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On bigraded regularities of Rees algebra

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Abstract. For any homogeneous ideal I in $K[x_1, \ldots, x_n]$ of analytic spread ℓ, we show that for the Rees algebra $R(I)$, $\text{reg}^{\text{syz}}_{(0,1)}(R(I)) = \text{reg}^{\text{T}}_{(0,1)}(R(I))$. We compute a formula for the (0, 1)-regularity of $R(I)$, which is a bigraded analog of Theorem 1.1 of Aramova and Herzog (*Am. J. Math.* **122(4)** (2000) 689–719) and Theorem 2.2 of Römer (*Ill. J. Math.* **45(4)** (2001) 1361–1376) to $R(I)$. We show that if the defect sequence, $e_k := \text{reg}(I^k) - k\rho(I)$, is weakly increasing for $k \geq \text{reg}^{\text{syz}}_{(0,1)}(R(I))$, then $\text{reg}(I^j) = j\rho(I) + e$ for $j \geq \text{reg}^{\text{syz}}_{(0,1)}(R(I)) + \ell$, where $\ell = \min\{\mu(J) \mid J \subseteq I$ a graded minimal reduction of $I\}$. This is an improvement of Corollary 5.9(i) of [16].

Keywords. Castelnuovo–Mumford regularity; syzygy; local cohomology; bigraded algebra.

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1. Introduction

The aim of this paper is to study various bigraded regularities of the Rees algebra of a homogeneous ideal in $K[x_1, \ldots, x_n]$, where K is an infinite field. The Castelnuovo–Mumford regularity of a standard graded algebra is well understood by the work of various authors. The multigraded analogs of the regularity in terms of free resolutions and that of the local cohomology were defined by various authors, see [4,7,9,10,12,14–16,19]. In [1], Aramova *et al*. defined bigraded regularities in terms of graded Betti numbers (or syzygies) of bigraded modules over the standard bigraded polynomial K-algebra. Later, in [18], multigraded regularities of graded modules over standard multigraded polynomial K-algebra were defined in terms of syzygies [18, Definition 5.4] (also see [10, Definition 1.1.3]). Since $R(I)$, the Rees algebra of a graded ideal I, need not be a standard bigrded K-algebra, in other words, $R(I)$ is a quotient of a non-standard bigraded polynomial K-algebra. We study about the bigraded regularities, $\text{reg}^{\text{syz}}_{(1,0)}(R(I))$, $\text{reg}^{\text{syz}}_{(0,1)}(R(I))$ defined in terms of syzygies in the sense of [1,18] in the non-standard graded setting. The multigraded analog of Castelnuovo–Mumford regularity in terms of local cohomology were defined in [12] (also see [14]). In their notion, it is a subset of the abelian group considered under grading. Whereas we define the local cohomology regularity in a different way (see Definition 2.3). We also define the Tor bigraded regularities of R; see

§ 2. We compare these regularities and the relations among them. Our approach unifies many known results in the past on regularities in the setting of standard graded or standard bigraded algebras by various authors. Let $R = \oplus_{i,j\geq 0} R_{(i,j)}$ be a bigraded K-algebra. Write $R = S/L$, where $S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ with the bigrading defined by $\deg(x_i) = (s_i, 0)$ and $\deg(y_j) = (d_j, 1)$ for some $s_i, d_j \in \mathbb{Z}_+$ and L is a bigraded ideal of S. The Rees algebra of a graded ideal in $K[x_1, \ldots, x_n]$ is a typical example of such a bigraded algebra. Throughout, we use this bigrading for the Rees algebra.

In [2], Aramova and Herzog gave a characterization for the regularity of a graded module over a standard graded K-algebra. Later Römer [16] gave a characterization for the $\mathrm{reg}^{\mathrm{syz}}_{(0,1)}(R)$, $\mathrm{reg}^{\mathrm{syz}}_{(1,0)}(R)$ for any standard bigraded K-algebra R. This characterization is a bigraded analog of [2]. For any homogeneous ideal I, the bigraded regularities of $R(I)$ have not been attempted so far in the past. We show that for any finitely generated bigraded R-module M, the $(0, 1)$-syzygy regularity and $(0, 1)$-Tor regularity of M are equal (Proposition 3.2). We prove a characterization for $\mathrm{reg}^{\mathrm{syz}}_{(0,1)}(R)$, which extends the work of [2,16], to any bigraded K-algebra R with the specified grading (see Theorem 3.5). As a consequence, we show that the regularity $\mathrm{reg}(R)$ in the sense of Trung [21] or Chardin *et al.* [9], $\mathrm{reg}^{\mathrm{syz}}_{(0,1)}(R)$ are connected by the relation

$$\mathrm{reg}(R) = \max\{r(R_+), \mathrm{reg}^{\mathrm{syz}}_{(0,1)}(R)\},$$

where $r(R_+)$ is the reduction number of R_+ (see Corollary 3.7), where $R_+ = \oplus_{i\geq 0, j\geq 1} R_{(i,j)}$. In § 4, we study the relation between the bigraded regularities $\mathrm{reg}^{\mathrm{syz}}_{(1,0)}(R(I))$, $\mathrm{reg}^{\mathrm{syz}}_{(0,1)}(R(I))$ and the stabilization index of $\mathrm{reg}(I^k)$ for $k \geq 1$. Let $I \subset K[x_1, \ldots, x_n]$ be a homogeneous ideal and $\ell = \min\{\mu(J) \mid J \subseteq I$ a graded minimal reduction of $I\}$, where $\mu(J)$ denotes the minimal number of generators of J. Let $e_m := \mathrm{reg}(I^m) - \rho(I)m$ (known as regularity defects), for $m \geq 1$. It is well known that there exists $e, m_0 \in \mathbb{Z}$ such that $\mathrm{reg}(I^j) = j\rho(I) + e$, for all $j \geq m_0$, see for example [13] or [6]. We show that if $e_m \leq e_{m+1}$ for $m \geq \mathrm{reg}^{\mathrm{syz}}_{(0,1)}(R(I))$, then $m_0 \leq \mathrm{reg}^{\mathrm{syz}}_{(0,1)}(R(I)) + \ell$ (see Corollary 4.2). That is, the stabilization index has the upper bound $\mathrm{reg}^{\mathrm{syz}}_{(0,1)}(R(I)) + \ell$ which is a better bound compared with that in [16, Corollary 5.9(i)]. Note that in [3], Berlekamp gave a family of bounds on the differences e_m, when I is $\mathfrak{m}$-primary (see also [8]). We show that $\mathrm{reg}^{T}_{(1,0)}(R(I))$ is a bound for the defect sequence of any graded ideal I (see Theorem 4.3). Finally in § 5, we define a^*, b^* invariants of a bigraded K-algebra, which are bigraded analogues of a^*-invariant defined in [22]. For a bigraded K-algebra R, with the bigraded maximal ideal $\mathfrak{M}$, we show that

$$b^*(R) + \mathrm{grade}(\mathfrak{M}) \leq \max\{\mathrm{reg}^{L}_{(0,1)}(R), \mathrm{reg}^{\mathrm{syz}}_{(0,1)}(R)\} \leq b^*(R) + \ell(\mathfrak{M})$$

(see Corollary 5.9).

2. Preliminaries

First, we fix the notation. Let $R = \oplus_{i,j\geq 0} R_{(i,j)} t^j$ be a finitely generated bigraded K-algebra (need not be standard bigraded) and $R_{(0,0)} = K$, any field. Then $\mathfrak{M} =$ the bihomogeneous maximal ideal of R generated by $R_{(a,0)}, R_{(0,b)}$ for $a > 0, b > 0$ and, $R_+ = \oplus_{i\geq 0, j\geq 1} R_{(i,j)} t^j$. For any homogeneous ideal I in $K[x_1, \ldots, x_n]$, denote $\theta(I) =$ the maximal degree of any minimal generator of I and $\rho(I) = \min\{\theta(J) : J \subset I$, a minimal reduction of $I\}$.

Let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ be a bigraded polynomial K-algebra. Let M be a finitely generated bigraded S-module. We define

$$\mathrm{end}_{\mathrm{x}}(M) := \sup\{a \mid M_{(a,b)} \neq 0 \text{ for some } b\},$$
$$\mathrm{end}_{\mathrm{y}}(M) := \sup\{b \mid M_{(a,b)} \neq 0 \text{ for some } a\},$$

if $M \neq (0)$, and both are defined to be $-\infty$, if $M = (0)$.

DEFINITION 2.1 (Syzygy regularity [18], Definition 5.4)

$$\mathrm{reg}^{\mathrm{syz}}_{(1,0)}(M) := \sup\{a - i : \beta^S_{i,(a,b)}(M) \neq 0 \text{ for some } b\},$$
$$\mathrm{reg}^{\mathrm{syz}}_{(0,1)}(M) := \sup\{b - i : \beta^S_{i,(a,b)}(M) \neq 0 \text{ for some } a\}.$$

Note that $\mathrm{reg}^{\mathrm{syz}}_{(1,0)}(M) = \sup\{a - i : \mathrm{Tor}^S_i(M, K)_{(a,b)} \neq 0 \text{ for some } b\}$ and $\mathrm{reg}^{\mathrm{syz}}_{(0,1)}(M) = \sup\{b - i : \mathrm{Tor}^S_i(M, K)_{(a,b)} \neq 0 \text{ for some } a\}$. This definition coincides with the definition of $\mathrm{reg}^S_x(M)$ and $\mathrm{reg}^S_y(M)$ in [1], for the standard bigrading on S, with $\deg(x_i) = (1, 0)$ and $\deg(y_j) = (0, 1)$ (see also [16]).

DEFINITION 2.2 (Tor-regularity)

$$\mathrm{reg}^{\mathrm{T}}_{(1,0)}(M) := \sup\{j - i \mid \mathrm{Tor}^S_i(K[y_1, \ldots, y_m], M)_{(j,*)} \neq 0\},$$
$$\mathrm{reg}^{\mathrm{T}}_{(0,1)}(M) := \sup\{j - i \mid \mathrm{Tor}^S_i(K[x_1, \ldots, x_n], M)_{(*,j)} \neq 0\}.$$

DEFINITION 2.3 (Local cohomology-regularity)

$$\mathrm{reg}^{L}_{(1,0)}(M) := \max\{\mathrm{end}_{\mathrm{x}}(H^i_{\mathfrak{M}}(M)) + i : i \geq 0\},$$
$$\mathrm{reg}^{L}_{(0,1)}(M) := \max\{\mathrm{end}_{\mathrm{y}}(H^i_{\mathfrak{M}}(M)) + i : i \geq 0\}.$$

For the bigaded K-algebra $R = \oplus_{i,j\geq 0} R_{(i,j)}$, define a $\mathbb{Z}$-grading as follows: let $R_j = \oplus_{i\geq 0} R_{(i,j)}$, then $R = \oplus_{j\geq 0} R_j$ is a $\mathbb{Z}$-graded algebra. Then the regularity of R in the sense of Trung [21] is defined as

$$\mathrm{reg}(R) = \{a(H^i_{R_+}(R)) + i \ : \ i \geq 0\},$$

where $R_+ = \oplus_{j\geq 1} R_j$ and $a(H^i_{R_+}(R))$ denote the largest non-zero degree of $H^i_{R_+}(R)$.

The following example shows that the Tor and local cohomology regularities need not be equal.

Example 2.4. Let $R = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$, where $\deg(x_i) = (1, 0)$ and $\deg(y_j) = (0, 1)$. Then R is a bigraded K-algebra. We have $H^i_{\mathfrak{M}}(R) = 0$ for $i < m + n$ and $H^{m+n}_{\mathfrak{M}}(R) = K[x_1^{i_1} \ldots x_n^{i_n} y_1^{j_1} \ldots y_m^{j_m} | i_1 < 0, \ldots, i_n < 0, j_1 < 0, \ldots, j_m < 0]$. This implies that $\mathrm{end}_{\mathrm{x}}(H^{m+n}_{\mathfrak{M}}(R)) = -n$ and $\mathrm{end}_{\mathrm{y}}(H^{m+n}_{\mathfrak{M}}(R)) = -m$. Therefore $\mathrm{reg}^L_{(1,0)}(R) = m$ and $\mathrm{reg}^L_{(0,1)}(R) = n$. Whereas $\mathrm{reg}^T_{(1,0)}(R) = 0 = \mathrm{reg}^{\mathrm{T}}_{(0,1)}(R)$.

Remark 2.5. Suppose R is a standard bigraded K-algebra and M, a finitely generated bigraded R-module. Then by using the proof of [15, Theorem 3.1(i)], one can prove that

(1) $\mathrm{reg}^L_{(0,1)}(M) \leq \mathrm{reg}^L_{(0,1)}(R) + \mathrm{reg}^{\mathrm{syz}}_{(0,1)}(M)$

(2) $\operatorname{reg}^L_{(1,0)}(M) \leq \operatorname{reg}^L_{(1,0)}(R) + \operatorname{reg}^{\mathrm{syz}}_{(1,0)}(M)$.

The inequalities in the above remark can be strict.

Example 2.6. Let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$, $\deg(x_i) = (1,0)$, $\deg(y_j) = (0,1)$ and $M = K$. Then we have $H^0_{(\underline{\mathbf{x}},\underline{\mathbf{y}})}(M) = K$ and $H^i_{(\underline{\mathbf{x}},\underline{\mathbf{y}})}(M) = 0$ for all $i > 0$. This implies that $\operatorname{reg}^L_{(0,1)}(K) = \operatorname{reg}^L_{(1,0)}(K) = 0$. Also by Example 2.4, we have $\operatorname{reg}^L_{(1,0)}(S) = m$ and $\operatorname{reg}^L_{(0,1)}(S) = n$. Since $\underline{\mathbf{x}}, \underline{\mathbf{y}}$ is a regular sequence, therefore $K(\underline{\mathbf{x}}, \underline{\mathbf{y}})$ is the minimal bigraded S-free resolution of K, where

$$[K(\underline{\mathbf{x}}, \underline{\mathbf{y}})]_k = S^{\binom{m}{k}}(-(0,k)) \oplus S^{n\binom{m}{k}}(-(1,k-1)) \oplus S^{\binom{n}{2}\binom{m}{k}} (-(2,k-2)) \oplus \cdots \oplus S^{\binom{n}{k}}(-(k,0)).$$

Therefore $\operatorname{reg}^{\mathrm{syz}}_{(0,1)}(K) = \operatorname{reg}^{\mathrm{syz}}_{(1,0)}(K) = 0$. Hence

$$\operatorname{reg}^L_{(0,1)}(K) = 0 < n = \operatorname{reg}^L_{(0,1)}(S) + \operatorname{reg}^{\mathrm{syz}}_{(0,1)}(K),$$
$$\operatorname{reg}^L_{(1,0)}(K) = 0 < m = \operatorname{reg}^L_{(1,0)}(S) + \operatorname{reg}^{\mathrm{syz}}_{(1,0)}(K).$$

3. Homological interpretation of $\operatorname{reg}^{\mathrm{syz}}_{(0,1)}(R(I))$

Let R be a standard graded R_0-algebra, where R_0 is any commutative ring with 1. Let R_+ be the ideal generated by homogeneous elements of R of positive degree. We need the notion of filter-regular sequence in the sequel. Recall from [20] that a sequence of elements $f_1, \ldots, f_\ell \in R_1$ is called a *filter-regular sequence* with respect to R_+ if $f_i \notin P$ for all prime ideals $P \in \operatorname{Ass}(R/(f_1, \ldots, f_{i-1}))$ such that $P \not\supseteq R_+$, for $i = 1, \ldots, \ell$. Equivalently, $((f_1, \ldots, f_{i-1}) : f_i)_n = (f_1, \ldots, f_{i-1})_n$ for $n \gg 0$ and $i = 1, \ldots, \ell$. Equivalently, the multiplication map $[\frac{R}{(f_1,\ldots,f_{i-1})}]_{n-1} \xrightarrow{f_i} [\frac{R}{(f_1,\ldots,f_{i-1})}]_n$ is injective for $n \gg 0$ and $i = 1, \ldots, \ell$.

Let $R = \oplus_{i,j\geq 0} R_{(i,j)} t^j$ be a bigraded K-algebra and $f_1 t, \ldots, f_\ell t \in R$, generates a minimal reduction of $R_+ = \oplus_{i \geq 0, j > 0} R_{(i,j)} t^j$.

DEFINITION 3.1

(1) For $i = 1, \ldots, \ell$, $s_i(R) := \max\{b : [\frac{((f_1 t, \ldots, f_{i-1} t) : f_i t)}{(f_1 t, \ldots, f_{i-1} t)}]_{(*,b)} \neq 0\}$.
(2) $s(R) := \max\{s_1(R), \ldots, s_\ell(R)\}$.
(3) $r_k(R) := \sup\{j - i : H_i(y_1, \ldots, y_k; R)_{(*,j)} \neq 0\}$, for $k = 1, \ldots, \ell$.

PROPOSITION 3.2

Let $(s_1, \ldots, s_m) \in \mathbb{N}^m$. Let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ be the bigraded K-algebra (need not be standard bigraded) with $\deg(x_i) = (s_i, 0)$. Let M be a finitely generated bigraded S-module. Then $\operatorname{reg}^T_{(0,1)}(M) = \operatorname{reg}^{\mathrm{syz}}_{(0,1)}(M)$.

Proof. First we show that $\text{reg}^T_{(0,1)}(M) \le \text{reg}^{\text{syz}}_{(0,1)}(M)$. Let $\mathbb{F}_\bullet$ be a minimal bigraded S-free resolution of M. Then we have

$$\begin{aligned} H_i(x_1, \ldots, x_n; M) &\cong Tor^S_i(S/(x_1, \ldots, x_n), M) \\ &\cong H_i(S/(x_1, \ldots, x_n) \otimes_S \mathbb{F}_\bullet) \end{aligned}$$

for each i. For $j > \text{reg}^{\text{syz}}_{(0,1)}(M)$, we have $\beta^S_{i,(*,j+i)}(M) = 0$ for each i. This implies that $\text{Tor}^S_i(S/(x_1, \ldots, x_n), M)_{(*,j+i)} = 0$. This implies that $H_i(x_1, \ldots, x_n; M)_{(*,j+i)} = 0$. This gives that $\text{reg}^T_{(1,0)}(M) \le \text{reg}^{\text{syz}}_{(1,0)}(M)$. Now we show that $\text{reg}^T_{(0,1)}(M) \ge \text{reg}^{\text{syz}}_{(0,1)}(M)$. Since $\underline{\mathbf{x}}, \underline{\mathbf{y}}$ is a regular sequece, therefore we have $\text{Tor}^S_i(K[\underline{\mathbf{x}}], M) = H^S_i(\underline{\mathbf{y}}; M)$, $\text{Tor}^S_i(K[\underline{\mathbf{y}}], M) = H^S_i(\underline{\mathbf{x}}; M)$ and $\text{Tor}^S_i(K, M) = H^S_i(\underline{\mathbf{x}}, \underline{\mathbf{y}}; M)$ for all i.

We prove by induction on n. Assume $n = 1$. We have an exact sequece

$$0 \to K_.(\underline{\mathbf{y}}) \to K_.(\underline{\mathbf{y}}, x_1) \to K_.(\underline{\mathbf{y}})[-1] \to 0.$$

This gives a long exact sequence in the Koszul homology

$$\begin{aligned} \cdots &\to H^S_i(\underline{\mathbf{y}}; M)(-s_1, 0) \xrightarrow{\pm x_1} H^S_i(\underline{\mathbf{y}}; M) \to H^S_i(x_1, \underline{\mathbf{y}}; M) \\ &\to H^S_{i-1}(\underline{\mathbf{y}}; M)(-1, 0) \xrightarrow{\pm x_1} \cdots . \end{aligned}$$

This implies the short exact sequence

$$0 \to \frac{H^S_i(\underline{\mathbf{y}}; M)}{x_1 H^S_i(\underline{\mathbf{y}}; M)} \to H^S_i(x_1, \underline{\mathbf{y}}; M) \to K_{i-1} \to 0 \tag{1}$$

for all i, where $K_i := \ker(H^S_i(\underline{\mathbf{y}}; M)(-s_1, 0) \xrightarrow{\pm x_1} H^S_i(\underline{\mathbf{y}}; M))$. From exact sequence (1), we have

$$\text{end}_{\text{y}}(H^S_i(x_1, \underline{\mathbf{y}}; M)) \le \max \left\{ \text{end}_{\text{y}}\left(\frac{H^S_i(\underline{\mathbf{y}}; M)}{x_1 H^S_i(\underline{\mathbf{y}}; M)}\right), \text{end}_{\text{y}}(K_{i-1}) \right\}.$$

Since K_{i-1} is a submodule of $H^S_{i-1}(\underline{\mathbf{y}}; M))(-s_1, 0)$, therefore $\text{end}_{\text{y}}(K_{i-1}) \le \text{end}_{\text{y}}(H^S_{i-1}(\underline{\mathbf{y}}; M)))$. On the other hand, the natural surjective map $H^S_i(\underline{\mathbf{y}}; M) \to \frac{H^S_i(\underline{\mathbf{y}}; M)}{x_1 H^S_i(\underline{\mathbf{y}}; M)}$ implies that $\text{end}_{\text{y}}(\frac{H^S_i(\underline{\mathbf{y}}; M)}{x_1 H^S_i(\underline{\mathbf{y}}; M)}) \le \text{end}_{\text{y}}(H^S_i(\underline{\mathbf{y}}; M))$. Therefore we have

$$\text{end}_{\text{y}}(H^S_i(x_1, \underline{\mathbf{y}}; M)) \le \max \left\{ \text{end}_{\text{y}}(H^S_i(\underline{\mathbf{y}}; M)), \text{end}_{\text{y}}(H^S_{i-1}(\underline{\mathbf{y}}; M)) \right\}$$

for all i. This implies that

$$\begin{aligned} \text{reg}^{\text{syz}}_{(0,1)}(M) &= \max\{\text{end}_{\text{y}}(H^S_i(x_1, \underline{\mathbf{y}}; M)) - i \mid i \ge 0\} \\ &\le \max\{\text{end}_{\text{y}}(H^S_i(\underline{\mathbf{y}}; M)) - i, \text{end}_{\text{y}}(H^S_{i-1}(\underline{\mathbf{y}}; M)) - i \mid i \ge 0\} \\ &\le \max\{\text{end}_{\text{y}}(H^S_i(\underline{\mathbf{y}}; M)) - i \mid i \ge 0\} \\ &= \text{reg}^T_{(0,1)}(M). \end{aligned}$$

Therefore we have $\mathrm{reg}^{\mathrm{T}}_{(0,1)}(M) = \mathrm{reg}^{\mathrm{syz}}_{(0,1)}(M)$. Thus the statement is true for $n = 1$. The proof is complete by repeating this argument to the following exact sequence of complexes and the induction hypothesis

$$0 \to K_{.}(\underline{\mathbf{y}}, x_1) \to K_{.}(\underline{\mathbf{y}}, x_1, x_2) \to K_{.}(\underline{\mathbf{y}}, x_1)[-1] \to 0$$

$$\vdots$$

$$0 \to K_{.}(\underline{\mathbf{y}}, x_1, \ldots, x_{n-1}) \to K_{.}(\underline{\mathbf{y}}, \underline{\mathbf{x}}) \to K_{.}(\underline{\mathbf{y}}, x_1, \ldots, x_{n-1})[-1] \to 0.$$

□

Now we prove a bigraded version of a theorem of Schenzel [17, (2.2), (4.1)].

PROPOSITION 3.3

Let $R = \oplus_{i,j\geq 0} R_{(i,j)} t^j = S/L$ be a bigraded K-algebra (need not be standard bigraded) with $\deg(x_i) = (s_i, 0)$ and $\deg(y_j) = (d_j, 1)$ for some $s_i, d_j \in \mathbb{N}$. Let $R_+ = \oplus_{i\geq 0, j\geq 1} R_{(i,j)} t^j$. Suppose J is a minimal reduction of R_+ minimally generated by bihomogeneous elements $f_1 t, \ldots, f_\ell t$ in R_+. Then

$$\mathrm{reg}^{T}_{(0,1)}(R) = \sup\{j - i \mid H_i(Jt; R)_{(*,j)} \neq 0\}.$$

Proof. Let R_+ be generated by $f_1 t, \ldots, f_\ell t, \ldots, f_m t$, for some $f_{\ell+1} t, \ldots, f_m t \in I$. By definition, we have

$$\begin{aligned}\mathrm{reg}^{T}_{(0,1)}(R) &= \sup\{j - i \mid H_i(f_1 t, \ldots, f_m t; R)_{(*,j)} \neq 0\}\\ &= \sup\{j - i \mid H_i(y_1, \ldots, y_m; R)_{(*,j)} \neq 0\}.\end{aligned}$$

Note that $\mathrm{reg}^{T}_{(0,1)}(R)$ does not depend on the generating set that we selected for R_+. Assume $m = \ell + 1$. Therefore $f^c_{\ell+1} \in J$ for some $c \in \mathbb{N}$. This implies that $y^c_{\ell+1} H_i(y_1, \ldots, y_\ell; R) = 0$ for all i. This implies that $R^c_+ H_i(y_1, \ldots, y_\ell; R) = 0$ for all i. This gives that

$$H_i(y_1, \ldots, y_\ell; R)_{(*,j)} = 0$$

for $j \gg 0$, for all i. Let $\mu_{i,k} := \sup\{j \mid H_i(y_1, \ldots, y_k; R)_{(*,j)} \neq 0\}$ for $k = \ell, \ell + 1$. We have an exact sequence of Koszul complexes

$$\begin{aligned}0 &\to K(y_1, \ldots, y_\ell; R) \to K(y_1, \ldots, y_{\ell+1}; R)\\ &\to K(y_1, \ldots, y_\ell; R)(-d_{\ell+1}, -1) \to 0.\end{aligned}$$

This gives a long exact sequence

$$\begin{aligned}\cdots \to H^S_i(y_1, \ldots, y_\ell; R)(-d_{\ell+1}, -1) &\xrightarrow{\psi_i = \pm y_{\ell+1}} H^S_i(y_1, \ldots, y_\ell; R)\\ \to H^S_i(y_1, \ldots, y_{\ell+1}; R) \to H^S_{i-1}(y_1, \ldots, y_\ell; R)(-d_{\ell+1}, -1) &\xrightarrow{\psi_{i-1} = \pm y_{\ell+1}} \cdots\end{aligned}$$

for all j. This gives the exact sequences

$$\begin{aligned}0 &\to \ker(\psi_i) \to H^S_i(y_1, \ldots, y_\ell; R)(-d_{\ell+1}, -1))\\ &\xrightarrow{\psi_i} H^S_i(y_1, \ldots, y_\ell; R \to \mathrm{coker}(\psi_i) \to 0 \qquad (2)\end{aligned}$$

and

$$0 \to \operatorname{coker}(\psi_i) \to H_i^S(y_1, \ldots, y_{\ell+1}; R) \to \ker(\psi_{i-1}) \to 0. \tag{3}$$

From (3), we have $\mu_{i,\ell+1} \le \max\{\operatorname{end}_y(\operatorname{coker}(\psi_i)), \operatorname{end}_y(\ker(\psi_{i-1}))\}$ and from (2), $\operatorname{end}_y(\ker(\psi_i)) \le \mu_{i,\ell} + 1$ and $\operatorname{end}_y(\operatorname{coker}(\psi_i)) \le \mu_{i,\ell}$. This implies that

$$\begin{aligned}\operatorname{reg}^T_{(0,1)}(R) &= \sup\{\mu_{i,\ell+1} - i \mid i \ge 0\} \\ &\le \sup\{\operatorname{end}_y(\operatorname{coker}(\psi_i)) - i, \operatorname{end}_y(\ker(\psi_{i-1})) - i \mid i \ge 0\} \\ &\le \sup\{\mu_{i,\ell} - i, \mu_{i-1,\ell} - (i-1) \mid i \ge 0\} \\ &= \sup\{\mu_{i,\ell} - i \mid i \ge 0\}.\end{aligned}$$

For $j > \operatorname{reg}^T_{(0,1)}(R)$, we have $H_i^S(y_1, \ldots, y_{\ell+1}; R)_{(*,j+i)} = 0$ for all i. Then the above long exact sequence gives that

$$H_i^S(y_1, \ldots, y_\ell; R)_{(*,j+i)} \to H_i^S(y_1, \ldots, y_\ell; R)_{(*,j+i+1)}$$

is injective. Since for $j \gg 0$, $H_i^S(y_1, \ldots, y_\ell; R)_{(*,j)} = 0$, therefore the above injectiveness implies that $H_i^S(y_1, \ldots, y_\ell; R)_{(*,j+i)} = 0$ for all $j > \operatorname{reg}^T_{(0,1)}(R)$ and for all i. This impies that $\mu_{i,\ell} - i \le \operatorname{reg}^T_{(0,1)}(R)$ for all i. This gives that $\sup\{\mu_{i,\ell} - i \mid i \ge 0\} \le \operatorname{reg}^T_{(0,1)}(R)$. The other inequality is already proved above. Therefore $\operatorname{reg}^T_{(0,1)}(R) = \sup\{\mu_{i,\ell} - i \mid i \ge 0\}$. The proof for any m, follows by repeating the above argument to the following exact sequences

$$\begin{aligned}0 &\to K(y_1, \ldots, y_j; R) \to K(y_1, \ldots, y_{j+1}; R) \\ &\to K(y_1, \ldots, y_j; R)(-d_{\ell+1}, -1) \to 0\end{aligned}$$

for $j = \ell, \ldots, m$. □

COROLLARY 3.4

Suppose I is a graded ideal generated in single degree and $\ell = \ell(I)$. Then

$$\operatorname{reg}^T_{(0,1)}(R(I)) = \sup\{j - i \mid H_i(y_1, \ldots, y_\ell; R(I))_{(*,j)} \neq 0\}.$$

Proof. It is known that there exists a graded minimal reduction J minimally generated by $f_1, \ldots, f_\ell$ which is a filter regular sequence for $R(I)$. Therefore the assertion follows from Proposition 3.3. □

Now the following theorem is an extension of [16, Theorem 2.2] to non-standard bigraded K-algebras.

Theorem 3.5. *Let $R = \oplus_{i,j\ge 0} R_{(i,j)} t^j$ be a bigraded K-algebra with the grading $\deg(x_i) = (s_i, 0)$, where $s_i \in \mathbb{N}$. Let $R_+ = \oplus_{i\ge 0, j\ge 1} R_{(i,j)} t^j$ and $f_1 t, \ldots, f_\ell t$ is a filter regular sequence for R_+ which generates a minimal reduction of R_+. Then $r_k(R) = \max\{s_1(R), \ldots, s_k(R)\}$, for $k = 1, \ldots, \ell$. In particular,*

$$\operatorname{reg}^{\text{syz}}_{(0,1)}(R) = s(R),$$

where $r_k(R), s_k(R), s(R)$ are as defined in Definition 3.1.

Proof. By Propositions 3.2 and 3.3, we have $\mathrm{reg}^{T}_{(0,1)}(R) = \mathrm{reg}^{\mathrm{syz}}_{(0,1)}(R) = r_\ell(R)$. Let $\deg(f_j) = d_j$ for j. Consider the long exact sequence

$$\begin{aligned}
\cdots &\to H_i(y_1, \ldots, y_{k-1}; R)(-d_k, -1) \xrightarrow{\pm y_k} H_i(y_1, \ldots, y_{k-1}; R) \\
&\to H_i(y_1, \ldots, y_k; R) \\
&\to H_{i-1}(y_1, \ldots, y_{k-1}; R)(-d_k, -1) \xrightarrow{\pm y_k} \cdots \\
&\to H_1(y_1, \ldots, y_k; R) \to \frac{((f_1 t, \ldots, f_{k-1} t) : f_k t)}{(f_1 t, \ldots, f_{k-1} t)}(-d_k, -1) \to 0
\end{aligned}$$

for any $k = 1, \ldots, \ell$. We prove the assertion by induction on k. Suppose $k = 1$. Then the above exact sequence implies that $0 \to H_1(y_1; R) \to (0 : f_1 t)(-d_k, -1) \to 0$ is exact. This gives that $r_1(R) = s_1(R)$ as required. Assume $k > 1$. The above long exact sequence implies that $r_k(R) \geq s_k(R)$. By definition of $r_{k-1}(R)$, there exists an i such that $H_i(y_1, \ldots, y_{k-1}; R)_{(*, r_{k-1}(R)+i)} \neq 0$. Now by taking the corresponding bigraded components in the above exact sequence we get $H_{i+1}(y_1, \ldots, y_k; R)_{(*, r_{k-1}(R)+i+1)} \neq 0$. This implies that $r_k(R) \geq r_{k-1}(R)$. Thus we show that $r_k(R) \geq \max\{r_{k-1}(R), s_k(R)\}$. Let $b > \max\{r_{k-1}(R), s_k(R)\}$. Then we have $H_i(y_1, \ldots, y_{k-1}; R)_{(*, b+i)} = 0$ for all i. This imples that the above long exact sequence gives $H_i(y_1, \ldots, y_k; R)_{(*, b+i)} \neq 0$ for all i. This implies that $r_k(R) \leq \max\{r_{k-1}(R), s_k(R)\}$. Therefore $r_k(R) = \max\{r_{k-1}(R), s_k(R)\}$. The induction completes the proof. □

COROLLARY 3.6

Let I be a homogeneous ideal in a graded K-algebra. Let $f_1, \ldots f_\ell \in I$ generate a minimal reduction of I such that $f_1 t, \ldots, f_\ell t$ is a filter regular sequence for $R(I)$. Then

$$\mathrm{reg}^{\mathrm{syz}}_{(0,1)}(R(I)) = \max\left\{ \mathrm{end}_{\mathrm{y}}\left(\frac{((f_1 t, \ldots, f_{i-1} t) : f_i t)}{(f_1 t, \ldots, f_{i-1} t)} \right) \;\middle|\; i \in \{1, \ldots, \ell\} \right\}.$$

The following corollary gives a connection between the regularity $\mathrm{reg}(R)$ in the sense of Trung [21] and $\mathrm{reg}^{\mathrm{syz}}_{(0,1)}(R)$.

COROLLARY 3.7

Let $R = \oplus_{i,j \geq 0} R_{(i,j)}$ be a bigraded K-algebra such that $R = \oplus_{j \geq 0} R_j$ is a standard graded R_0-algebra, where $R_j := \oplus_{i \geq 0} R_{(i,j)}$ for $j \geq 0$. Let $R_+ = \oplus_{j \geq 1} R_j$ and $f_1 t, \ldots, f_\ell t$ is a filter regular sequence for R_+ which generates a minimal reduction of R_+ with reduction number $r(R_+)$. Then

$$\mathrm{reg}(R) = \max\{r(R_+), \mathrm{reg}^{\mathrm{syz}}_{(0,1)}(R)\}.$$

Proof. By [21, Theorem 2.4], we have that

$$\begin{aligned}
\mathrm{reg}(R) &= \max\left\{ r(R_+), \mathrm{end}_{\mathrm{y}}\left(\frac{(f_1 t, \ldots, f_i t : f_{i+1} t)}{(f_1 t, \ldots, f_i t)} \right) \middle| i = 0, \ldots, \ell - 1 \right\} \\
&= \max\{r(R_+), \mathrm{reg}^{\mathrm{syz}}_{(0,1)}(R)\} \text{ by using the Theorem 3.5.}
\end{aligned}$$

□

4. Regularity stabilization

Let $R := K[x_1, \ldots, x_n]$ and $J \subset R$ be a graded ideal of R. Let $\theta(J) :=$ the maximal degree of any minimal generator of J. Define

$$\rho(I) := \min\{\theta(J) : J \subset I, \text{ a minimal reduction of } I\}.$$

The following Theorem 4.1(2), is not only an improvement but also an extenstion of [16, Corollary 5.9(i)] to any graded ideal.

Theorem 4.1. *Let $I \subset K[x_1, \ldots, x_n]$ be a homogeneous ideal. Let $J \subset I$ be a graded minimal reduction of I with $\rho(I) = \theta(J)$ and ℓ denote the minimal number of generators of J. Then*

(1) *For $j > \operatorname{reg}^{\mathrm{syz}}_{(0,1)}(R(I))$,*

$$\operatorname{reg}(Z_i(K_\bullet(Jt; R(I))_{*,j+i})) \leq \max\nolimits_{0\leq k\leq i} \left\{\operatorname{reg}(I^{j+k}) + \rho(I)(i-k) + k\right\}$$

for all $i \geq 0$.

(2) *For $j > \operatorname{reg}^{\mathrm{syz}}_{(0,1)}(R(I)) + \ell$,*

$$\operatorname{reg}(I^j) \leq \max\{\operatorname{reg}(I^{j-s}) + \rho(I)s - s + 1 \mid s = 1, \ldots, \ell\}.$$

Proof. For simplicity, let $d = \rho(I)$. Let $J \subset I$ be a graded minimal reduction of I, which is minimally generated in degrees $d_1 \leq \cdots \leq d_\ell = d$.

(1) Let $j > \operatorname{reg}^{\mathrm{syz}}_{(0,1)}(R(I))$. From Proposition 3.2, we have $\operatorname{reg}^{T}_{(0,1)}(R(I)) = \operatorname{reg}^{\mathrm{syz}}_{(0,1)}(R(I))$. This gives $j > \operatorname{reg}^{\mathrm{T}}_{(0,1)}(R(I))$. This implies that $H_i(y_1, \ldots, y_\ell; R(I))_{(*,j+i)} = 0$ for all i. Consider the Koszul complex $K_\bullet(y_1, \ldots, y_\ell; R(I))$:

$$\begin{aligned}0 &\to R(I)(-d_1 - \cdots - d_\ell, -\ell) \to \cdots \to \oplus_{k_1<k_2} R(I)(-d_{k_1} - d_{k_2}, -2)\\ &\to \oplus_k R(I)(-d_k, -1) \to R(I) \to 0,\end{aligned}$$

where its $(* + d(j+i), j+i)$-graded piece $K_\bullet(y_1, \ldots, y_\ell; R(I))_{(*+d(j+i),j+i)}$ gives the exact sequence of $K[x_1, \ldots, x_n]$-modules:

$$\begin{aligned}0 &\to (Z_i)_{*+d(j+i)} \to I^j(-d_1 - \cdots - d_i)_{*+d(j+i)} \to \cdots\\ &\to \oplus_{k<l} I^{j+i-2}(-d_k - d_l)_{*+d(j+i)} \to \oplus_k I^{j+i-1}(-d_k)_{*+d(j+i)}\\ &\to I^{j+i}_{*+d(j+i)} \to 0\end{aligned}$$

because $H_i(y_1, \ldots, y_\ell; R(I))_{(*,j+i)} = 0$ for all $i \geq 0$, where Z_i denotes the i-th Koszul cycle of $K_\bullet(y_1, \ldots, y_\ell; R(I))_{(*,j+i)}$. This implies that

$$\begin{aligned}0 &\to (Z_i)(d(j+i)) \to \oplus_{k_1<\cdots<k_i} I^j(-d_{k_1} - \cdots - d_{k_i} + d(j+i))\\ &\to \cdots \to \oplus_{k<l} I^{j+i-2}(-d_k - d_l + d(j+i))\\ &\to \oplus_k I^{j+i-1}(-d_k + d(j+i)) \to I^{j+i}(d(j+i)) \to 0\end{aligned}$$

is exact. Now by using the regularity lemma and each $d_k \leq d$,

$$\begin{aligned}\operatorname{reg}(Z_i) - d(j+i) \leq \max\{&\operatorname{reg}(I^j) - dj, \operatorname{reg}(I^{j+1}) - d(j+1)\\ &+1, \ldots, \operatorname{reg}(I^{j+i}) - d(j+i) + i\}.\end{aligned}$$

This implies that

$$\operatorname{reg}(Z_i) \leq \max\{\operatorname{reg}(I^j) + di, \operatorname{reg}(I^{j+1}) + d(i-1) \\ +1, \ldots, \operatorname{reg}(I^{j+i}) + i\}.$$

This proves (1).
(2) Let $j > \operatorname{reg}^{\text{syz}}_{(0,1)}(R(I)) + \ell$. Then we have $H_i(y_1, \ldots, y_\ell; R(I))_{(*,j)} = 0$ for all i. This implies that $K_\bullet(y_1, \ldots, y_\ell; R(I))_{(*,j)}$ is an exact sequence. This gives

$$0 \to I^{j-\ell}(-\ell d) \to \cdots \to \oplus_{k<l} I^{j-2}(-2d) \\ \to \oplus_k I^{j-1}(-d) \to I^j \to 0$$

is exact. By regularity lemma, this implies that

$$\operatorname{reg}(I^j) \leq \max\{\operatorname{reg}(I^{j-1}) + d, \operatorname{reg}(I^{j-2}) + 2d - 1, \ldots, \operatorname{reg}(I^{j-\ell}) \\ +\ell d - \ell + 1\},$$

as required. □

COROLLARY 4.2

Let $I \subset K[x_1, \ldots, x_n]$ be a homogeneous ideal. Let $J \subset I$ be a graded minimal reduction of I, minimally generated by ℓ elements and $\rho(I) = \theta(J)$. Let $e_m := \operatorname{reg}(I^m) - \rho(I)m$, for $m \geq 1$. Suppose $e_m \leq e_{m+1}$ for $m \geq \operatorname{reg}^{\text{syz}}_{(0,1)}(R(I))$. Then for $j \geq \operatorname{reg}^{\text{syz}}_{(0,1)}(R(I)) + \ell$, $\operatorname{reg}(I^j) = j\rho(I) + e$.

Proof. From [13], it is known that $\operatorname{reg}(I^k) = j\rho(I) + e$, for some $e \in \mathbb{Z}$, for $k \gg 0$. From the hypothesis we have $\operatorname{reg}(I^m) + \rho(I) \leq \operatorname{reg}(I^{m+1})$, for all $m \geq 1$. This implies that $\operatorname{reg}(I^{j-i}) + \rho(I)i - i + 1 \leq \operatorname{reg}(I^{j-1}) + \rho(I)$ for $i = 1, \ldots, \ell$. From Theorem 4.1(2), this implies that for $j > \operatorname{reg}^{\text{syz}}_{(0,1)}(R(I)) + \ell$, we have $\operatorname{reg}(I^j) \leq \operatorname{reg}(I^{j-1}) + \rho(I)$. By the assumption, this implies that for $j > \operatorname{reg}^{\text{syz}}_{(0,1)}(R(I)) + \ell$, we have $\operatorname{reg}(I^j) = \operatorname{reg}(I^{j-1}) + \rho(I)$. This implies the required assertion. □

See section 4.2 of [3], for examples of $\mathfrak{m}$-primary ideals I, for which the defect sequences e_m are weakly increasing.

Theorem 4.3. *Let I be a homogeneous ideal of $K[x_1, \ldots, x_n]$. Then $\operatorname{reg}(I^j) - j\rho(I) \leq \operatorname{reg}^T_{(1,0)}(R(I))$, for all $j \geq 1$.*

Proof. Let $j > \operatorname{reg}^T_{(1,0)}(R(I))$. Then we have

$$H_i(x_1, \ldots, x_n; R(I))_{(j+i,*)} = Tor^S_i(S/(x_1, \ldots, x_n), R(I))_{(j+i,*)} = 0$$

for all i. This implies that

$$H_i(x_1, \ldots, x_n; R(I))_{(j+i+k\rho(I),k)} = Tor^S_i(S/(x_1, \ldots, x_n), R(I))_{(j+i+k\rho(I),k)} = 0$$

for all $i \geq 0, k \geq 1$. By [6, Lemma 3.3], this implies that $\mathrm{Tor}_i^{K[x_1,\dots,x_n]}(K, I^k)_{j+i+k\rho(I)} = 0$ for all i, k. Thus we have $\mathrm{Tor}_i^{K[x_1,\dots,x_n]}(K, I^k)_{j+i+k\rho(I)} = 0$ for all $i \geq 0, k \geq 1$ and $j > \mathrm{reg}^T_{(1,0)}(R(I))$. This implies that $\mathrm{reg}(I^k) \leq \mathrm{reg}^T_{(1,0)}(R(I)) + k\rho(I)$ for all $k \geq 1$. □

COROLLARY 4.4 [16, Theorem 5.3(i)]

Let I be a homogeneous ideal in $K[x_1, \dots, x_n]$ generated in degree d. Then $\mathrm{reg}(I^j) - jd \leq \mathrm{reg}^{\mathrm{syz}}_{(1,0)}(R(I))$, *for all* $j \geq 1$.

Proof. Since I is generated in single degree d, therefore $\rho(I) = d$ and $R(I) = S/L$ is a standard bigraded K-algebra with $\deg(x_i) = (1, 0)$ and $\deg(y_j) = (0, 1)$, for some bigraded ideal L of S. By Proposition 3.2, we have $\mathrm{reg}^{\mathrm{T}}_{(1,0)}(R(I)) = \mathrm{reg}^{\mathrm{syz}}_{(1,0)}(R(I))$. The corollary follows from Theorem 4.3. □

The following example shows that the reduction number is a sharp bound for the regularity stabilization index.

Example 4.5. For any integer $d \geq 3$, let $I = (x^d, y^d, x^{d-1}y, xy^{d-1}) \subset K[x, y]$ and $J = (x^d, y^d)$. Then J is a minimal reduction of I of reduction number equal to $d - 2$. By using [5], we have $\mathrm{reg}(I^k) = kd$, for all $k \geq d - 2$ and the stabilization index is equal to $d - 2$.

5. a^*, b^* Invariants of bigraded algebras

In this section we define a^* and b^* invariants of a bigraded K-algebra and we give characterizations for them.

Let R be a bigraded K-algebra (need not be standard) and $\mathfrak{M}$ its homogeneous maximal ideal. Let M be any finitely generated bigraded R-module.

DEFINITION 5.1

(1) $a^*(M) := \sup\{\mathrm{end}_{\mathrm{x}}(H^i_{\mathfrak{M}}(M)) \mid i \geq 0\}$.
(2) $b^*(M) := \sup\{\mathrm{end}_{\mathrm{y}}(H^i_{\mathfrak{M}}(M)) \mid i \geq 0\}$.

From the argument in the proof of [22, Lemma 2.1], we have that if $R_{(*,j)} = 0$, for $j \gg 0$, then $b^*(R) = \max\{j \mid R_{(*,j)} \neq 0\}$ and if $R_{(j,*)} = 0$ for $j \gg 0$, then $a^*(R) = \max\{j \mid R_{(j,*)} \neq 0\}$.

Remark 5.2. By using the notation and the discussion before Lemma 1.1 in [11], if $f_1t, \dots, f_\ell t$ is a filter regular sequence which generates a minimal reduction of R_+, then it follows that

$$\mathrm{end}_{\mathrm{y}}\left(H^0_{\mathfrak{M}}\left(\frac{R}{(f_1t, \dots, f_\ell t)}\right)\right) = \mathrm{end}\left(H^0_{\mathfrak{M}^\varphi}\left(\left(\frac{R}{(f_1t, \dots, f_\ell t)}\right)^\varphi\right)\right)$$
$$= \mathrm{end}\left(H^0_{R_+}\left(\frac{R}{(f_1t, \dots, f_\ell t)}\right)\right),$$

where $\varphi : \mathbb{Z}^2 \to \mathbb{Z}$, $\varphi((a, b)) = b$, $\mathfrak{M}^\varphi = (R_+, \mathfrak{m}_{R_0})$. In the above, we also use the fact that $\frac{R}{(f_1t,\dots,f_\ell t)})_{(*,j)} = 0$ for $j \gg 0$ and $H^0_{R_+}(\frac{R}{(f_1t,\dots,f_\ell t)}) = ((f_1t, \dots, f_\ell t) :$

$R_+^{\infty})/(f_1t, \dots, f_\ell t)$. These imply that $\mathrm{end}_y(H^0_{R_+}(\frac{R}{(f_1t,\dots,f_\ell t)})) = \mathrm{end}_y(((f_1t, \dots, f_\ell t) : R_+)/(f_1t, \dots, f_\ell t))$. This implies that

$$\mathrm{end}_y(\frac{(f_1t, \dots, f_\ell t) : \mathfrak{M})}{(f_1t, \dots, f_\ell t)}) = r_{(\mathbf{f}t)}(R_+) = \mathrm{end}_y(\frac{(f_1t, \dots, f_\ell t) : R_+)}{(f_1t, \dots, f_\ell t)}).$$

The following proposition is a bigraded analog of [20, Lemma 2.3]. For the sake of completeness we give the proof here.

PROPOSITION 5.3 [20], Lemma 2.3].

Let $R = \oplus_{i,j\geq 0} R_{(i,j)}t^j$ be a bigraded K-algebra and $\mathfrak{M}$ denote its bigraded maximal ideal. Suppose ft is a bihomogeneous filter-regular element of R such that $\deg(ft) = (d, 1)$. *Then*

(1) $a^*(R/ftR) \leq a^*(R) + d$;
(2) $b^*(R/ftR) \leq b^*(R) + 1$.

Proof. Let ℓ be the analytic spread of $\mathfrak{M}$. We have a short exact sequence

$$0 \to (0 : ft) \to R \to R/(0 : ft) \to 0.$$

Since $(0 : ft)_k = 0$ for $k \gg 0$, therefore the above short exact sequence gives that $H^i_{\mathfrak{M}}(R) \cong H^i_{\mathfrak{M}}(R/(0 : ft))$ for all $i \geq 1$. Now by using these isomorphisms and the fact $H^\ell_{\mathfrak{M}}(R/ftR) = 0$, the exact sequence

$$0 \to R/(0 : ft)(-d, -1) \to R \to R/ftR \to 0$$

gives the long exact sequence

$$\begin{aligned} 0 &\to H^0_{\mathfrak{M}}(R/(0 : ft))(-d, -1) \to H^0_{\mathfrak{M}}(R) \to H^0_{\mathfrak{M}}(R/ftR) \\ &\to H^1_{\mathfrak{M}}(R)(-d, -1) \to \cdots \to H^{\ell-1}_{\mathfrak{M}}(R/ftR) \\ &\to H^\ell_{\mathfrak{M}}(R)(-d, -1) \to H^\ell_{\mathfrak{M}}(R) \to 0. \end{aligned}$$

This implies that

$$\begin{aligned} \mathrm{end_x}(H^\ell_{\mathfrak{M}}(R)) &\leq \mathrm{end_x}(H^{\ell-1}_{\mathfrak{M}}(R/ftR)) - d, \\ \mathrm{end_y}(H^\ell_{\mathfrak{M}}(R)) &\leq \mathrm{end_y}(H^{\ell-1}_{\mathfrak{M}}(R/ftR)) - 1, \\ \mathrm{end_x}(H^j_{\mathfrak{M}}(R/ftR)) &\leq \max\{\mathrm{end_x}(H^j_{\mathfrak{M}}(R)), \mathrm{end_x}(H^{j+1}_{\mathfrak{M}}(R)) + d\}, \\ \mathrm{end_y}(H^j_{\mathfrak{M}}(R/ftR)) &\leq \max\{\mathrm{end_y}(H^j_{\mathfrak{M}}(R)), \mathrm{end_y}(H^{j+1}_{\mathfrak{M}}(R)) + 1\} \end{aligned}$$

for all $0 \leq j \leq \ell - 1$. This implies that $a^*(R/ftR) \leq a^*(R) + d$ and $b^*(R/ftR) \leq b^*(R) + 1$. □

COROLLARY 5.4

$$\max\{\mathrm{end}_y(0 : ft), \mathrm{reg}^L_{(0,1)}(R/ftR)\} \leq \mathrm{reg}^L_{(0,1)}(R).$$

Proof. By Proposition 5.3, we have

$$\text{end}_y(H^j_{\mathfrak{M}}(R/fR)) \leq \max\{\text{end}_y(H^j_{\mathfrak{M}}(R)), \text{end}_y(H^{j+1}_{\mathfrak{M}}(R)) + 1\}$$

for all $0 \leq j \leq \ell - 1$. This implies that $\text{reg}^L_{(0,1)}(R/ftR) \leq \text{reg}^L_{(0,1)}(R)$. This gives that $\max\{\text{end}_y(0 : ft), \text{reg}^L_{(0,1)}(R/ftR)\} \leq \text{reg}^L_{(0,1)}(R)$. □

The following theorem is an extension of [22, Theorem 2.2] to bigraded algebras.

Theorem 5.5. *Let $R = \oplus_{i,j\geq 0} R_{(i,j)}t^j$ be a bigraded K-algebra. Let $R_+ = \oplus_{i\geq 0\,\cdot\, j\geq 1} R_{(i,j)}t^j$ have a minimal reduction generated by $f_1t, \ldots, f_\ell t$ which is a superficial sequence for R with reduction number $r(R_+)$. Then*

$$b^*(R) = \max\left\{r(R_+) - \ell, \text{end}_y\left(\frac{(f_1t, \ldots, f_it : f_{i+1}t)}{(f_1t, \ldots, f_it)}\right) - i \;\middle|\; i = 0, \ldots, \ell - 1\right\}.$$

Proof. Let $f_1t, \ldots, f_\ell t \in R_+$ generate a minimal reduction of R_+ of reduction number $r(R_+)$, which is a filter-regular sequence for R. Suppose $\ell = 0$. Then $R_{(*,j)} = 0$ for $j \gg 0$. This gives that $b^*(R) = \text{end}_{\text{y}}(R)$. It is easy to see that $\text{end}_{\text{y}}(R) = r(R_+)$. Therefore the statment is true for $\ell = 0$. Assume $\ell \geq 1$. Let $j \geq b^*(R/f_1tR)$. Then we have $H^{i-1}_{\mathfrak{M}}(R/f_1tR)_{(*,j+1)} = 0$ for all i. Since $H^i_{\mathfrak{M}}(R)_{(*,u)} = 0$ for $u \gg 0$, therefore from the long exact sequence of local cohomology in Proposition 5.3, we get $H^i_{\mathfrak{M}}(R)_{(*,j)} = 0$. Thus $H^i_{\mathfrak{M}}(R)_{(*,j)} = 0$ for all $j \geq b^*(R/f_1tR)$ and for all $i \geq 1$. This implies that $b^*(R) \leq \max\{\text{end}_{\text{y}}(0 : f_1t), b^*(R/f_1tR) - 1\}$. Now by Proposition 5.3(2) and by Remark 5.2, $\text{end}_{\text{y}}(0 : f_1t) = \text{end}_{\text{y}}(0 : R_+) = \text{end}_{\text{y}}(0 : \mathfrak{M})$. This implies that

$$\begin{aligned} b^*(R) &\leq \max\{\text{end}_{\text{y}}(0 : f_1t), b^*(R/f_1tR) - 1\} \\ &\leq \max\{\text{end}_{\text{y}}(0 : f_1t), b^*(R)\} = b^*(R). \end{aligned}$$

This gives $b^*(R) = \max\{\text{end}_y(0 : f_1t), b^*(R/f_1tR) - 1\}$. Since $r(R_+) = r(R_+/f_1tR)$, therefore the assertion follows from this by induction on ℓ. □

Theorem 5.6. *Let $R = \oplus_{i,j\geq 0} R_{(i,j)}t^j$ be a bigraded K-algebra. Let $R_+ = \oplus_{i\geq 0, j\geq 1} R_{(i,j)}t^j$ and $f_1t, \ldots, f_\ell t$ is a filter regular sequence for R. Then*

$$\max\{\text{end}_y(H^i_{R_+}(R)) \mid i = 0, \ldots, \ell\} = \max\left\{\text{end}_y\left(\frac{(f_1t, \ldots, f_it : R_+)}{(f_1t, \ldots, f_it)}\right) - i \;\middle|\; i = 0, \ldots, \ell\right\}.$$

Proof. From the argument in the proof of [22, Lemma 2.4] it is easy to see that if $f_1t \in R_{(d_1,1)}$ is a filter regular element for R, then $\text{end}_y(H^0_{R_+}(R)) = \text{end}_y(0 : f_1t) = \text{end}_y(0 : R_+)$. From the proof of Proposition 5.3 and the long exact sequece of local cohomology with base R_+, we have $\text{end}_y(H^{j+1}_{R_+}(R)) + 1 \leq \text{end}_y(H^j_{R_+}(R/ftR)) \leq \max\{\text{end}_y(H^j_{R_+}(R)), \text{end}_y(H^{j+1}_{R_+}(R)) + 1\}$ for all $0 \leq j \leq \ell - 1$. By repeating this argument we can show that

$$\begin{aligned}
&\operatorname{end}_y\left(H^{j+1}_{R_+}\left(\frac{R}{(f_1t,\ldots,f_{i-1}t)}\right)\right)+1\\
&\quad\le \operatorname{end}_y\left(H^{j}_{R_+}\left(\frac{R}{(f_1t,\ldots,f_{i}t)}\right)\right)\\
&\quad\le \max\left\{\operatorname{end}_y\left(H^{j}_{R_+}\left(\frac{R}{(f_1t,\ldots,f_{i-1}t)}\right)\right),\operatorname{end}_y\left(H^{j+1}_{R_+}\left(\frac{R}{(f_1t,\ldots,f_{i-1}t)}\right)\right)+1\right\}
\end{aligned}$$

for all $i=1,\ldots,\ell$. From these we can derive that

$$\begin{aligned}
\operatorname{end}_y(H^i_{R_+}(R))+i &\le \operatorname{end}_y\left(H^0_{R_+}\left(\frac{R}{(f_1t,\ldots,f_it)}\right)\right)\\
&\le \max\{\operatorname{end}_y(H^j_{R_+}(R))+j \mid 0\le j\le i\}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\max\{\operatorname{end}_y(H^i_{R_+}(R)) \mid i=0,\ldots,\ell\}\\
&\quad\le \max\{\operatorname{end}_y(H^0_{R_+}(R/(f_1t,\ldots,f_it)))-i \mid i=0,\ldots,\ell\}\\
&\quad\le \max\{\operatorname{end}_y(H^j_{R_+}(R))+j-i \mid 0\le j\le i, i=0,\ldots,\ell\}\\
&\quad\le \max\{\operatorname{end}_y(H^i_{R_+}(R)) \mid i=0,\ldots,\ell\}.
\end{aligned}$$

This gives $\max\{\operatorname{end}_y(H^i_{R_+}(R)) \mid i=0,\ldots,\ell\} = \max\{\operatorname{end}_y(H^0_{R_+}(R/(f_1t,\ldots, f_it)))-i \mid i=0,\ldots,\ell\}$. The proof is complete because $\operatorname{end}_y(H^0_{R_+}(R/(f_1t,\ldots, f_it))) = \operatorname{end}_y(\frac{((f_1t,\ldots,f_it):R_+)}{(f_1t,\ldots,f_it)})$. □

COROLLARY 5.7

$$\max\{\operatorname{end}_y(H^i_{R_+}(R)) \mid i\ge 0\} = b^*(R).$$

Proof. Let $f_1t,\ldots,f_\ell t \in R_+$ generates a minimal reduction of R_+ of reduction number $r(R_+)$, which is a filter-regular sequence for R. Since $\operatorname{end}_y(\frac{(f_1t,\ldots,f_it:R_+)}{(f_1t,\ldots,f_it)}) = \operatorname{end}_y(\frac{(f_1t,\ldots,f_it:f_{i+1}t)}{(f_1t,\ldots,f_it)})$ for any i, therefore by Theorem 5.6, we have

$$\begin{aligned}
&\max\{\operatorname{end}_y(H^i_{R_+}(R)) \mid i=0,\ldots,\ell\}\\
&\quad= \max\left\{\operatorname{end}_y\left(\frac{(f_1t,\ldots,f_it:f_{i+1}t)}{(f_1t,\ldots,f_it)}\right)-i \;\middle|\; i=0,\ldots,\ell\right\}\\
&\qquad \max\left(\left\{\operatorname{end}_y\left(\frac{(f_1t,\ldots,f_it:f_{i+1}t)}{(f_1t,\ldots,f_it)}\right)-i \;\middle|\; i=0,\ldots,\ell-1\right\}\cup\{r(R_+)-\ell\}\right)\\
&\quad= b^*(R).
\end{aligned}$$

The last equality follows from Theorem 5.5. □

Remark 5.8. Note that the above corollary also follows from [11, Lemma 2.3], which uses the spectral sequences. By using the notation and the discussion before Lemma 1.1 in [11], we get

$$\oplus_{a\in\mathbb{Z}}(H^i_{\mathfrak{M}}(M))_{(a,b)} = [(H^i_{\mathfrak{M}}(M))^{\varphi}]_b = [H^i_{\mathfrak{M}^{\varphi}}(M^{\varphi})]_b$$
$$= [H^i_{(R_+,\mathfrak{m}_{R_0})}(M)]_b,$$

where $\varphi : \mathbb{Z}^2 \to \mathbb{Z}$, $\varphi((a,b)) = b$, $R^{\varphi} = \oplus_{j\geq 0} R_j$, $R_j = \oplus_{i\geq 0} R_{(i,j)}$, $\mathfrak{M}^{\varphi} = (R_+, \mathfrak{m}_{R_0})$. By [11, Lemma 2.3], this implies that for any $b_0 \in \mathbb{Z}$, $[H^i_{R_+}(M)]_b = 0$ for all $b \geq b_0$, $i \geq 0$ if and only if $(H^i_{\mathfrak{M}}(M))_{(a,b)} = 0$ for all $a \in \mathbb{Z}$, $i \geq 0$ and $b \geq b_0$.

COROLLARY 5.9

$b^*(R) + \mathrm{grade}(\mathfrak{M}) \leq \max\{\mathrm{reg}^L_{(0,1)}(R), \mathrm{reg}^{\mathrm{syz}}_{(0,1)}(R)\} \leq b^*(R) + \ell(\mathfrak{M})$. *In particular, if* $\mathfrak{M}$ *is an equimultiple ideal, then*

$$\max\{\mathrm{reg}^L_{(0,1)}(R), \mathrm{reg}^{\mathrm{syz}}_{(0,1)}(R)\} = b^*(R) + \ell(\mathfrak{M}).$$

Proof. First inequality follows from Theorems 3.5 and 5.5 and the fact that $\mathrm{grade}(R_+) \leq \ell$. The last inequality follows directly from Theorems 5.6, 3.5 and Corollary 5.7. □

Note that Example 2.4 satisfies the inequality in Corollary 5.9.

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