,2} & \longrightarrow & \cdots & \longrightarrow & A^{1,n+2} \\
\downarrow & & \downarrow & & & & \downarrow \\
A^{2,1} & \longrightarrow & A^{2,2} & \longrightarrow & \cdots & \longrightarrow & A^{2,n+2} \\
\downarrow & & \downarrow & & & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & & & \downarrow \\
A^{n+2,1} & \to & A^{n+2,2} & \to & \cdots & \to & A^{n+2,n+2}
\end{array}$$

be a commuting diagram in $\mathcal{A}$ such that all columns are n-exact sequences. Then $n+1$ of the $n+2$ rows are n-exact sequences and implies the remainder.

Proof. By Lemma 2.11, Corollary 3.3 and Lemma 3.4, the remainder row is an exact sequence of $\operatorname{mod}\mathcal{P}$ under the functor F. Then the remainder row in $\mathcal{A}$ is an n-exact sequence by Corollary 3.3 again. □

Theorem 3.6 **(5-lemma).** *Let $\mathcal{A}$ be a small projectively generated n-abelian category, $\mathcal{P}$ the category of projective objects in $\mathcal{A}$. Let*

$$\begin{array}{ccccccccc}
X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & X^2 & \xrightarrow{d_X^2} & X^3 & \xrightarrow{d_X^3} & X^4 \\
\downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^2} & & \downarrow{\scriptstyle f^3} & & \downarrow{\scriptstyle f^4} \\
Y^0 & \xrightarrow{d_Y^0} & Y^1 & \xrightarrow{d_Y^1} & Y^2 & \xrightarrow{d_Y^2} & Y^3 & \xrightarrow{d_Y^3} & Y^4
\end{array}$$

be a commuting diagram. If f^1 and f^3 are isomorphisms, f^0 is an epimorphism, f^4 is a monomorphism and one of the following conditions holds, then f^2 is also an isomorphism.

(*i*) *d_X^i and d_Y^i are weak cokernels of d_X^{i-1} and d_Y^{i-1} respectively for $i=1,2,3,4$.*

(*ii*) *d_X^i and d_Y^i are weak kernels of d_X^{i+1} and d_Y^{i+1} respectively for $i=0,1,2,3$.*

Proof. We only prove (i). By axiom (A1), there exists a weak cokernel $d_X^4 : X^4 \to X^5$ of d_X^3, inductively, there exist morphisms $d_X^i : X^i \to X^{i+1}$ for $i=5,\dots,n$ such that $(d_X^1, d_X^2,\dots,d_X^n)$ is an n-cokernel of d_X^0 by Lemma 2.6. Similarly, there exist morphisms $d_Y^i : Y^i \to Y^{i+1}$ for $i=4,\dots,n$ such that $(d_Y^1, d_Y^2,\dots,d_Y^n)$ is an n-cokernel of d_Y^0. By the property of weak cokernels we obtain a commutative diagram

$$\begin{array}{ccccccccccc}
X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & \cdots & \longrightarrow & X^{n-1} & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & X^{n+1} \\
\downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} \\
Y^0 & \xrightarrow{d_Y^0} & Y^1 & \xrightarrow{d_Y^1} & \cdots & \longrightarrow & Y^{n-1} & \xrightarrow{d_Y^{n-1}} & Y^n & \xrightarrow{d_Y^n} & Y^{n+1}
\end{array},$$

Applying F, by Corollary 3.3, we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccccccc}
FX^0 & \xrightarrow{Fd_X^0} & FX^1 & \xrightarrow{Fd_X^1} & \cdots & \longrightarrow & FX^{n-1} & \xrightarrow{Fd_X^{n-1}} & FX^n & \xrightarrow{Fd_X^n} & FX^{n+1} & \longrightarrow & 0 \\
\downarrow{\scriptstyle Ff^0} & & \downarrow{\scriptstyle Ff^1} & & & & \downarrow{\scriptstyle Ff^{n-1}} & & \downarrow{\scriptstyle Ff^n} & & \downarrow{\scriptstyle Ff^{n+1}} & & \\
FY^0 & \xrightarrow{Fd_Y^0} & FY^1 & \xrightarrow{Fd_Y^1} & \cdots & \longrightarrow & FY^{n-1} & \xrightarrow{Fd_Y^{n-1}} & FY^n & \xrightarrow{Fd_Y^n} & FY^{n+1} & \longrightarrow & 0
\end{array},$$

We need to show that Ff^0 is an epimorphism in $\operatorname{mod}\mathcal{P}$, Ff^4 is a monomorphism in $\operatorname{mod}\mathcal{P}$ and Ff^1, Ff^3 are isomorphisms in $\operatorname{mod}\mathcal{P}$. Indeed, let $u : FY^0 \to G$ be a morphism in $\operatorname{mod}\mathcal{P}$ such that $uFf^0 = 0$. There exists an object $M \in \mathcal{A}$ and a monomorphism left $F\mathcal{M}$-approximation $v : G \to FM$ since $F\mathcal{A}$ is a n-cluster tilting subcategory of $\operatorname{mod}\mathcal{P}$. Therefore, since F is a fully faithful functor, there exists a morphism $s : Y^0 \to M$ such that $Fs = vu$. We have that $FsFf^0 = 0$, thus $sf^0 = 0$. $s = 0$ since f^0 is a epimorphism, so is Fs. This implies $u = 0$ since v is a monomorphism. It follows that Ff^0 is an epimorphism. Similarly, if Ff^4 is a monomorphism, Ff^1, Ff^3 are isomorphisms.

By 5-lemma of abelian categories, Ff^3 is an isomorphism, so is f^3 since F is fully faithful. □

4. (Co)homology of *n*-abelian categories

In this section, we introduce the right(resp., left) derived functors of covariant or contravariant left(resp., right) n-exact functors and study their basic properties.

4.1 *n-Exact functors and n-resolutions*

Let $\mathcal{A}$ be an n-abelian category and $\mathcal{B}$ an abelian category, and let $G : \mathcal{A} \to \mathcal{B}$ be a covariant additive functor. Let $X :\ X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}$ in $\mathsf{Ch}^n(\mathcal{A})$ be an n-exact sequence. We say that G is

(i) *Left n-exact* if $0 \to GX^0 \to GX^1 \to \cdots \to GX^n \to GX^{n+1}$ is an exact sequence of $\mathcal{B}$.
(ii) *Right n-exact* if $GX^0 \to GX^1 \to \cdots \to GX^n \to GX^{n+1} \to 0$ is an exact sequence of $\mathcal{B}$.
(iii) *n-Exact* if $0 \to GX^0 \to GX^1 \to \cdots \to GX^n \to GX^{n+1} \to 0$ is an exact sequence of $\mathcal{B}$.

The notions of contravariant additive left n-exact functors, right n-exact functors and n-exact functors are defined dually.

For example, the hom-functors $\mathcal{A}(M, -)$ and $\mathcal{A}(-, M)$ are covariant left n-exact functor and contravariant left n-exact functor respectively by the definitions of n-kernel and n-cokernel.

We say that an n-abelian category $\mathcal{A}$ *has enough projectives* if for every object $M \in \mathcal{A}$, there exists projective objects $P_1, P_2, \ldots, P_n \in \mathcal{A}$ and an n-exact sequence

$$N \to P_n \to \cdots \to P_1 \to M$$

The notion of *having enough injectives* is defined dually.

Let $\mathcal{A}$ be an n-abelian category which has enough projectives, $M \in \mathcal{A}$. We have n-exact sequences

$$\Omega_n M \xrightarrow{j_1} P_n \xrightarrow{d_n} \cdots \to P_1 \to M$$
$$\Omega_n^2 M \xrightarrow{j_2} P_{2n} \xrightarrow{d_{2n}} \cdots \to P_{n+1} \xrightarrow{\pi_1} \Omega_n M$$
$$\Omega_n^3 M \xrightarrow{j_3} P_{3n} \xrightarrow{d_{3n}} \cdots \to P_{2n+1} \xrightarrow{\pi_2} \Omega_n^2 M$$
$$\vdots$$

Connecting them, let $d_{in+1} = j_i \pi_i$. We call the sequence

$$\cdots \to P_{3n} \xrightarrow{d_{3n}} \cdots \to P_{2n+1} \xrightarrow{d_{2n+1}} P_{2n} \xrightarrow{d_{2n}} \cdots \to P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} \cdots \to P_1 \xrightarrow{d_1} M \quad (2)$$

a *projective n-resolution of* M, also denoted simply as $P_\bullet \xrightarrow{d_1} M$. We call $\Omega_n^k M$ the k-th n-syzygy of M for $k \geq 0$. It is easily seen that Ω_n defines a functor between stable categories $\Omega_n : \underline{\mathcal{A}} \to \underline{\mathcal{A}}$, and moreover, if $\mathcal{A}$ is Frobenious, $\Omega_n : \underline{\mathcal{A}} \to \underline{\mathcal{A}}$ is self-equivalence [13, Proposition 5.8]. The notions of *injective n-coresolution* and *k-th n-cosyzygy* $\Omega_n^{-k} M$ of M are defined dually.

We cannot see that d_{in+1} is a weak kernel (resp. weak cokernel) of d_{in} (resp. d_{in+2}) for any i. But, it is easy to see that d_{in} is a weak cokernel of d_{in+1}, d_{in+2} is a weak kernel of d_{in+1}.

4.2 *Right(left) n-derived functors*

Let $\mathcal{A}$ be an n-abelian category and $\mathcal{B}$ an abelian category, and let $G : \mathcal{A} \to \mathcal{B}$ be a contravariant left n-exact functor. We can construct the *right n-derived functors* $\mathsf{nR}^i G$ for $i \geq 0$ as follows. Let $M \in \mathcal{A}$. Choose a projective n-resolution $P_\bullet \to M$ as (2) and define

$$\begin{aligned} \mathsf{nR}^i G(M) &:= H_{in+1}(GP_\bullet) \\ &:= \mathsf{Ker}\, Gd_{in+2}/\mathsf{Im}\, Gd_{in+1} \;\; \text{for } i = 0, 1, \ldots. \end{aligned}$$

(Observe that $GP_\bullet$ is exact at P_j for any $j \neq in + 1$.) Note that $\mathsf{nR}^0 G(M) \simeq GM$.

Lemma 4.1. *Let $\mathcal{A}$ be an n-abelian category and $\mathcal{B}$ an abelian category, and let $G : \mathcal{A} \to \mathcal{B}$ be a contravariant left n-exact functor. Then*

(*i*) *The objects $\mathsf{nR}^i G(M)$ of $\mathcal{B}$ are well defined up to natural isomorphism. That is, if $Q_\bullet \to M$ is a second projective n-resolution, then there is a canonical isomorphism*:

$$\mathsf{nR}^i G(M) := H^{in+1}(GP_\bullet) \simeq H^{in+1}(GQ_\bullet).$$

(*ii*) *If $f : M \to N$ is any map in $\mathcal{A}$, there is a natural map $\mathsf{nR}^i G(f) : \mathsf{nR}^i G(N) \to \mathsf{nR}^i G(M)$ for each $i \geq 0$.*
(*iii*) *$\mathsf{nR}^i G(P) = 0$ for all projective object P and $i > 0$.*
(*iv*) *Each $\mathsf{nR}^i G$ is an additive functor from $\mathcal{A}$ to $\mathcal{B}$.*

Proof. We can prove (i), (ii), and (iv) similar to that of [19, proof of Lemma 2.4.1, Lemma 2.4.4, Theorem 2.4.5] by using Comparison Lemma 2.1.

For (iii), the assertion follows by choosing the projective n-resolution of P to be $\cdots \to 0 \to P \xrightarrow{1} P$. □

The notions of *right n-derived functors* nR^i *for covariant left n-exact functors* by using injective n-resolutions, *left n-derived functors* nL_i *for covariant right n-exact functors* by using projective n-coresolutions, and *left n-derived functors* nL_i *for contravariant right n-exact functors* by using injective n-coresolutions are defined dually.

Specially, for contravariant (resp., covariant) left n-exact functor $\mathcal{A}(-, B)$ (resp., $\mathcal{A}(A, -)$), we have

DEFINITION 4.2 (nExt functors)

Let $\mathcal{A}$ be an n-abelian category, the hom-functor $\mathcal{A}(-, B)$(resp., $\mathcal{A}(A, -)$) is a contravariant (resp., covariant) left n-exact additive functor. We define the right n-derived functors

$$\mathsf{nExt}^i_{\mathcal{A}}(-, B) = \mathsf{nR}^i\mathcal{A}(-, B) \quad \text{resp., } \mathsf{nExt}^i_{\mathcal{A}}(A, -) = \mathsf{nR}^i\mathcal{A}(A, -)$$

In particular, $\mathsf{nExt}^0_{\mathcal{A}}(-, B) = \mathcal{A}(-, B)$, $\mathsf{nExt}^0_{\mathcal{A}}(A, -) = \mathcal{A}(A, -)$.

PROPOSITION 4.3

Let $\mathcal{A}$ be an n-abelian category which has enough projectives and enough injectives, $A, B \in \mathcal{A}$. We have

(*i*) $\mathsf{nExt}^i_{\mathcal{A}}(A, -)(B) \simeq \mathsf{nExt}^i_{\mathcal{A}}(-, B)(A) = \mathsf{nExt}^i_{\mathcal{A}}(A, B)$.

(*ii*) *If $\mathcal{A}$ is an n-cluster tilting subcategory of a projectively generated injectivity cogenerated abelian category $\mathcal{D}$ and $\mathcal{B}$ is an abelian category, then for any left n-exact functor $G : \mathcal{D} \to \mathcal{B}$ and any right n-exact functor $S : \mathcal{D} \to \mathcal{B}$, we have*

$$\mathsf{nR}^m G(A) \simeq \mathsf{R}^{mn} G(A), \quad \mathsf{nL}_m S(A) \simeq \mathsf{L}_{mn} S(A) \;\; \forall A \in \mathcal{A}, m \geq 0,$$

$$\mathsf{R}^{mn+i} G(A) = 0, \quad \mathsf{L}_{mn+i} S(A) = 0 \;\; \forall A \in \mathcal{A}, m \geq 0, 1 \leq i \leq n-1,$$

where $\mathsf{R}^i G$ and $\mathsf{L}_i S$ are the classical right and left derived functors respectively. In particular,

$$\mathsf{nExt}^m_{\mathcal{A}}(A, B) \simeq \mathsf{Ext}^{mn}_{\mathcal{D}}(A, B), \quad \mathsf{Ext}^{mn+i}_{\mathcal{D}}(A, B) = 0 \; \forall A, B \in \mathcal{A}, m \geq 0, 1 \leq i \leq n-1.$$

(*iii*) $\mathsf{nExt}^m_{\mathcal{A}}(A, \bigoplus_{i\in I} B_i) \simeq \bigoplus_{i\in I} \mathsf{nExt}^m_{\mathcal{A}}(A, B_i)$ *and* $\mathsf{nExt}^m_{\mathcal{A}}(\bigoplus_{i\in I} A_i, B) \simeq \prod_{i\in I} \mathsf{nExt}^m_{\mathcal{A}}(A_i, B)$ *for any $m \geq 0$.*

Proof. (i) follows from [19, Theorem 2.7.6]. (ii) follows immediately from the definition of right(resp. left) n-derived functors and Proposition 3.2, and (iii) follows from [19, proof of Corollary 2.6.11] and its dual. □

The following proposition is a generalization of the classical 'Horseshoes lemma' of projective generated abelian categories.

PROPOSITION 4.4 (n-Horseshoes lemma)

Let $\mathcal{A}$ be a small n-abelian category which has enough projectives, $\mathcal{P}$ the category of projective objects in $\mathcal{A}$. The $X : X^0 \xrightarrow{\alpha^0} X^1 \xrightarrow{\alpha^1} \cdots \xrightarrow{\alpha^n} X^{n+1}$ is an n-exact sequence of $\mathcal{A}$. Then, there exist projective n-resolutions $P^i_\bullet \to X^i$ for $i = \{0, 1, \ldots, n+1\}$ such that the following diagram commute:

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & P^0_\bullet & \longrightarrow & P^1_\bullet & \longrightarrow & \cdots & \longrightarrow & P^n_\bullet & \longrightarrow & P^{n+1}_\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\ & & X^0 & \longrightarrow & X^1 & \longrightarrow & \cdots & \longrightarrow & X^n & \longrightarrow & X^{n+1} & & \end{array}, \tag{3}$$

where the upper rows is a split exact sequence of complexes.

Proof. Applying F to X, by Corollary 3.3, $0 \to FX^0 \to FX^1 \to \cdots \to FX^{n+1} \to 0$ is an exact sequence of mod $\mathcal{P}$, we split it to exact sequences of mod $\mathcal{P}$,

$$0 \to L^i \to FX^i \to L^{i+1} \ \text{ for } \ i \in \{0, 1, \ldots, n\},$$

where $L^0 = FX^0$, $L^{n+1} = FX^{n+1}$.

Step 1. $F\mathcal{A}$ is closed under n-th syzygy, dually, closed under n-th cosyzygy. Indeed, for any $Y \in F\mathcal{A}$, there is an object $X \in \mathcal{A}$ such that $FX \simeq Y$. Since $\mathcal{A}$ is a small n-abelian category which has enough projectives, there is an n-exact sequence $\Omega_n X \to P_{n-1} \to \cdots \to P_0 \to X$ such that $0 \to F\Omega_n X \to FP_{n-1} \to \cdots \to FP_0 \to FX \simeq Y \to 0$ is an exact sequence in mod $\mathcal{P}$ (see Corollary 3.3) where P_i are projective objects. This proves that $F\mathcal{A}$ closed under n-th syzygy.

Step 2. Giving two projective n-resolutions $P^0_\bullet \to X^0$ and $P^{n+1}_\bullet \to X^{n+1}$, by Corollary 3.3, $FP^0_\bullet \to FX^0$ and $FP^{n+1}_\bullet \to FX^{n+1}$ are projective resolutions in mod $\mathcal{P}$. For any L^i, $i \in \{1, 2, \ldots, n\}$, take a projective resolution

$$FP^i_\bullet \to L^i : \ \cdots \to FP^i_2 \xrightarrow{Fd^i_2} FP^i_1 \xrightarrow{Fd^i_1} FP^i_0 \xrightarrow{Fd^i_0} L^i \to 0,$$

where $P^i_j \in \mathcal{P}$ $j \in \{0, 1, \ldots\}$. By Horseshoes lemma of abelian category mod $\mathcal{P}$, there exists a projective resolution

$$FQ^i_\bullet \xrightarrow{Ff^i_0} FX^i : \ \cdots \xrightarrow{f^i_3} F(P^i_2 \oplus P^{i+1}_2) \xrightarrow{Ff^i_2}$$
$$F(P^i_1 \oplus P^{i+1}_1) \xrightarrow{Ff^i_1} F(P^i_0 \oplus P^{i+1}_0) \xrightarrow{Ff^i_0} FX^i \to 0$$

such that $0 \to FP^i_\bullet \to FQ^i_\bullet \to FP^{i+1}_\bullet \to 0$ is a split short exact sequence of projective complexes, so is $0 \to P^i_\bullet \to Q^i_\bullet \to P^{i+1}_\bullet \to 0$. Connecting them, we have $0 \to P^0_\bullet \to Q^1_\bullet \to Q^2_\bullet \to \cdots \to Q^n_\bullet \to P^{n+1}_\bullet \to 0$ is a split exact sequence of projective complexes.

Step 3. $Q^i_\bullet \to X^i$ is a projective n-resolution for any $i \in \{1, 2, \ldots, n\}$. Indeed, we can split $FQ^i_\bullet \to FX^i$ to

$$S^j : \ 0 \to T^i_j \xrightarrow{k^i_j} FQ^i_{jn-1} \to FQ^i_{jn-2}$$
$$\to \cdots \to FQ^i_{(j-1)n} \xrightarrow{\pi^i_{j-1}} T^i_{j-1} \to 0,$$

where $T^i_0 = FX^i$ for $j \in \{1, 2, \ldots\}$. For $j = 1$, since F is fully faithful, there is an epimorphism $Q^i_0 \xrightarrow{f^i_0} X^i$ in $\mathcal{A}$ such that $Ff^i_0 = \pi^i_0$. We have $\mathsf{Ext}^k_{\mathrm{mod}\,\mathcal{P}}(T^i_1, F\mathcal{A}) \simeq \mathsf{Ext}^{n+k}_{\mathrm{mod}\,\mathcal{P}}(FX^i, F\mathcal{A}) = 0$ for $k = 1, 2, \ldots, n-1$ (see Proposition 4.3 (ii)). It provides that $T^i_1 \in F\mathcal{A}$ (see Lemma 2.11 and Step 1). Then there exists an object $G_1 \in \mathcal{A}$ such that $T^i_1 \simeq FG_1$. Then there is an n-exact sequence $G_1 \to Q^i_{n-1} \to \cdots \to Q^i_0 \to X^i$ (see Corollary 3.3). We can discuss $S^2, S^3, \ldots,$ inductively. This proves that $Q^i_\bullet \to X^i$ is a projective n-resolution. □

Theorem 4.5 (Long n-exact sequence theorem). *Let $\mathcal{A}$ be a small n-abelian category, $\mathcal{P}$ the category of projective objects in $\mathcal{A}$. $\mathcal{B}$ is an abelian category, $G : \mathcal{A} \to \mathcal{B}$ is an additive functor, and $X : X^0 \xrightarrow{\alpha^0} X^1 \xrightarrow{\alpha^1} \cdots \xrightarrow{\alpha^n} X^{n+1}$ an n-exact sequence of $\mathcal{A}$. Then we have*

(*i*) *If $\mathcal{A}$ has enough injectives and G is a covariant left n-exact functor, we have an exact sequence*

$$0 \to GX^0 \to GX^1 \to \cdots \to GX^{n+1} \xrightarrow{\partial_n} \mathsf{nR}^1 G(X^0)$$
$$\to \cdots \to \mathsf{nR}^1 G(X^{n+1}) \xrightarrow{\partial_n^1} \cdots \xrightarrow{\partial_n^{i-1}} \mathsf{nR}^i G(X^0) \to \cdots \to \mathsf{nR}^i G(X^{n+1}) \xrightarrow{\partial_n^i} \cdots .$$

(*ii*) *If $\mathcal{A}$ has enough projectives and G is a contravariant left n-exact functor, we have an exact sequence*

$$0 \to GX^{n+1} \to GX^n \to \cdots \to GX^0 \xrightarrow{\partial_n} \mathsf{nR}^1 G(X^{n+1})$$
$$\to \cdots \to \mathsf{nR}^1 G(X^0) \xrightarrow{\partial_n^1} \cdots \xrightarrow{\partial_n^{i-1}} \mathsf{nR}^i G(X^{n+1}) \to \cdots \to \mathsf{nR}^i G(X^0) \xrightarrow{\partial_n^i} \cdots .$$

(*iii*) *If $\mathcal{A}$ has enough projectives and G is a covariant right n-exact functor, we have an exact sequence*

$$\cdots \xrightarrow{\partial_n^i} \mathsf{nL}_i G(X^0) \to \cdots \to \mathsf{nL}_i G(X^{n+1}) \xrightarrow{\partial_n^{i-1}} \cdots \xrightarrow{\partial_n^1} \mathsf{nL}_1 G(X^0)$$
$$\to \cdots \to \mathsf{nL}_1 G(X^{n+1}) \xrightarrow{\partial_n} GX^0 \to \cdots \to GX^{n+1} \to 0.$$

(*iv*) *If $\mathcal{A}$ has enough injectives and G is a contravariant right n-exact functor, we have an exact sequence*

$$\cdots \xrightarrow{\partial_n^i} \mathsf{nL}_i G(X^{n+1}) \to \cdots \to \mathsf{nL}_i G(X^0) \xrightarrow{\partial_n^{i-1}} \cdots \xrightarrow{\partial_n^1} \mathsf{nL}_1 G(X^{n+1})$$
$$\to \cdots \to \mathsf{nL}_1 G(X^0) \xrightarrow{\partial_n} GX^{n+1} \to \cdots \to GX^0 \to 0.$$

Proof. If $n = 1$, then the result is the classical long exact sequence theorem. Let $n \geq 2$, we only prove (ii).

Applying G to the first arrow of (3) gives a split exact sequence of complexes

$$GP_\bullet^{n+1} \to GP_\bullet^n \to \cdots \to GP_\bullet^1 \to GP_\bullet^0 \tag{4}$$

since the first arrow of (3) is a split exact sequence of complexes. Since $\mathsf{Ch}(\mathcal{B})$ is an abelian category, we can split (4) to short split exact sequences of complexes

$$0 \to L^{i+1} \to GP_\bullet^i \to L^i \to 0 \quad (1 \leq i \leq n),$$

where $L^{n+1} = GP_\bullet^{n+1}$ and $L^1 = GP_\bullet^0$. Thus $H_{kn+i}(GP_\bullet^j) = 0$ for $i \in \{2, 3, \ldots, n\}$, $k \in \{0, 1, 2, \ldots\}$ and all $j \in \{0, 1, \ldots, n+1\}$ by the definition of left n-exact functor.

For exact sequence $0 \to L^{n+1} \to GP_\bullet^n \to L^n \to 0$, by the long exact sequence theorem, we have exact sequences

$$0 \to H_1(L^{n+1}) \to H_1(GP_\bullet^n) \to H_1(L^n) \to 0 \tag{5}$$
$$0 \to H_{kn}(L^n) \to H_{kn+1}(L^{n+1}) \to H_{kn+1}(GP_\bullet^n)$$
$$\to H_{kn+1}(L^n) \to 0 \qquad \text{for } k = 1, 2, \ldots \tag{6}$$

and

$$H_{kn+i}(L^n) = 0 \quad \text{for } i = 2, \dots, n-1. \tag{7}$$

Inductively, let $s \in \{2, 3, \dots, n-1\}$ and suppose that for all $i \geq s+1$ we have exact sequences

$$0 \to H_1(L^{i+1}) \to H_1(GP^i_\bullet) \to H_1(L^i) \to 0, \tag{8}$$

$$\begin{aligned} &0 \to H_{kn}(L^i) \to H_{kn+1}(L^{i+1}) \to H_{kn+1}(GP^i_\bullet) \\ &\quad \to H_{kn+1}(L^i) \to 0 \text{ for } k = 1, 2, \dots \end{aligned} \tag{9}$$

and

$$H_{kn+j}(L^i) \simeq H_{kn+j+1}(L^{i+1}) \text{ for } j = 2, \dots, n-1. \tag{10}$$

For $i = s$, i.e., the exact sequence $0 \to L^{s+1} \to GP^s_\bullet \to L^s \to 0$, by the long exact sequence theorem, we have exact sequences

$$\begin{aligned} &0 \to H_1(L^{s+1}) \to H_1(GP^s_\bullet) \to H_1(L^s) \to H_2(L^{s+1}) \to 0, \\ &0 \to H_{kn}(L^s) \to H_{kn+1}(L^{s+1}) \to H_{kn+1}(GP^s_\bullet) \\ &\quad \to H_{kn+1}(L^s) \to H_{kn+2}(L^{s+1}) \to 0 \quad \text{for } k = 1, 2, \dots \end{aligned}$$

and

$$H_{kn+j}(L^s) \simeq H_{kn+j+1}(L^{s+1}) \quad \text{for } j = 2, \dots, n-1.$$

But,

$$H_2(L^{s+1}) \overset{(10)}{\simeq} H_3(L^{s+2}) \overset{(10)}{\simeq} \cdots \overset{(10)}{\simeq} \begin{cases} H_{n-s+1}(L^n) \overset{(7)}{=} 0 & (s > 2) \\ H_{n-1}(L^n) \overset{(7)}{=} 0 & (s = 2), \end{cases}$$

$$\begin{aligned} H_{nk+2}(L^{s+1}) &\overset{(10)}{\simeq} H_{nk+2}(L^{s+2}) \\ &\overset{(10)}{\simeq} \cdots \overset{(10)}{\simeq} \begin{cases} H_{nk+n-s+1}(L^n) \overset{(7)}{=} 0 & (s > 2) \\ H_{nk+n-1}(L^n) \overset{(7)}{=} 0 & (s = 2). \end{cases} \end{aligned}$$

Thus we have exact sequences

$$\begin{aligned} &0 \to H_1(L^{s+1}) \to H_1(GP^s_\bullet) \to H_1(L^s) \to 0, \\ &0 \to H_{kn}(L^s) \to H_{kn+1}(L^{s+1}) \\ &\quad \to H_{kn+1}(GP^s_\bullet) \to H_{kn+1}(L^s) \to 0 \quad \text{for } k = 1, 2, \dots, \end{aligned}$$

This finishes the induction step.

For exact sequence $0 \to L^2 \to GP^1_\bullet \to L^1 \to 0$, by the long exact sequence theorem, we have exact sequences

$$\begin{aligned}&0 \to H_1(L^2) \to H_1(GP^1_\bullet) \to H_1(L^1) \to H_2(L^2) \to 0,\\ &0 \to H_{kn+1}(L^2) \to H_{kn+1}(GP^1_\bullet)\\ &\quad \to H_{kn+1}(L^1) \to \mathsf{H}_{kn+2}(L^2) \to 0 \quad \text{for } k = 1, 2, \ldots\end{aligned}$$

and

$$H_{kn+i}(L^2) = 0 \quad \text{for } i = 3, 4, \ldots, n. \tag{11}$$

But

$$H_{kn+2}(L^2) \overset{(10)}{\simeq} H_{kn+3}(L^3) \overset{(10)}{\simeq} \cdots \overset{(10)}{\simeq} H_{(k+1)n}(L^n) \quad \text{for } k = 0, 1, \ldots,$$

$$\begin{aligned}H_{kn}(L^i) &\overset{(10)}{\simeq} H_{kn-1}(L^{i-1}) \overset{(10)}{\simeq} \cdots \overset{(10)}{\simeq} H_{kn-i+2}(L^2)\\ &\overset{(11)}{\simeq} 0 \quad \text{for } k = 1, 2, \ldots, \quad i = 1, 2, \ldots, n-1\end{aligned}$$

Thus we have exact sequences

$$0 \to H_1(L^2) \to H_1(GP^1_\bullet) \to H_1(L^1) \to H_n(L^n) \to 0, \tag{12}$$

$$0 \to H_{kn+1}(L^{i+1}) \to H_{kn+1}(GP^i_\bullet) \to H_{kn+1}(L^i) \to 0 \quad \text{for } k = 1, 2, \ldots, \quad i = 1, \ldots, n-1, \tag{13}$$

$$\begin{aligned}&0 \to H_{kn+1}(L^2) \to H_{kn+1}(GP^1_\bullet) \to H_{kn+1}(L^1)\\ &\quad \to H_{(k+1)n}(L^n) \to 0 \quad \text{for } k = 1, 2, \ldots.\end{aligned} \tag{14}$$

Connecting them in order (5)(8)(12)(6)(13)(14)(6)(13)(14)(6)(13)(14)..., we have the desired exact sequence. □

The morphisms ∂_n^j induced in Theorem 4.5 are called the *j-th n-connecting morphisms*. Specially, ∂_n is called the *n-connecting morphism*.

COROLLARY 4.6

Let $\mathcal{A}$ be a small n-abelian category which has enough projective objects, $\mathcal{P}$ the category of projective objects in $\mathcal{A}$. If $\mathsf{nExt}^1_{\mathcal{A}}(A, B) = 0$, then every n-exact sequence starting at A and ending with B is contractible.

Proof. Given an n-exact sequence $\xi : B \overset{d^0_\xi}{\to} X^1 \to \cdots \to X^n \to A$. By Theorem 4.5, there is an exact sequence of groups

$$\begin{aligned}&0 \to \mathcal{A}(A, B) \to \mathcal{A}(X^n, B) \to \cdots \to \mathcal{A}(X^1, B)\\ &\quad \to \mathcal{A}(B, B) \xrightarrow{\partial_n} \mathsf{nExt}^1_{\mathcal{A}}(A, B) = 0.\end{aligned}$$

So the identity morphism 1_B lifts to a morphism $\sigma : X^1 \to B$ and $\sigma d^0_\xi = 1_B$. This proves that ξ is a contractible n-exact sequence. □

COROLLARY 4.7

Let $\mathcal{A}$ be an n-abelian category which has enough projectives. The following are equivalent:

(*i*) *A is a projective object.*
(*ii*) *$\mathcal{A}(A, -)$ is an n-exact functor.*
(*iii*) $\mathsf{nExt}^i_{\mathcal{A}}(A, B) = 0$ *for all* $i \neq 0$ *and all B.*
(*iv*) $\mathsf{nExt}^1_{\mathcal{A}}(A, B) = 0$ *for all B.*

Proof. It is clear that (i) ⇔ (ii), (iii) ⇒ (iv). (i) ⇒ (iii), (iv) holds by Lemma 4.1(iii). If we are giving an n-resolution $P_\bullet \to A$, applying $\mathcal{A}(-, B)$, we have

$$0 \to \mathcal{A}(A, B) \to \mathcal{A}(P_1, B)$$
$$\to \cdots \to \mathcal{A}(P_n, B) \xrightarrow{d^*_{n+1}} \mathcal{A}(P_{n+1}, B) \xrightarrow{d^*_{n+2}} \mathcal{A}(P_{n+2}, B) \cdot$$
$$\mathcal{A}(P_n, B) \xrightarrow{i^*} \mathcal{A}(\Omega_n A, B) \xrightarrow{\pi^*} \mathcal{A}(P_{n+1}, B)$$

i^* is surjective since the upper row is an exact sequence and π^* is an injective. Let $B = \Omega_n A$, we have that $i : \Omega_n A \to P_n$ is a split monomorphism. It follows that A is a direct summand of P_1 by [13, Proposition 2.6], so (iv) ⇒ (i). □

By the definition of n-derived functors, we have

PROPOSITION 4.8 (*Dimension shifting theorem*)

Let $\mathcal{A}$ be an n-abelian category which has enough projectives and enough injectives. Then, for $i \geq 1$, $j \geq 0$, *we have*

(*i*) *If G is a covariant left n-exact functor* $\mathsf{nR}^i G(\Omega_n^{-j} A) \simeq \mathsf{nR}^{i+j} G(A)$.
(*ii*) *If G is a contravariant left n-exact functor*, $\mathsf{nR}^i G(\Omega_n^j A) \simeq \mathsf{nR}^{i+j} G(A)$.
(*iii*) *If G is a covariant right n-exact functor*, $\mathsf{nL}^i G(\Omega_n^j A) \simeq \mathsf{nL}^{i+j} G(A)$.
(*iv*) *If G is a contravariant right n-exact functor*, $\mathsf{nL}^i G(\Omega_n^{-j} A) \simeq \mathsf{nL}^{i+j} G(A)$.
(*v*) *Specially*,

$$\mathsf{nExt}^i_{\mathcal{A}}(\Omega_n^j A, B) \simeq \mathsf{nExt}^{i+j}_{\mathcal{A}}(A, B) \simeq \mathsf{nExt}^i_{\mathcal{A}}(A, \Omega_n^{-j} B).$$

4.3 *n-Homological dimension*

DEFINITION 4.9

Let $\mathcal{A}$ be an n-abelian category which has enough projective objects, $M \in \mathcal{A}$. The *n-projective dimension* $\mathsf{npd}\, M$ is the minimum integer m (if it exists) such that there is a projective n-resolution of M,

$$\cdots \to 0 \to P_{mn+1} \to P_{mn} \to \cdots \to P_2 \to P_1 \to M. \tag{15}$$

The notion of *n-injective dimension* $\mathsf{nid}\, M$ is defined dually.

PROPOSITION 4.10

The following are equivalent for any n-abelian category $\mathcal{A}$.

(*i*) $\mathsf{npd}A \le m$.
(*ii*) $\mathsf{nExt}^d_{\mathcal{A}}(A, B) = 0$ *for all* $d > m$ *and all* $B \in \mathcal{A}$.
(*iii*) $\mathsf{nExt}^{m+1}_{\mathcal{A}}(A, B) = 0$ *for all* $B \in \mathcal{A}$.
(*iv*) *If* $A' \to P_{mn} \to \cdots \to P_1 \to A$ *is any m-fold n-exact sequence with P_i projective, then A' is also projective.*

Proof. Since $\mathsf{nExt}^*(A, B)$ may be computed using a projective n-resolution of A, it is clear that (iv) $\Rightarrow$ (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). If we give a projective n-resolution as (iv), then $\mathsf{nExt}^{m+1}(A, B) \simeq \mathsf{nExt}^1(A', B)$ by Proposition 4.8, and A' is projective if and only if $\mathsf{nExt}^1(A', B) = 0$ for all $B \in \mathcal{A}$ by Corollary 4.7, so (iii) $\Rightarrow$ (iv). □

5. *n*-Extensions and *m*-fold *n*-extensions

We assume that m, n be two positive integers in the rest of this section.

In any abelian category $\mathcal{A}$, we can define $\mathsf{Ext}^1_{\mathcal{A}}(A, B)$ even if it has no projectives and no injectives, to be the set of equivalence classes of extensions under Baer sum (R. Baer in 1934) by using pushout and pullback [19, Definition 3.4.4]. Baer's description of $\mathsf{Ext}^1_{\mathcal{A}}(A, B)$ as extension $E(A, B)$ has been generalized by Yoneda [15] to a description of $\mathsf{Ext}^m_{\mathcal{A}}(A, B)$ for all $m \ge 1$. Elements of Yonedas $\mathsf{Ext}^m_{\mathcal{A}}(A, B)$ are a certain equivalence classes of m-fold exact sequences and add them by a generalized Baer sum ([5], [14, p. 82–p. 87], [19, Vista 3.4.6], [15,16]).

In this section, let $\mathcal{A}$ be an n-abelian category which has enough projectives. We prove

$$\mathsf{nE}^m(A, B) \simeq \mathsf{nExt}^m_{\mathcal{A}}(A, B)$$

under n-Baer sum where $\mathsf{nE}^m(A, B)$ is the equivalence class of *m-fold n-extensions* of A by B.

5.1 *n-Extension groups*

DEFINITION 5.1

Let $\mathcal{A}$ be an n-abelian category, $A, B \in \mathcal{A}$. An *n-extension* ξ of A by B is an n-exact sequence $B \to X^1 \to \cdots \to X^n \to A$ in $\mathcal{A}$. And two n-extensions ξ, ξ' of A by B are *equivalent* if there is a commutative diagram

$$\begin{array}{lccccccccccc} \xi: & B & \xrightarrow{d^0_\xi} & X^1 & \longrightarrow & \cdots & \longrightarrow & X^{n-1} & \longrightarrow & X^n & \longrightarrow & A \\ & \| & & \downarrow f^1 & & & & \downarrow f^{n-1} & & \downarrow f^n & & \| \\ \xi': & B & \xrightarrow{d^0_{\xi'}} & Y^1 & \longrightarrow & \cdots & \longrightarrow & Y^{n-1} & \longrightarrow & Y^n & \longrightarrow & A \end{array} . \tag{16}$$

ξ is *contractible* if it is equivalent to an n-exact sequence ξ' with $d^0_{\xi'}$ a split monomorphism. We simplify the 0 n-exact sequence $0 \to 0 \to \cdots \to 0$ by 0 if no confusion appears.

Thus $d^0_{\xi'}$ is a split monomorphism if and only if ξ' is a contractible n-exact sequence [13, Proposition 2.6]. The following lemma shows the equivalent to be an equivalence relation.

Lemma 5.2 (*Proposition* 4.10 *of* [13]). *Let $\mathcal{A}$ be an n-abelian category. If there exists an equivalence of n-exact sequences $f : \xi \to \xi'$, then there exists an equivalence of n-exact sequences $g : \xi' \to \xi$ such that f and g are mutually inverse isomorphisms in $\mathsf{H}(\mathcal{A})$.*

We denote the equivalence class of an n-exact sequence ξ by $[\xi]$, and we define

$$\mathsf{nE}(A, B) = \{[\xi] | \xi \text{ is an } n\text{-extension of } A \text{ by } B\} \quad \text{and} \quad \xi \equiv \xi' \text{ if } [\xi] = [\xi'].$$

Before giving the additive group structure of $\mathsf{nE}(A, B)$ with addition of n-Baer sum, we give some properties of the equivalence classes of n-exact sequences.

PROPOSITION 5.3

Let $\mathcal{A}$ be an n-abelian category, $\xi : \ B \overset{d^0_\xi}{\to} B^1 \overset{d^1_\xi}{\to} \cdots \overset{d^{n-1}_\xi}{\to} B^n \overset{d^n_\xi}{\to} A$ an n-extension of A by B and $f^0 : B \to C$ a morphism in $\mathcal{A}$. Taking an n-pushout along f^0, by Proposition 2.7, there is a morphism between n-exact sequences

$$\begin{array}{ccccccccccc}
\xi : & & B & \xrightarrow{d^0_\xi} & B^1 & \xrightarrow{d^1_\xi} & \cdots & \longrightarrow & B^n & \xrightarrow{d^n_\xi} & A \\
\downarrow f & & \downarrow f^0 & & \downarrow f^1 & & & & \downarrow f^n & & \| \\
\xi_{\text{po}} : & & C & \xrightarrow{d^0_{\xi_{\text{po}}}} & T^1 & \xrightarrow{d^1_{\xi_{\text{po}}}} & \cdots & \longrightarrow & T^n & \xrightarrow{d^n_{\xi_{\text{po}}}} & A
\end{array}.$$

Then, $[\xi_{\text{po}}]$ is unique as determined by $[\xi]$ and f^0.

Proof. Let $g : \xi' \to \xi$ be an equivalence and $t : \xi' \to \xi'_{\text{po}}$ is a morphism induced by n-pushout of ξ along $f^0 : B \to C$. By [13, Proposition 2.13] of n-pushout, there exists a morphism

$$\begin{array}{ccccccccccc}
\xi'_{\text{po}} : & & C & \xrightarrow{d^0_{\xi'_{\text{po}}}} & T'^1 & \xrightarrow{d^1_{\xi'_{\text{po}}}} & \cdots & \longrightarrow & T'^n & \xrightarrow{d^n_{\xi'_{\text{po}}}} & A \\
\downarrow p & & \| & & \downarrow p^1 & & & & \downarrow p^n & & \\
\xi_{\text{po}} : & & C & \xrightarrow{d^0_{\xi_{\text{po}}}} & T^1 & \xrightarrow{d^1_{\xi_{\text{po}}}} & \cdots & \longrightarrow & T^n & \xrightarrow{d^n_{\xi_{\text{po}}}} & A
\end{array}$$

such that there exists homotopy $h : fg \to pt$ with $h^1 = 0$. We have

$$\begin{aligned}
(d^n_{\xi'_{\text{po}}} - d^n_{\xi_{\text{po}}} p^n) t^n &= d^n_{\xi'} - d^n_{\xi_{\text{po}}} (f^n g^n + d^{n-1}_{\xi_{\text{po}}} h^n) = d^n_{\xi'} - d^n_\xi g^n = 0, \\
(d^n_{\xi'_{\text{po}}} - d^n_{\xi_{\text{po}}} p^n) d^{n-1}_{\xi'_{\text{po}}} &= 0,
\end{aligned}$$

but $(t^n \ d^{n-1}_{\xi'_{\text{po}}})$ is an epimorphism, so that $d^n_{\xi'_{\text{po}}} = d^n_{\xi_{\text{po}}} p^n$. This proves $[\xi_{\text{po}}] = [\xi'_{\text{po}}]$. □

The following property is a generalization of [14, Lemma 1.3].

PROPOSITION 5.4

Let $\mathcal{A}$ be an n-abelian category. A morphism $f : X \to Y$ of two n-exact sequences X, Y factors over an n-exact sequence Z,

$$\begin{array}{ccccccccccc}
X & & X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & \cdots & \longrightarrow & X^n & \xrightarrow{d_X^n} & X^{n+1} \\
\downarrow g & & \downarrow f^0 & & \downarrow g^1 & & & & \downarrow g^n & & \| \\
Z & f & Y^0 & \xrightarrow{d_Z^0} & Z^1 & \xrightarrow{d_Z^1} & \cdots & \longrightarrow & Z^n & \xrightarrow{d_Z^n} & X^{n+1} \\
\downarrow p & & \| & & \downarrow p^1 & & & & \downarrow p^n & & \downarrow f^{n+1} \\
Y & & Y^0 & \xrightarrow{d_Y^0} & Y^1 & \xrightarrow{d_Y^1} & \cdots & \longrightarrow & Y^n & \xrightarrow{d_Y^n} & Y^{n+1}
\end{array}$$

in such a way that the upper-left n-squares and lower right n-squares are n-exact diagrams. $[Z]$ does not depend on the choices of n-extensions in the equivalence classes $[X]$, $[Y]$.

Proof. Taking a good n-pushout of X along f;

$$\begin{array}{ccccccccccc}
X & & X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & \cdots & \longrightarrow & X^n & \xrightarrow{d_X^n} & X^{n+1} \\
\downarrow g & & \downarrow f^0 & & \downarrow g^1 & & & & \downarrow g^n & & \| \\
Z & & Y^0 & \xrightarrow{d_Z^0} & Z^1 & \xrightarrow{d_Z^1} & \cdots & \longrightarrow & Z^n & \overset{d_Z^n}{\dashrightarrow} & X^{n+1}
\end{array} .$$

Then, by Proposition 2.7, there exists a cokernel $d_Z^n : Z^n \to X^{n+1}$ such that Z is an n-exact sequence. It is an n-exact diagram since d_X^0 is a monomorphism. By the definition of the good n-pushout, there exists a morphism $p : Z \to Y$ such that $f = pg$. The lower right n-squares is an n-exact diagram by the dual of [13, Proposition 4.8] since d_Y^n is an epimorphism.

The last sentence follows from Proposition 5.3. □

Notation 5.5. Denote the n-exact sequence ξ_{po} in Proposition 5.3 by $f \cdot \xi$. By Proposition 5.3, $[f \cdot \xi]$ does not depend on the choice of n-pushout of ξ along f and nor on the choice of n-extensions in the equivalence class $[\xi]$. Thus we can define $f \cdot [\xi] := [f \cdot \xi]$. The notions $\xi \cdot g$, $[\xi] \cdot g := [\xi] \cdot g$ for $g : A' \to A$ are defined dually by the dual of Proposition 5.3.

The following proposition shows that any n-pushout (resp., n-pullback) along the first (resp., last) morphism of an n-exact sequence yields a contractible n-exact sequence, which is a generalization of [14, Lemma 1.7].

Lemma 5.6. *Let $\mathcal{C}$ be an additive category, $\xi : X^0 \overset{d^0}{\to} X^1 \to \cdots \to X^n \overset{d^n}{\to} X^{n+1}$ an n-exact sequence. Then $d^0 \cdot \xi$ and $\xi \cdot d^n$ are contractible.*

Proof. The diagram

$$\begin{array}{ccccccccc}
X^0 & \xrightarrow{d^0} & X^1 & \xrightarrow{d^1} & X^2 & \xrightarrow{d^2} & \cdots & \longrightarrow & X^{n+1} \\
\downarrow d^0 & & \downarrow (1\ d^1)^T & & \| & & & & \| \\
X^1 & \xrightarrow{(1\ 0)^T} & X^1 \oplus X^2 & \xrightarrow{(0\ 1)} & X^2 & \xrightarrow{d^2} & \cdots & \longrightarrow & X^{n+1}
\end{array}$$

is commutative. Then, the properties follow from Proposition 5.3 and its dual. □

PROPOSITION 5.7

Let $\mathcal{A}$ be an n-abelian category, $\xi : B \to X^1 \to \cdots \to X^n \to A$ an n-extension of A by B. Then, for any morphisms $B \xrightarrow{f} B' \xrightarrow{f'} B''$, and any morphism $A'' \xrightarrow{g'} A' \xrightarrow{g} A$, we have

(*i*) $(f'f) \cdot [\xi] = f' \cdot [f \cdot \xi] = [f' \cdot (f \cdot \xi)]$.
(*ii*) $[\xi] \cdot (gg') = [\xi \cdot g] \cdot g' = [(\xi \cdot g) \cdot g']$.
(*iii*) $f \cdot [\xi] \cdot g := (f \cdot [\xi]) \cdot g = f \cdot ([\xi] \cdot g)$.

Proof. (i), (ii) follow immediately from Proposition 2.4,2.7,5.3 and their dual. For (iii), we only need to show $(f \cdot \xi) \cdot g \equiv f \cdot (\xi \cdot g)$ by Proposition 5.3 and its dual. There is a morphism $(f, f_1, \ldots, f_n, g) : \xi \cdot g \to f \cdot \xi$. By Proposition 5.4, $(f, f_1, \ldots, f_n, g)$ has factorizations

$$\xi \cdot g \to (f \cdot \xi) \cdot g \to f \cdot \xi \quad \text{or} \quad \xi \cdot g \to f \cdot (\xi \cdot g) \to f \cdot \xi$$

By Proposition 5.3, $(f \cdot \xi) \cdot g \equiv f \cdot (\xi \cdot g)$. □

For any object $M \in \mathcal{A}$, the diagonal map is

$$\nabla_M = \begin{pmatrix} 1_M & 1_M \end{pmatrix} : M \oplus M \to M$$

and the sum map is

$$\Delta_M = \begin{pmatrix} 1_M & 1_M \end{pmatrix}^T : M \to M \oplus M.$$

Note that the fact that the equivalent class $[(\nabla(\xi \oplus \xi'))\Delta]$ is well defined does not depend on the choices of the representable elements of equivalent classes of $[\xi]$ and $[\xi']$, and does not depend on the choices of the n-pushout and n-pullback by Proposition 5.7(iii).

Theorem 5.8. *Let $\mathcal{A}$ be an n-abelain category, A, B two objects of $\mathcal{A}$. Then $n\mathsf{E}(A, B)$ is an abelian group under n-Baer sum*

$$[\xi] + [\xi'] = [(\nabla(\xi \oplus \xi'))\Delta]$$

with zero being the class of contractible n-extensions, the inverse of any $[\xi]$ is the n-extension $(-1_B) \cdot [\xi]$.

For homomorphisms $\alpha, \alpha_i : A \to A'$ and $\gamma, \gamma_i : B \to B'$ for $i = 1, 2$, we have

$$\begin{aligned} ([\xi] + [\xi']) \cdot \alpha &= [\xi] \cdot \alpha + [\xi'] \cdot \alpha, \qquad \gamma \cdot ([\xi] + [\xi']) \\ &= \gamma \cdot [\xi] + \gamma \cdot [\xi'], \\ [\xi] \cdot (\alpha_1 + \alpha_2) &= [\xi] \cdot \alpha_1 + [\xi] \cdot \alpha_2, \quad (\gamma_1 + \gamma_2) \cdot [\xi] \\ &= \gamma_1 \cdot [\xi] + \gamma_2 \cdot [\xi]. \end{aligned}$$

Proof. The proof of this theorem is same as [14, Theorem 2.1]. □

Theorem 5.9. *Let $\mathcal{A}$ be an n-abelian category which has enough projectives, $A, B \in \mathcal{A}$. There is a group isomorphism of abelian groups*

$$\Theta : n\mathsf{E}(A, B) \to n\mathsf{Ext}^1_{\mathcal{A}}(A, B)$$

in which the contractible n-extensions correspond to the element $0 \in n\mathsf{Ext}^1_{\mathcal{A}}(A, B)$.

Proof. See Theorem 5.17. □

5.2 *m-Fold n-extension groups*

In the rest of this section, we show that $\mathsf{nE}^m(A, B) \simeq \mathsf{nExt}_{\mathcal{A}}^m(A, B)$, where $\mathsf{nE}^m(A, B)$ is the equivalent classes of m-fold n-extensions of A by B.

Let $\mathcal{A}$ be an n-abelian category, and

$$\xi : C \to E_{2n} \to \cdots \to E_{n+1} \to B,$$

and

$$\xi' : B \to E_n \to \cdots \to E_1 \to A$$

are two n-exact sequences of $\mathcal{A}$. Then, splice them together as

$$\xi \circ \xi' : C \to E_{2n} \to \cdots \to E_{n+1} \to E_n \to \cdots \to E_1 \to A$$

$\xi \circ \xi'$ may be false, a $2n$-exact sequence. We call a sequence

$$\begin{aligned} S:\ & A_m \to E_{mn} \xrightarrow{d_{mn}} \cdots \to E_{(m-1)n+1} \xrightarrow{d_{(m-1)n+1}} \\ & \cdots \xrightarrow{d_{n+1}} E_n \xrightarrow{d_n} \cdots \to E_1 \xrightarrow{d_1} A_0 \end{aligned} \tag{17}$$

an *m-fold n-exact sequence* starting at A_m and ending at A_0 if d_{in+1} can be written as a composite $E_{in+1} \to A_i \to E_{in}$ for each $i \in \{1, 2, \ldots, m-1\}$ such that $\xi_j : A_j \to E_{jn} \to \cdots \to E_{(j-1)n+1} \to A_{j-1}$ are n-exact sequences for $j \in \{1, 2, \ldots, m\}$. Conventionally, we write $S = \xi_m \circ \xi_{m-1} \circ \cdots \circ \xi_1$. Similarly, we write

$$S_d \circ S_{d-1} \circ \cdots \circ S_1$$

for n_i-fold n-exact sequences S_i.

Remark 5.10. We cannot see that d_{in+1} is a weak kernel(resp. weak cokernel) of d_{in}(resp. d_{in+2}) for any i. But, it is easily to see that, d_{in} is a weak cokernel of d_{in+1}, d_{in+2} is a weak kernel of d_{in+1}.

DEFINITION 5.11

Let $\mathcal{A}$ be an n-abelian category. An *m-fold n-extension* of A_0 by A_m is an m-fold n-exact sequence (17) in $\mathcal{A}$. The m-fold n-extension S of A_0 by A_m is *similar* to the m-fold n-extension S' if there is a commutative diagram

$$\begin{array}{lccccccccc} S: & A_m & \longrightarrow & E_{mn} & \longrightarrow & \cdots & \longrightarrow & E_1 & \longrightarrow & A_0 \\ & \| & & \downarrow{\scriptstyle f_{mn}} & & & & \downarrow{\scriptstyle f_1} & & \| \\ S': & A_m & \longrightarrow & E'_{mn} & \longrightarrow & \cdots & \longrightarrow & E'_1 & \longrightarrow & A_0 \end{array} .$$

The m-fold n-exact sequence S is *equivalent* to S' or $S \equiv_m S'$ if there exists a finite sequences of m-fold n-exact sequences $S_0, S_1, \ldots, S_r$ such that $S = S_0$, $S' = S_r$ and S_i is similar to S_{i+1} or S_{i+1} is similar to S_i for $i = 0, 1, \ldots, r-1$. Specially, we write $\equiv$ for $\equiv_1$.

It is easy to see that '$\equiv_m$' is an equivalence relation. We call S a *contractible* m-fold n-exact sequence if S is equivalent to $\xi_m \circ 0 \circ \cdots \circ 0 \circ \xi_1$ with $\xi_m : A_m \xrightarrow{1_{A_m}} A_m \to 0 \to \cdots \to 0$, $\xi_1 : 0 \to \cdots \to 0 \to A_0 \xrightarrow{1_{A_0}} A_0$ for $m \geq 2$.

We denote the equivalence class of an m-fold n-exact sequence S by $[S]$, and we define

$$\mathsf{nE}^m(A, B) = \{[S] | S \text{ is an } m\text{-fold } n\text{-extension of } A \text{ by } B\}.$$

There are some basic properties of m-fold n-exact sequences. We list them in the following lemma without proof.

Lemma 5.12. *Let $\mathcal{A}$ be an n-abelian category.*

(*i*) *For any n-extensions $\xi : A \to E_{2n} \to \cdots \to E_{n+1} \to C$ and $\xi' : D \to E_n \to \cdots \to E_1 \to B$, and for any morphism $\sigma : D \to C$, we have*

$$\xi\sigma \circ \xi' \equiv_2 \xi \circ \sigma\xi'.$$

(*ii*) *Let $S = \xi_m \circ \xi_{m-1} \circ \cdots \circ \xi_1$ be an m-fold n-exact sequence. Assume $\xi_i \circ \cdots \circ \xi_{i-j} \equiv_{j+1} \mu_i \circ \cdots \circ \mu_{i-j}$ for some $i = 1, 2, \ldots, m-1$, $i > j \geq 0$. Then $S \equiv_m \xi_m \circ \cdots \circ \xi_{i+1} \circ \mu_i \circ \cdots \circ \mu_{i-j} \circ \xi_{i-j-1} \circ \cdots \circ \xi_1$*

(*iii*) *Each morphism $\gamma : S \to S'$ of m-fold n-extensions starting at α and ending with β yields $\alpha S \equiv_m S'\beta$.*

Similar to Notation 5.5, we define

$$\begin{aligned} \alpha \cdot [S] &:= [\alpha \cdot S] = [\alpha \cdot (\xi_m \circ \xi_{m-1} \circ \cdots \circ \xi_1)] \\ &:= [(\alpha \cdot \xi_m) \circ \xi_{m-1} \circ \cdots \circ \xi_1], \qquad (18) \\ [S] \cdot \gamma &:= [S \cdot \gamma] = [(\xi_m \circ \xi_{m-1} \circ \cdots \circ \xi_1) \cdot \gamma] \\ &:= [\xi_m \circ \cdots \circ \xi_2 \circ (\xi_1 \cdot \gamma)]. \qquad (19) \end{aligned}$$

Similar to Proposition 5.7, we have

PROPOSITION 5.13

Let $\mathcal{A}$ be an n-abelian category, S an m-fold n-extension of A by B. Then, for any morphisms $B \xrightarrow{f} B' \xrightarrow{f'} B''$, and any morphism $A'' \xrightarrow{g'} A' \xrightarrow{g} A$, we have

(*i*) $(f'f) \cdot [S] = f' \cdot [f \cdot S] = [f' \cdot (f \cdot S)]$.
(*ii*) $[S] \cdot (gg') = [S \cdot g] \cdot g' = [(S \cdot g) \cdot g']$.
(*iii*) *If $S : B \xrightarrow{i} E_{mn} \xrightarrow{d_{mn}} \cdots \xrightarrow{d_2} E_1 \xrightarrow{\pi} A$, then $i \cdot S$ and $S \cdot \pi$ are contractible.*
(*iv*) $f \cdot [S] \cdot g := (f \cdot [S]) \cdot g = f \cdot ([S] \cdot g)$.
(*v*) *For $[R] \in \mathsf{nE}^k(C, A)$, $[T] \in \mathsf{nE}^l(B, M)$,*

$$[T] \circ [S] = [T \circ S], \quad ([T] \circ [S]) \circ [R] = [T] \circ ([S] \circ [R]).$$

Proof.

(i), (ii) follow immediately from Proposition 5.7.

(iii) By Lemma 5.6, there is a morphism $S \to i \cdot S$ and an equivalence $i \cdot S \equiv_m S'$ in the following diagram:

$$\begin{array}{ccccccccccccccc}
S: & & B & \xrightarrow{i} & E_{mn} & \xrightarrow{d_{mn}} & E_{mn-1} & \xrightarrow{d_{mn-1}} & E_{mn-2} & \longrightarrow & \cdots & \longrightarrow & E_1 & \xrightarrow{\pi} & A \\
\downarrow & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle (1\ d_{mn})^T} & & \| & & \| & & & & \| & & \| \\
i \cdot S: & & E_{mn} & \xrightarrow{(1\ 0)^T} & E_{mn} \oplus E_{mn-1} & \xrightarrow{(0\ 1)} & E_{mn-1} & \xrightarrow{d_{mn-1}} & E_{mn-2} & \longrightarrow & \cdots & \longrightarrow & E_1 & \xrightarrow{\pi} & A \\
\downarrow{\scriptstyle \equiv_m} & & \| & & \downarrow{\scriptstyle (1\ 0)} & & \downarrow & & \downarrow & & & & \downarrow{\scriptstyle \pi} & & \| \\
S': & & E_{mn} & \xrightarrow{1} & E_{mn} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & A & \xrightarrow{1_A} & A
\end{array}$$

so $i \cdot S$ is contractible, and $S \cdot \pi$ is contractible by dual.
(iv) If $m = 1$, it is trivial by Proposition 5.7. If $m \geq 2$, $(f \cdot S) \cdot g \equiv_m f \cdot (S \cdot g)$ follows from Proposition 5.12 and there is a map $S \cdot g \to f \cdot S$ starting at f and ending with g. If $S \equiv_m S'$, by (18), we have $S' \cdot g \equiv_m S \cdot g$ and $f \cdot S \equiv_m f \cdot S'$. Then, by (18),

$$f \cdot (S \cdot g) \equiv_m (f \cdot S) \cdot g \equiv_m (f \cdot S') \cdot g.$$

(v) The first formula follows from Lemma 5.12(ii). The second formula follows from the first. □

Proposition 5.13 shows that $[(\nabla(S \oplus S'))\Delta]$ is well defined for any equivalence classes $[S], [S'] \in \mathsf{nE}^m(A, B)$.

Theorem 5.14. *Let $\mathcal{A}$ be an n-abelain category, A, B be two objects of $\mathcal{A}$. Then $\mathsf{nE}^m(A, B)$ is an abelian group under n-Baer sum*

$$[S] + [S'] = [(\nabla(S \oplus S'))\Delta]$$

with zero being the class of contractible m-fold n-extensions, the inverse of any $[S]$ is the n-extension $(-1_B) \cdot [S]$.

For $[R] \in \mathsf{nE}^k(C, A)$, $[T] \in \mathsf{nE}^l(B, M)$,

$$([S] + [S']) \circ [R] = [S] \circ [R] + [S'] \circ [R], \tag{20}$$
$$[T] \circ ([S] + [S']) = [T] \circ [S] + [T] \circ [S']. \tag{21}$$

Proof. The proof of this theorem is same as [14, Theorem 5.3]. □

5.3 *m-Fold n-extensions and nExt^m*

In the rest of this section, we simplify $\mathcal{A}(f, M)$ by f^* for any morphism f.

Let

$$S: \ B = A_m \to E_{mn} \xrightarrow{\alpha_{mn}} \cdots \to E_{(m-1)n+1} \xrightarrow{\alpha_{(m-1)n+1}} \cdots \xrightarrow{\alpha_{n+1}} E_n \xrightarrow{\alpha_n} \cdots \xrightarrow{\alpha_2} E_1 \xrightarrow{\alpha_1} A_0 = A$$

be an m-fold n-extension of A by B with $m \geq 1$, taking an n-resolution $P_\bullet \to A$ of A. Then we have a commutative diagram lifting 1_A,

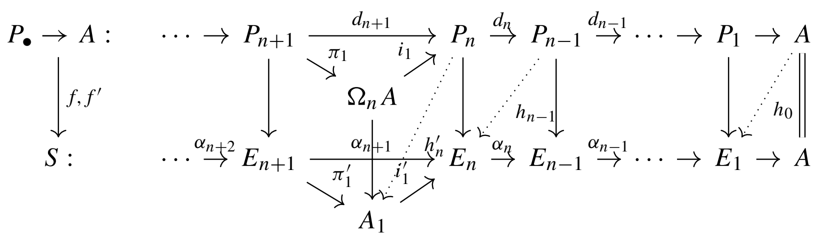
$$\begin{array}{lccccccccc}
P_\bullet \to A: & \cdots \xrightarrow{d_{mn+2}} & P_{mn+1} & \xrightarrow{d_{mn+1}} & P_{mn} & \xrightarrow{d_{mn}} \cdots \longrightarrow & P_1 & \xrightarrow{d^1} & A \\
\quad \downarrow f & & \downarrow f_{mn+1} & & \downarrow f_{mn} & & \downarrow f_1 & & \| \\
S: & & B & \xrightarrow{\alpha_{mn+1}} & E_{mn} & \xrightarrow{\alpha_{mn}} \cdots \longrightarrow & E_1 & \xrightarrow{\alpha_1} & A
\end{array}.$$

We have $f_{mn+1}d_{mn+2} = 0$. Applying $\mathcal{A}(-, B)$ to the diagram, define

$$\Theta_m : \mathsf{nE}^m(A, B) \to \mathsf{nExt}^m_{\mathcal{A}}(A, B) \text{ by } [S] \mapsto f^*_{mn+1}(1_B) + \operatorname{Im} d^*_{mn+1}.$$

Lemma 5.15. *Θ_m is well-defined.*

Proof. We show that Θ_m does not depend on the chain map $f : (P_\bullet \to A) \to S$ nor on the choice of extensions in the equivalence class $[S]$. Since $f_{mn+1}d_{mn+2} = 0$, $f_{mn+1} \in \operatorname{Ker} d_{mn+2}$. Replace f by any other chain map $f' : (P_\bullet \to A) \to S$ lifting 1_A,



There exists a morphism $g_1, g'_1 : \Omega_n A \to A_1$ lifting 1_A such that the above diagram commute (solid line). Comparison lemma 2.1 says that there exist $h_0, h_1, \ldots, h_{n-1}, h'_n$ in the above diagram such that $g_1 - g'_1 = h'_n i_1$. Since P_n is projective, there is a morphism $h_n : P_n \to E_{n+1}$ with $h'_n = \pi'_1 h_n$. Then, we have

$$\begin{aligned}
f_n - f'_n &= i'_1 h'_n + h_{n-1}d_n = i'_1\pi'_1 h_n + h_{n-1}d_n \\
&= \alpha_{n+1}h_n + h_{n-1}d_n \\
\alpha_{n+1}[(f_{n+1} - f'_{n+1}) - h_n d_{n+1}] &= (f_n - f'_n)d_{n+1} \\
&\qquad - (f_n - f'_n - h_{n-1}d_n)d_n + 1 = 0.
\end{aligned}$$

Then there is a morphism $h_{n+1} : P_{n+1} \to E_{n+2}$ such that $f_{n+1} - f'_{n+1} = \alpha_{n+2}h_{n+1} - h_n d_{n+1}$. Inductively, we can construct morphisms $h_i : P_i \to E_{i+1}$ for $i \in \{0, 1, \ldots, h_{mn}\}(h_0 = 0$ and $E_{mn+1} := B)$ such that $f_j - f'_j = \alpha_{j+1}h_j - h_{j-1}d_j$ for $j \in \{1, 2, \ldots, mn\}$. Then, we have

$$\begin{aligned}
&\alpha_{mn+1}[(f_{mn+1} - f'_{mn+1}) - h_{mn}d_{mn+1}] = (f_{mn} - f'_{mn})d_{mn+1} \\
&\quad - [(f_{mn} - f'_{mn}) - h_{mn-1}d_{mn}]d_{mn+1} = 0.
\end{aligned}$$

This implies $f_{mn+1} - f'_{mn+1} = h_{mn}d_{mn+1}$ since α_{mn+1} is a monomorphism. It follows that $f^*_{mn+1}(1_B) + \operatorname{Im} d^*_{mn+1} = f'^*_{mn+1}(1_B) + \operatorname{Im} d^*_{mn+1}$.

Next, replace S by any equivalent m-fold n-exact sequence S'. It suffice to consider the case when there is a morphism $\gamma : S \to S'$ and the case when there is a morphism $\beta : S' \to$

S, where γ and β are morphisms starting and ending with 1. If there is a morphism $\gamma = (1_B, k_{mn}, \ldots, k_1, 1_A) : S \to S'$, we have $[S'] \mapsto f^*_{mn+1}(1_B) + \operatorname{Im} d^*_{mn+1}$ by the definition of Θ_m. Thus Θ_m is well defined. If there is a morphism $\beta = (1_B, \beta_{mn}, \ldots, \beta_1, 1_A) : S' \to S$, there is a chain map $\phi : (P_\bullet \to A) \to S$ lifting 1_A. By comparison lemma and the above paragraph, there exists a morphism $h_{mn} : P_{mn} \to B$ such that $\phi_{mn+1} - f_{mn+1} = h_{mn} d_{mn+1}$. This proves $\phi^*_{mn+1} + \operatorname{Im} d^*_{mn+1} = f^*_{mn+1} + \operatorname{Im} d^*_{mn+1}$. □

Lemma 5.16. *Let $\mathcal{A}$ be an n-abelain category, $[S], [R] \in n\mathsf{E}^m(A, B)$, $[S'] \in n\mathsf{E}^m(C, D)$, $h \in \mathcal{A}(B, B')$. Then we have*

(*i*) $\Theta_m[S \oplus S'] = \Theta_m[S] \oplus \Theta_m[S]$;
(*ii*) $\Theta_m[h \cdot S] = h\Theta_m[S]$;
(*iii*) $\Theta_m([S \oplus R] \cdot \Delta_A) = \Theta_m([S \oplus R]) \cdot \Delta$.

Proof.

(i) Let $P_\bullet \to A$, $P'_\bullet \to C$ be projective n-resolutions, and $f : (P_\bullet \to A) \to S$, $f' : (P'_\bullet \to C) \to S'$ are chain maps lifting 1_A and 1_C respectively. Then $P_\bullet \oplus P'_\bullet \to A \oplus C$ is a projective n-resolution, $f \oplus f'$ is a chain map lifting $1_{A\oplus C}$.
(ii) Let $P_\bullet \to A$ be a projective n-resolution, and $f : (P_\bullet \to A) \to S$ a chain map lifting 1_A. There is a chain map $\overline{h} := (h, \ldots, 1) : S \to h \cdot S$. Then we have $\overline{h} f : P_\bullet \to h \cdot S$. This implies $\Theta_m[h \cdot S] = h\Theta_m[S]$.
(iii) Let $P_\bullet \to A$ be a projective n-resolution, and $f : (P_\bullet \to A) \to (S \oplus R) \cdot \Delta_A$ a chain map lifting 1_A, $\sigma : (S \oplus R) \cdot \Delta_A \to (S \oplus R)$ induced by the n-pullback of $(S \oplus R)$ along Δ_A. Thus, the left most morphism of σ is $1_{B\oplus B}$. There is a chain map $t : (P_\bullet \oplus P_\bullet \to A \oplus A) \to S \oplus R$ lifting $1_{A\oplus A}$, and there is a diagonal chain map $\Delta_{P_\bullet} : P_\bullet \oplus P_\bullet \to P_\bullet$. Thus we have

$$\Theta_m([S \oplus R] \cdot \Delta_A) = f^*_{mn+1} + \operatorname{Im} d^*_{mn+1}$$

and

$$\begin{aligned}\Theta_m([S \oplus R])\Delta_{P_{mn+1}} &= (t^*_{mn+1} + \operatorname{Im}(d^*_{mn+1} \oplus d^*_{mn+1}))\Delta_{P_{mn+1}} \\ &= (t_{mn+1}\Delta_{P_{mn+1}})^* + \operatorname{Im} d^*_{mn+1}.\end{aligned}$$

By Comparison lemma, $t_{mn+1}\Delta_{P_{mn+1}} - f_{mn+1} \in \operatorname{Im} d^*_{mn+1}$. Thus we have $\Theta_m([S \oplus R] \cdot \Delta_A) = \Theta_m([S \oplus R])\Delta_{P_{mn+1}}$. □

Theorem 5.17. *Let $\mathcal{A}$ be an n-abelain category, A, B two objects of $\mathcal{A}$. Then the map $\Theta_m : n\mathsf{E}^m(A, B) \to n\mathsf{Ext}^m_{\mathcal{A}}(A, B)$ is an isomorphism of abelian groups.*

Proof. The proof of this theorem is an adoption of [14, Proof of Theorem 6.4]. First, Θ_m is a group homomorphism. Indeed, for any $[S], [S'] \in n\mathsf{E}^m(A, B)$,

$$\begin{aligned}\Theta_m([S] + [S']) &= \Theta_m[(\nabla(S \oplus S'))\Delta] = \nabla(\Theta_m[S \oplus S'])\Delta \\ &= \nabla(\Theta_m[S] \oplus \Theta_m[S'])\Delta = \Theta_m[S] + \Theta_m[S'].\end{aligned}$$

Second, we construct the inverse of Θ_m. Given an n-resolution $P_\bullet \to A$, factor $d_{mn+1} : P_{mn+1} \xrightarrow{\pi} \Omega^m A \xrightarrow{i} P_{mn}$, with π the surjection and i the injection. For any f_{mn+1} :

$P_{mn+1} \to B$ such that $f_{mn+1}d_{mn+2} = 0$ but $f_{mn+1} \neq rd_{mn+1}$ $\forall r \in \mathcal{A}(P_{mn}, B)$. There is a morphism $h : \Omega_n^m A \to B$ such that $f_{mn+1} = h\pi$ since π is a cokernel of d_{mn+2}. Taking an n-pushout along h, we have a commutative diagram

$$\begin{array}{lccccccccccc}
\cdots \to P_{mn+2} \to P_{mn+1} & & & & & & & & & & \\
 & \downarrow{\scriptstyle \pi} & \searrow & & & & & & & & \\
S_n: & \Omega_n^m A & \xrightarrow{i} & P_{mn} & \to \cdots \to & P_{(m-1)n+1} & \to & P_{(m-1)n} & \to \cdots \to & A & \cdot \\
 & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f_{mn}} & & \downarrow & & \| & & \| & \\
S: & B & \xrightarrow{\alpha} & E_{mn} & \to \cdots \to & E_{(m-1)n+1} & \to & P_{(m-1)n} & \to \cdots \to & A &
\end{array}$$

Define

$$\Psi_m : \mathsf{nExt}_{\mathcal{A}}^m(A, B) \to \mathsf{nE}^m(A, B) \quad \text{by} \quad f_{mn+1}^* + \operatorname{Im} d_{mn+1}^* \mapsto [S].$$

We show that Ψ_m is well defined. Indeed, let $f_{mn+1}^* + \operatorname{Im} d_{mn+1}^* = g_{mn+1}^* + \operatorname{Im} d_{mn+1}^*$. Then there is a morphism $p : P_{mn} \to B$ such that $f_{mn+1} - g_{mn+1} = pd_{mn+1}$. There is a morphism $k : \Omega_n^m A \to B$ such that $g_{mn+1} = k\pi$ since π is a cokernel of d_{mn+2}. We have $(h - k)\pi = f_{mn+1} - g_{mn+1} = pd_{mn+1} = pi\pi$, so $h - k = pi$. Then, $(h - k) \cdot [S_n] = p \cdot [iS_n]$ is contractible and follows from Proposition 5.13(iii). This proves that $\Psi_m(f_{mn+1}^* + \operatorname{Im} d_{mn+1}^*) = \Psi_m(g_{mn+1}^* + \operatorname{Im} d_{mn+1}^*)$. Next, if $\Psi_m(f_{mn+1}^* + \operatorname{Im} d_{mn+1}^*)$ is a contractible m-fold n-exact sequence, then there is a morphism $\beta : E_{mn} \to B$ such that $\beta\alpha = 1_B$. Then, $f_{mn+1} = \beta f_{mn} d_{mn+1}$, it follows that $f_{mn+1}^* \in \operatorname{Im} d_{mn+1}^*$.

To prove Ψ_m is an isomorphism. It is enough to show that Ψ_m is a homomorphism. Let $\Psi_m(f_{mn+1}^* + \operatorname{Im} d_{mn+1}^*) = [S]$, $\Psi_m(g_{mn+1}^* + \operatorname{Im} d_{mn+1}^*) = [S']$. Then

$$\begin{aligned}
&\Psi_m((f_{mn+1} + g_{mn+1})^* + \operatorname{Im} d_{mn+1}^*) \\
&= \Psi_m[(\nabla(f_{mn+1} \oplus g_{mn+1})\Delta)^* + \operatorname{Im} d_{mn+1}^*] \\
&= \Psi_m[\nabla((f_{mn+1} \oplus g_{mn+1})^* + \operatorname{Im}(d_{mn+1} \oplus d_{mn+1})^*)\Delta] \\
&= \nabla\Psi_m((f_{mn+1} \oplus g_{mn+1})^* + \operatorname{Im}(d_{mn+1} \oplus d_{mn+1})^*)\Delta \\
&= \nabla([S] \oplus [S'])\Delta = [S] + [S'].
\end{aligned}$$

□

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