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Twisting formula of epsilon factors

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Abstract. For characters of a non-Archimedean local field we have explicit formula for epsilon factors. But in general, we do not have any generalized twisting formula of epsilon factors. In this paper, we give a generalized twisting formula of epsilon factors via local Jacobi sums.

Keywords. Local field; Gauss sum; Jacobi sum; epsilon factor; conductor.

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1. Introduction

By Langlands, we can associate a local epsilon factor (also known as local constant) with each character $\chi : F^\times \to \mathbb{C}^\times$ of a non-Archimedean local field F of characteristic zero. In general, we do not have any explicit formula of epsilon factor of a twisted character (by a ramified character). Let χ_1 and χ_2 be two characters of $F^\times$. Let $\psi : F \to \mathbb{C}^\times$ be a nontrivial additive character of F. If any one of these characters is unramified, then we have a formula for $\epsilon(\chi_1\chi_2, \psi)$ due to Tate. Also if conductor $a(\chi_1) \geqslant 2a(\chi_2)$, then by Deligne (cf. [2], Lemma 4.16) we have a formula for $\epsilon(\chi_1\chi_2, \psi)$. In this article, we give a generalized twisting formula for $\epsilon(\chi_1\chi_2, \psi)$, when both χ_1 and χ_2 are ramified via the following local Jacobi sums.

Let U_F be the group of units in O_F (ring of integers of F). For characters χ_1, χ_2 of $F^\times$ and a positive integer n, we define the local Jacobi sum

$$J_t(\chi_1, \chi_2, n) = \sum_{\substack{x \in \frac{U_F}{U_F^n} \\ t-x \in U_F}} \chi_1^{-1}(x)\chi_2^{-1}(t-x). \tag{1.1}$$

In our twisting formula, conductors of characters play an important role and the formula (cf. Theorem 3.5) is

$$\epsilon(\chi_1\chi_2, \psi) = \begin{cases} \dfrac{q^{\frac{n}{2}}\epsilon(\chi_1, \psi)\epsilon(\chi_2, \psi)}{J_1(\chi_1, \chi_2, n)} & \text{when } n = m = r, \\ \dfrac{q^{\frac{r}{2}}\chi_1\chi_2(\pi_F^{r-n})\epsilon(\chi_1, \psi)\epsilon(\chi_2, \psi)}{J_1(\chi_1, \chi_2, n)} & \text{when } n = m > r, \\ \dfrac{q^{n-\frac{m}{2}}\epsilon(\chi_1, \psi)\epsilon(\chi_2, \psi)}{J_1(\chi_1, \chi_2, n)} & \text{when } n = r > m, \end{cases} \tag{1.2}$$

where $n = a(\chi_1), m = a(\chi_2), r = a(\chi_1\chi_2)$ and q is the cardinality of the residue field of the field F.

2. Notations and preliminaries

Let F be a non-Archimedean local field of characteristic zero, i.e., a finite extension of the field $\mathbb{Q}_p$ (field of p-adic numbers), where p is a prime. Let O_F be the ring of integers in local field F and $P_F = \pi_F O_F$ is the unique prime ideal in O_F and π_F is a uniformizer, i.e., an element in P_F whose valuation is one i.e., $\nu_F(\pi_F) = 1$. The cardinality of the residue field $\mathbb{F}_q = O_F/P_F$ of F is q, i.e., $|\mathbb{F}_q| = q$. Let $U_F = O_F - P_F$ be the group of units in O_F. Let $P_F^i = \{x \in F : \nu_F(x) \geqslant i\}$ and for $i \geqslant 0$ define $U_F^i = 1 + P_F^i$ (with proviso $U_F^0 = U_F = O_F^\times$).

DEFINITION 2.1 (Conductor of characters)

The conductor of any nontrivial additive character ψ of a field is an integer $n(\psi)$ if ψ is trivial on $P_F^{-n(\psi)}$, but nontrivial on $P_F^{-n(\psi)-1}$. We also consider $a(\chi)$ as the conductor of nontrivial character $\chi : F^\times \to \mathbb{C}^\times$, i.e., $a(\chi)$ is the smallest integer $m \geqslant 0$ such that χ is trivial on U_F^m. We say χ is *unramified* if the conductor of χ is zero and otherwise *ramified*. We also recall here that for two characters χ_1 and χ_2 of $F^\times$ we have $a(\chi_1\chi_2) \leq \max(a(\chi_1), a(\chi_2))$ with equality if $a(\chi_1) \neq a(\chi_2)$.

2.1 *Classical Gauss sums and Jacobi sums*

Let k_q be a finite field of order q. Let χ, ψ be a multiplicative and an additive character respectively of k_q. Then the classical Gauss sum $G(\chi, \psi)$ is defined by

$$G(\chi, \psi) = \sum_{x \in k_q^\times} \chi(x)\psi(x). \tag{2.1}$$

Let χ_1 and χ_2 be two multiplicative characters of k_q. The classical Jacobi sum $J_1(\chi_1, \chi_2)$ is defined by

$$J_1(\chi_1, \chi_2) = \sum_{x \in k_q^\times} \chi_1(x)\chi_2(1 - x). \tag{2.2}$$

The relation between classical Gauss sums and Jacobi sums is as follows (cf. [1], p. 59, Theorem 2.1.3)

(1) Let χ_1, χ_2 be two multiplicative characters of k_q and ψ a nontrivial additive character of k_q. If $\chi_1\chi_2$ is nontrivial, then

$$J_1(\chi_1, \chi_2) = \frac{G(\chi_1, \psi) \cdot G(\chi_2, \psi)}{G(\chi_1\chi_2, \psi)}. \tag{2.3}$$

(2) If χ_1, χ_2 and $\chi_1\chi_2$ are all nontrivial, then we have

$$|J_1(\chi_1, \chi_2)| = q^{\frac{1}{2}}. \tag{2.4}$$

2.2 *Epsilon factors*

For a nontrivial multiplicative character χ of $F^\times$ and nontrivial additive character ψ of F, we have (cf. [4], p. 4)

$$\epsilon(\chi,\psi,c)=\chi(c)\frac{\int_{U_F}\chi^{-1}(x)\psi(x/c)dx}{|\int_{U_F}\chi^{-1}(x)\psi(x/c)dx|}, \tag{2.5}$$

where the Haar measure dx on F is normalized such that the Haar measure m' of O_F is 1, i.e., $m'(O_F)=1$ and $c\in F^\times$ with valuation $n(\psi)+a(\chi)$. To get modified *summation formula* of epsilon factor from the integral formula (2.5), we need the next lemma which can be found in [3].

Lemma 2.2. *Let F be a non-Archimedean local field with q as the cardinality of the residue field of F. Let χ be a nontrivial character of $F^\times$ with conductor $a(\chi)$. Let ψ be an additive character of F with conductor $n(\psi)$. We define the integration for an integer $m\in\mathbb{Z}$ as*

$$I(m)=\int_{U_F}\chi^{-1}(x)\psi\left(\frac{x}{\pi_F^{l+m}}\right)dx,\quad \textit{where } l=a(\chi)+n(\psi). \tag{2.6}$$

Then

$$|I(m)|=\begin{cases}q^{-\frac{a(\chi)}{2}} & \textit{when } m=0,\\ 0 & \textit{otherwise}.\end{cases} \tag{2.7}$$

Again since $m'(O_F)=1$, we have $m'(U_F^n)=q^{-n}$ for any positive integer n. By using the above Lemma 2.2, the formula (2.5) can be reduced to

$$\begin{aligned}\epsilon(\chi,\psi,c)&=\chi(c)q^{a(\chi)/2}\sum_{x\in\frac{U_F}{U_F^{a(\chi)}}}\chi^{-1}(x)\psi(x/c)m'(U_F^{a(\chi)})\\ &=\chi(c)q^{-a(\chi)/2}\sum_{x\in\frac{U_F}{U_F^{a(\chi)}}}\chi^{-1}(x)\psi(x/c),\end{aligned} \tag{2.8}$$

where $c=\pi_F^{a(\chi)+n(\psi)}$.

Now if $u\in U_F$ is an unit and we replace $c=cu$, then we have

$$\epsilon(\chi,\psi,cu)=\chi(c)q^{-\frac{a(\chi)}{2}}\sum_{x\in\frac{U_F}{U_F^{a(\chi)}}}\chi^{-1}(x/u)\psi(x/cu)=\epsilon(\chi,\psi,c). \tag{2.9}$$

Therefore $\epsilon(\chi,\psi,c)$ *depends* only on the exponent $\nu_F(c)=a(\chi)+n(\psi)$. Therefore we can simply write $\epsilon(\chi,\psi,c)=\epsilon(\chi,\psi)$, because c is determined by $\nu_F(c)=a(\chi)+n(\psi)$ up to a unit u which has *no influence on* $\epsilon(\chi,\psi,c)$. If χ is unramified, i.e., $a(\chi)=0$, therefore $\nu_F(c)=n(\psi)$. Then from the formula of $\epsilon(\chi,\psi,c)$, we can write

$$\epsilon(\chi, \psi, c) = \chi(c), \tag{2.10}$$

and therefore $\epsilon(1, \psi, c) = 1$ if $\chi = 1$ is the trivial character.

2.3 *Known twisting formula of abelian epsilon factors*

(1) If χ_1 and χ_2 are two unramified characters of $F^\times$ and ψ be a nontrivial additive character of F, then we have from eq. (2.10),

$$\epsilon(\chi_1\chi_2, \psi) = \epsilon(\chi_1, \psi)\epsilon(\chi_2, \psi). \tag{2.11}$$

(2) Let χ_1 and χ_2 be ramified and unramified characters of $F^\times$ respectively, then (cf. [5], (3.2.6.3))

$$\epsilon(\chi_1\chi_2, \psi) = \chi_2(\pi_F)^{a(\chi_1)+n(\psi)} \cdot \epsilon(\chi_1, \psi). \tag{2.12}$$

(3) We also have a twisting formula of epsilon factor by Deligne (cf. [2], Lemma 4.16) under some special conditions which are as follows:
Let α and β be two multiplicative characters of a local field F such that $a(\alpha) \geqslant 2 \cdot a(\beta)$. Let ψ be an additive character of F. Let $y_{\alpha,\psi}$ be an element of $F^\times$ such that

$$\alpha(1 + x) = \psi(y_{\alpha,\psi} x)$$

for all $x \in F$ with valuation $\nu_F(x) \geqslant \frac{a(\alpha)}{2}$ (if $a(\alpha) = 0$, $y_{\alpha,\psi} = \pi_F^{-n(\psi)}$). Then

$$\epsilon(\alpha\beta, \psi) = \beta^{-1}(y_{\alpha,\psi}) \cdot \epsilon(\alpha, \psi). \tag{2.13}$$

3. Generalized twisting formula of epsilon factors

3.1 *Local Gauss sum*

Let m be a nonzero positive integer. Let χ be a nontrivial multiplicative character of F with conductor $a(\chi)$ and $\psi : F \to \mathbb{C}^\times$ be an additive character of F with conductor $n(\psi)$. We define the local character sum of a character χ as

$$G(\chi, \psi, m) = \sum_{x \in \frac{U_F}{U_F^m}} \chi^{-1}(x)\psi(x/c), \tag{3.1}$$

where $c = \pi_F^{a(\chi)+n(\psi)}$. When $m = a(\chi)$, we call $G(\chi, \psi, a(\chi))$ as the *local Gauss sum* of character χ.

Now let conductor of χ be 1, i.e., $\chi : F/U_F^1 \to \mathbb{C}^\times$. Hence $\chi|_{U_F}$ is a character of U_F/U_F^1. If the conductor of $n(\psi)$ is -1, i.e., $\psi : F/P_F \to \mathbb{C}^\times$, then $\psi|_{O_F}$ is an additive character of O_F/P_F. Moreover, when $a(\chi) = 1$ and $n(\psi) = -1$, we have $c = \pi_F^{a(\chi)+n(\psi)} = \pi_F^{1-1} = 1$, and hence eq. (3.1) reduces to the classical Gauss sum, i.e., $G(\chi, \psi, 1) = G(\tilde{\chi}, \psi')$, where $\tilde{\chi} := \chi^{-1}|_{U_F}$ and $\psi' := \psi|_{O_F}$.

PROPOSITION 3.1

The definition of local Gauss sum $G(\chi,\psi,a(\chi))$ does not depend on the choice of the coset representatives of U_F mod $U_F^{a(\chi)}$.

Proof. It is very easy to see from the definition of local Gauss sum. If we change one of the coset representatives x to xu where $u\in U_F^{a(\chi)}$ in $G(\chi,\psi,a(\chi))$, then we have

$$\begin{aligned} G(\chi,\psi,a(\chi)) &= \sum_{x\in\frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x)\psi(x/c)\\ &= \sum_{x\in\frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(xu)\psi(xu/c) \quad (\text{replacing } x \text{ by } xu,\, u\in U_F^{a(\chi)})\\ &= \sum_{x\in\frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x)\psi(x/c)\psi\left(\frac{x}{c}(u-1)\right)\\ &= \sum_{x\in\frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x)\psi(x/c). \end{aligned}$$

Since $P_F^{-n(\psi)}$ is a fractional ideal of O_F, then $\frac{x}{c}(u-1)\in P_F^{-n(\psi)}$ for $x\in U_F$. Therefore $\psi(\frac{x}{c}(u-1)) = 1$ for all $x\in U_F$ and $u\in U_F^{a(\chi)}$. This proves the local Gauss sum is independent of the choice of the coset representatives of U_F mod $U_F^{a(\chi)}$. Therefore definition of local Gauss sum does not depend on the choice of the coset representatives of x. □

In the next proposition we compute the absolute value of $G(\chi,\psi,a(\chi))$ by using Lemma 2.2.

PROPOSITION 3.2

If χ is a ramified character of $F^\times$, then

$$|G(\chi,\psi,a(\chi))| = q^{\frac{a(\chi)}{2}}.$$

Proof. We can write

$$\int_{U_F}\chi^{-1}(x)\psi(x/c)dx = \sum_{x\in\frac{U_F}{U_F^{a(\chi)}}}\chi^{-1}(x)\psi(x/c)\times m'(U_F^{a(\chi)}), \tag{3.2}$$

where $c = \pi_F^{a(\chi)+n(\psi)}$ and m' is the Haar measure which is normalized so that $m'(O_F) = 1$. Now from eq. (3.2) we have

$$\left|\int_{U_F} \chi^{-1}(x)\psi(x/c)dx\right| = \left|\sum_{x\in\frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x)\psi(x/c)\right| \times |m'(U_F^{a(\chi)})|$$
$$= |G(\chi, \psi, a(\chi))|q^{-a(\chi)},$$

since $m'(U_F^{a(\chi)}) = q^{-a(\chi)}$. Therefore from Lemma 2.2, we have

$$|G(\chi, \psi, a(\chi))| = q^{\frac{a(\chi)}{2}}. \tag{3.3}$$

□

Furthermore, from Lemma 2.2, it can be proved that

$$\sum_{x\in\frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x)\psi\left(\frac{x}{\pi_F^{l+m}}\right) = 0, \tag{3.4}$$

for nonzero integers $m \neq 0$ and $l = a(\chi) + n(\psi)$.

In the next lemma, we see the relation between two local character sums $G(\chi, \psi, n_1)$, $G(\chi, \psi, n_2)$, where $n_1 > n_2$ and here we mean

$$G(\chi, \psi, n_i) = \sum_{x\in\frac{U_F}{U_F^{n_i}}} \chi^{-1}(x)\psi(x/c), \quad i = 1, 2,$$

where $c = \pi_F^{a(\chi)+n(\psi)}$.

Lemma 3.3.

$$G(\chi, \psi, n_1) = q^m G(\chi, \psi, n_2). \tag{3.5}$$

where $m = n_1 - n_2$.

Proof. From Lemma 2.2, it is straight forward. We have

$$\int_{U_F} \chi^{-1}(x)\psi\left(\frac{x}{c}\right)dx = \sum_{x\in\frac{U_F}{U_F^{n_1}}} \chi^{-1}(x)\psi\left(\frac{x}{c}\right) \times m'(U_F^{n_1})$$
$$= q^{-n_1}\sum_{x\in\frac{U_F}{U_F^{n_1}}} \chi^{-1}(x)\psi\left(\frac{x}{c}\right), \quad \text{since } m'(U_F^{n_1}) = q^{-n_1}$$
$$= q^{-n_1}G(\chi, \psi, n_1).$$

Similarly, we can express

$$\int_{U_F} \chi^{-1}(x)\psi\left(\frac{x}{c}\right) dx = q^{-n_2} G(\chi, \psi, n_2).$$

Comparing these two above equations, we obtain

$$G(\chi, \psi, n_1) = q^{n_1 - n_2} G(\chi, \psi, n_2) = q^m G(\chi, \psi, n_2).$$

□

3.2 *Local Jacobi sum*

Let χ_1 and χ_2 be two nontrivial characters of $F^\times$. For any $t \in \frac{U_F}{U_F^n}$, where $n \geqslant 1$, we define the local Jacobi sum for characters χ_1 and χ_2 as

$$J_t(\chi_1, \chi_2, n) = \sum_{\substack{x \in \frac{U_F}{U_F^n} \\ t - x \in U_F}} \chi_1^{-1}(x)\chi_2^{-1}(t - x). \tag{3.6}$$

When $n = 1$, $t = 1$, and conductors $a(\chi_1) = a(\chi_2) = 1$, this local Jacobi sum is nothing but the classical Jacobi sum for the characters χ_1^{-1} and χ_2^{-1}, i.e., $J_1(\chi_1, \chi_2, 1) = J_1(\chi_1^{-1}, \chi_2^{-1})$.

PROPOSITION 3.4

$$J_1(\chi_1, \chi_2, n) = \chi_1\chi_2(t) \cdot J_t(\chi_1, \chi_2, n), \quad \textit{for any } t \in \frac{U_F}{U_F^n}. \tag{3.7}$$

Proof. For any $t \in \frac{U_F}{U_F^n}$, from the definition of Jacobi sum, we have

$$\begin{aligned} J_t(\chi_1, \chi_2, n) &= \sum_{\substack{x \in \frac{U_F}{U_F^n} \\ t - x \in U_F}} \chi_1^{-1}(x)\chi_2^{-1}(t - x) \\ &= \sum_{\substack{x/t \in \frac{U_F}{U_F^n} \\ 1 - x/t \in U_F}} (\chi_1\chi_2)^{-1}(t)\chi_1^{-1}(x/t)\chi_2^{-1}(1 - x/t) \\ &= (\chi_1\chi_2)^{-1}(t) \sum_{\substack{s = x/t \in \frac{U_F}{U_F^n} \\ 1 - s \in U_F}} \chi_1^{-1}(s)\chi_2^{-1}(1 - s) \\ &= (\chi_1\chi_2)^{-1}(t) J_1(\chi_1, \chi_2, n). \end{aligned}$$

Therefore,

$$J_1(\chi_1, \chi_2, n) = \chi_1\chi_2(t) \cdot J_t(\chi_1, \chi_2, n), \quad \text{for any } t \in \frac{U_F}{U_F^n}.$$

□

In the following theorem, we give a generalized twisting formula of epsilon factors via the above local Jacobi sums.

Theorem 3.5. *Let F be a non-Archimedean local field with q as the cardinality of the residue field of F. Let ψ be a nontrivial additive character of F. Let χ_1 and χ_2 be two ramified characters of $F^\times$ with conductors n and m respectively. Let r be the conductor of character $\chi_1\chi_2$. Then*

$$\epsilon(\chi_1\chi_2,\psi)=\begin{cases}\dfrac{q^{\frac{n}{2}}\epsilon(\chi_1,\psi)\epsilon(\chi_2,\psi)}{J_1(\chi_1,\chi_2,n)} & \text{when } n=m=r,\\ \dfrac{q^{\frac{r}{2}}\chi_1\chi_2(\pi_F^{r-n})\epsilon(\chi_1,\psi)\epsilon(\chi_2,\psi)}{J_1(\chi_1,\chi_2,n)} & \text{when } n=m>r,\\ \dfrac{q^{n-\frac{m}{2}}\epsilon(\chi_1,\psi)\epsilon(\chi_2,\psi)}{J_1(\chi_1,\chi_2,n)} & \text{when } n=r>m.\end{cases} \tag{3.8}$$

Proof. We know that the formula of epsilon of a character χ of $F^\times$ is

$$\begin{aligned}\epsilon(\chi,\psi)&=\chi(c)q^{-\frac{a(\chi)}{2}}\sum_{x\in\frac{U_F}{U_F^{a(\chi)}}}\chi^{-1}(x)\psi(x/c)\\ &=\chi(c)q^{-\frac{a(\chi)}{2}}G(\chi,\psi,a(\chi)),\end{aligned} \tag{3.9}$$

where $c=\pi_F^{a(\chi)+n(\psi)}$.

Now we divide this proof into three cases.

Case 1: *When $n=m=r$*. By the definition of epsilon factor, in this case we have

$$\epsilon(\chi_1,\psi)=\chi_1(c_1)q^{-\frac{n}{2}}\sum_{x\in\frac{U_F}{U_F^n}}\chi_1^{-1}(x)\psi(x/c_1) \tag{3.10}$$

and

$$\epsilon(\chi_2,\psi)=\chi_2(c_2)q^{-\frac{m}{2}}\sum_{y\in\frac{U_F}{U_F^m}}\chi_2^{-1}(y)\psi(y/c_2), \tag{3.11}$$

where $c_1=\pi_F^{n+n(\psi)}$ and $c_2=\pi_F^{m+n(\psi)}$. Since $n=m$, then we consider $c=c_1=c_2$.

Now from equations (3.10) and (3.11) we have

$$\begin{aligned}\epsilon(\chi_1,\psi)\epsilon(\chi_2,\psi)&=q^{-n}\chi_1\chi_2(c)\sum_{x\in\frac{U_F}{U_F^n}}\chi_1^{-1}(x)\psi(x/c)\times\sum_{y\in\frac{U_F}{U_F^m}}\chi_2^{-1}(y)\psi(y/c)\\ &=q^{-n}\chi_1\chi_2(c)\sum_{x,y\in\frac{U_F}{U_F^n}}\chi_1^{-1}(x)\chi_2^{-1}(y)\psi(x/c)\psi(y/c)\end{aligned}$$

$$= q^{-n}\chi_1\chi_2(c)\sum_{x,y\in\frac{U_F}{U_F^n}}\chi_1^{-1}(x)\chi_2^{-1}(y)\psi\left(\frac{x+y}{c}\right)$$

$$= q^{-n}\chi_1\chi_2(c)\sum_{\substack{x\in\frac{U_F}{U_F^n}\\ t-x\in\frac{U_F}{U_F^n}}}\chi_1^{-1}(x)\chi_2^{-1}(t-x)\psi\left(\frac{t}{c}\right)$$

$$= q^{-n}\chi_1\chi_2(c)\sum_{a=0}^{n-1}\left\{\sum_{u\in\frac{U_F}{U_F^{n-a}}}\left\{\sum_{x\in\frac{U_F}{U_F^n}}\chi_1^{-1}\left(\frac{x}{\pi_F^a u}\right)\chi_2^{-1}\right.\right.$$

$$\left.\left.\times(1-\frac{x}{\pi_F^a u})\right\}(\chi_1\chi_2)^{-1}\left(\pi_F^a u\right)\psi\left(\frac{\pi_F^a u}{c}\right)\right\}$$

$$= q^{-n}\chi_1\chi_2(c)\sum_{a=0}^{n-1}\left\{\sum_{u\in\frac{U_F}{U_F^{n-a}}}\left\{\sum_{s=x/u\in\frac{U_F}{U_F^{n-a}}}\chi_1^{-1}\left(\frac{s}{\pi_F^a}\right)\chi_2^{-1}\right.\right.$$

$$\left.\left.\times\left(1-\frac{s}{\pi_F^a}\right)\right\}(\chi_1\chi_2)^{-1}(\pi_F^a u)\psi\left(\frac{\pi_F^a u}{c}\right)\right\}$$

$$= q^{-n}\chi_1\chi_2(c)\sum_{a=0}^{n-1}\left\{\sum_{s\in\frac{U_F}{U_F^{n-a}}}\chi_1^{-1}\left(\frac{s}{\pi_F^a}\right)\chi_2^{-1}\left(1-\frac{s}{\pi_F^a}\right)\right.$$

$$\left.\times\sum_{u\in\frac{U_F}{U_F^{n-a}}}(\chi_1\chi_2)^{-1}(\pi_F^a u)\psi\left(\frac{\pi_F^a u}{c}\right)\right\}$$

$$= q^{-n}\chi_1\chi_2(c)\sum_{a=0}^{n-1}\left\{J_1^{'}(\chi_1,\chi_2,a)\times G'(\chi_1\chi_2,\psi,a)\right\}, \tag{3.12}$$

where

$$G'(\chi_1\chi_2,\psi,a)=\sum_{u\in\frac{U_F}{U_F^{n-a}}}(\chi_1\chi_2)^{-1}(\pi_F^a u)\psi\left(\frac{\pi_F^a u}{c}\right) \tag{3.13}$$

and

$$J_1^{'}(\chi_1, \chi_2, a) = \sum_{s\in \frac{U_F}{U_F^{n-a}}} \chi_1^{-1}\left(\frac{s}{\pi_F^a}\right)\chi_2^{-1}\left(1-\frac{s}{\pi_F^a}\right). \tag{3.14}$$

In the above calculations, we assume $t = x + y$, where both x and y are in $\frac{U_F}{U_F^n}$ and this t can be written as $t = \pi_F^a u$, where a varies over $\{0, 1, \dots, n-1\}$ and $u \in \frac{U_F}{U_F^{n-a}}$. Furthermore, by using Lemma 2.2, for $a \neq 0$, we have

$$\sum_{u\in \frac{U_F}{U_F^{n-a}}} (\chi_1\chi_2)^{-1}(u)\psi\left(\frac{\pi_F^a u}{c}\right) = 0.$$

Therefore, for $a \neq 0$, we can write

$$\begin{aligned} G'(\chi_1\chi_2, \psi, a) &= \sum_{u\in \frac{U_F}{U_F^{n-a}}} (\chi_1\chi_2)^{-1}(\pi_F^a u)\psi\left(\frac{\pi_F^a u}{c}\right) \\ &= \chi_1\chi_2(\pi_F^{-a}) \sum_{u\in \frac{U_F}{U_F^{n-a}}} (\chi_1\chi_2)^{-1}(u)\psi\left(\frac{\pi_F^a u}{c}\right) \\ &= 0. \end{aligned}$$

Therefore, we have to take $a = 0$, because the left side of eq. (3.12) is *nonzero*, therefore $t = u \in \frac{U_F}{U_F^n}$. Now, put $a = 0$ in eq. (3.13), then we have from eq. (3.12)

$$\begin{aligned} \epsilon(\chi_1, \psi)\epsilon(\chi_2, \psi) &= q^{-n}\chi_1\chi_2(c)J_1^{'}(\chi_1\chi_2, a) \sum_{\alpha\in \frac{U_F}{U_F^r}} (\chi_1\chi_2)^{-1}(\alpha)\psi\left(\frac{\alpha}{c}\right) \\ &= q^{-n}\chi_1\chi_2(c)J_1(\chi_1, \chi_2, n)G(\chi_1\chi_2, \psi, n) \\ &= q^{-\frac{n}{2}}J_1(\chi_1, \chi_2, n)\epsilon(\chi_1\chi_2, \psi). \end{aligned}$$

Therefore, in this case we have

$$\epsilon(\chi_1\chi_2, \psi) = \frac{q^{\frac{n}{2}}\epsilon(\chi_1, \psi)\epsilon(\chi_2, \psi)}{J_1(\chi_1, \chi_2, n)}. \tag{3.15}$$

Case 2: *When* $n = m > r$. Like Case 1, in this case, it can be showed that $t = x + y \in \frac{U_F}{U_F^n}$ when $x, y \in \frac{U_F}{U_F^n}$. Since $c_1 = c_2$, let $c = c_1 = c_2$. In this situation, we have

$$\begin{aligned}
\epsilon(\chi_1,\psi)\epsilon(\chi_2,\psi) &= q^{-n}\chi_1\chi_2(c)\sum_{x\in\frac{U_F}{U_F^n}}\chi_1^{-1}(x)\psi(x/c)\sum_{y\in\frac{U_F}{U_F^m}}\chi_2^{-1}(y)\psi(y/c)\\
&= q^{-n}\chi_1\chi_2(c)\sum_{x,y\in\frac{U_F}{U_F^n}}\chi_1^{-1}(x)\chi_2^{-1}(y)\psi(x/c)\psi(y/c)\\
&= q^{-n}\chi_1\chi_2(c)\sum_{x,y\in\frac{U_F}{U_F^n}}\chi_1^{-1}(x)\chi_2^{-1}(y)\psi\left(\frac{x+y}{c}\right)\\
&= q^{-n}\chi_1\chi_2(c)\sum_{t,x\in\frac{U_F}{U_F^n}}\chi_1^{-1}(x)\chi_2^{-1}(t-x)\psi\left(\frac{t}{c}\right)\\
&= q^{-n}\chi_1\chi_2(c)\sum_{t\in\frac{U_F}{U_F^n}}\left\{\sum_{x\in\frac{U_F}{U_F^n}}\chi_1^{-1}(x/t)\chi_2^{-1}(1-x/t)\right\}(\chi_1\chi_2)^{-1}(t)\psi\left(\frac{t}{c}\right)\\
&= q^{-n}\chi_1\chi_2(c)\sum_{s=x/t\in\frac{U_F}{U_F^n}}\chi_1^{-1}(s)\chi_2^{-1}(1-s)\sum_{t\in\frac{U_F}{U_F^n}}(\chi_1\chi_2)^{-1}(t)\psi\left(\frac{t}{c}\right)\\
&= q^{-n}\chi_1\chi_2(c)J_1(\chi_1,\chi_2,n)G(\chi_1\chi_2,\psi,n)\\
&= q^{-n}\chi_1\chi_2(c)J_1(\chi_1,\chi_2,n)\,q^{n-r}G(\chi_1\chi_2,\psi,r)\quad \text{using Lemma 4.3}\\
&= q^{-\frac{r}{2}}\chi_1\chi_2(\pi_F^{n-r})J_1(\chi_1,\chi_2,n)\chi_1\chi_2(\pi_F^{r+n(\psi)})q^{-\frac{r}{2}}G(\chi_1\chi_2,\psi,r)\\
&= q^{-\frac{r}{2}}\chi_1\chi_2(\pi_F^{n-r})J_1(\chi_1,\chi_2,n)\epsilon(\chi_1\chi_2,\psi),\quad \text{since } a(\chi_1\chi_2)=r.
\end{aligned}$$

Therefore, in this condition we have

$$\epsilon(\chi_1\chi_2,\psi)=\frac{q^{\frac{r}{2}}\chi_1\chi_2(\pi_F^{r-n})\epsilon(\chi_1,\psi)\epsilon(\chi_2,\psi)}{J_1(\chi_1,\chi_2,n)}. \tag{3.16}$$

Case 3: *When* $n=r>m$. If conductor $a(\chi_1)>a(\chi_2)$, then conductor $a(\chi_1\chi_2)=\max(a(\chi_1),a(\chi_2))=a(\chi_1)$. Therefore, we are in this situation: $n=r>m$. In this case c_1 can be written as $c_1=c_2\pi_F^{n-m}$. If $x,z\in\frac{U_F}{U_F^n}$, then $x+\pi_F^{n-m}z\in\frac{U_F}{U_F^n}$. We have

$$\begin{aligned}
&\epsilon(\chi_1,\psi)\epsilon(\chi_2,\psi)\\
&\quad= q^{-\frac{m+n}{2}}\chi_1(c_1)\chi_2(c_2)\sum_{x\in\frac{U_F}{U_F^n}}\chi_1^{-1}(x)\psi(x/c_1)\times\sum_{z\in\frac{U_F}{U_F^m}}\chi_2^{-1}(z)\psi(z/c_2)\\
&\quad= q^{-\frac{m+n}{2}}\chi_2(\pi_F^{m-n})\chi_1\chi_2(c_1)\sum_{x\in\frac{U_F}{U_F^n}}\chi_1^{-1}(x)\psi(x/c_1)\sum_{z\in\frac{U_F}{U_F^m}}\chi_2^{-1}(z)\psi\left(\frac{z\pi_F^{n-m}}{c_1}\right)
\end{aligned}$$

$$= q^{-\frac{m+n}{2}} \chi_2\left(\pi_F^{m-n}\right) \chi_1\chi_2(c_1) \sum_{x\in\frac{U_F}{U_F^n}} \chi_1^{-1}(x)\psi(x/c_1)$$

$$\times q^{m-n} \sum_{z\in\frac{U_F}{U_F^n}} \chi_2^{-1}(z)\psi\left(\frac{z\pi_F^{n-m}}{c_1}\right)$$

$$= q^{\frac{m}{2}-\frac{3n}{2}} \chi_2\left(\pi_F^{m-n}\right) \chi_1\chi_2(c_1) \sum_{x,z\in\frac{U_F}{U_F^n}} \chi_1^{-1}(x)\chi_2^{-1}(z)\psi(x/c_1)\psi\left(\frac{z\pi_F^{n-m}}{c_1}\right)$$

$$= q^{\frac{m}{2}-\frac{3n}{2}} \chi_2\left(\pi_F^{m-n}\right) \chi_1\chi_2(c_1) \sum_{x,z\in\frac{U_F}{U_F^n}} \chi_1^{-1}(x)\chi_2^{-1}(z)\psi\left(\frac{x+z\pi_F^{n-m}}{c_1}\right)$$

$$= q^{\frac{m}{2}-\frac{3n}{2}} \chi_2\left(\pi_F^{m-n}\right) \chi_1\chi_2(c_1)\chi_2\left(\pi_F^{n-m}\right) \sum_{x,t\in\frac{U_F}{U_F^n}} \chi_1^{-1}(x)\chi_2^{-1}(t-x)\psi\left(\frac{t}{c_1}\right)$$

$$= q^{\frac{m}{2}-\frac{3n}{2}} \chi_1\chi_2(c_1) \sum_{t\in\frac{U_F}{U_F^n}} \left\{\sum_{x\in\frac{U_F}{U_F^n}} \chi_1^{-1}(x/t)\chi_2^{-1}(1-x/t)\right\} (\chi_1\chi_2)^{-1}(t)\psi\left(\frac{t}{c_1}\right)$$

$$= q^{\frac{m}{2}-\frac{3n}{2}} \chi_1\chi_2(c_1) \sum_{s=x/t\in\frac{U_F}{U_F^n}} \chi_1^{-1}(s)\chi_2^{-1}(1-s) \times \sum_{t\in\frac{U_F}{U_F^n}} (\chi_1\chi_2)^{-1}(t)\psi\left(\frac{t}{c_1}\right)$$

$$= q^{\frac{m}{2}-\frac{3n}{2}} \chi_1\chi_2(c_1) J_1(\chi_1,\chi_2,n) \times \sum_{t\in\frac{U_F}{U_F^n}} (\chi_1\chi_2)^{-1}(t)\psi\left(\frac{t}{c_1}\right)$$

$$= q^{\frac{m}{2}-n} J_1(\chi_1,\chi_2,n)\chi_1\chi_2(c_1) q^{-\frac{n}{2}} \times G(\chi_1\chi_2,\psi,n)$$

$$= q^{\frac{m}{2}-n} J_1(\chi_1,\chi_2,n)\epsilon(\chi_1\chi_2,\psi).$$

Therefore we have

$$\epsilon(\chi_1\chi_2,\psi) = \frac{q^{n-\frac{m}{2}}\epsilon(\chi_1,\psi)\epsilon(\chi_2,\psi)}{J_1(\chi_1,\chi_2,n)}. \tag{3.17}$$

□

Remark 3.6. Let χ_1, χ_2 be two characters of $F^\times$ with conductors $a(\chi_1) = a(\chi_2) = 1$. Let ψ be a nontrivial additive character of F. If the conductor of $\chi_1\chi_2$ is 1, then from using the above Theorem 3.5 and eq. (2.4) we can say

$$\frac{\epsilon(\chi_1\chi_2,\psi)}{\epsilon(\chi_1,\psi)\epsilon(\chi_2,\psi)} = \gamma, \tag{3.18}$$

where γ is a root of unity for which $J_1(\chi_1,\chi_2,1) = q^{\frac{1}{2}} \cdot \gamma^{-1}$.

Now let μ be the group of roots of unity which contains γ. Then for this special case (i.e., $a(\chi_1) = a(\chi_2) = a(\chi_1\chi_2) = 1$), we can write

$$\epsilon(\chi_1\chi_2) \equiv \epsilon(\chi_1, \psi) \cdot \epsilon(\chi_2, \psi) \mod \mu. \tag{3.19}$$

We also observe that our local Jacobi sum $J_1(\chi_1, \chi_2, n)$ is the generalization of the classical Jacobi sum. But explicit computation of this local Jacobi sums are difficult. When $n = 1 = a(\chi_1) = a(\chi_2)$, from eq. (2.4) we can say that $|J_1(\chi_1, \chi_2, 1)| = q^{\frac{1}{2}}$. If we can compute this local Jacobi sums explicitly, then by using Theorem 3.5, one can give more explicit twisting formula of epsilon factors.

By using our twisting formula (3.8) and Deligne's formula (2.13), for the following case we can give the explicit formula for local Jacobi sum $J_1(\chi_1, \chi_2, a(\chi_1))$, when $a(\chi_1) > a(\chi_2) = 1$.

PROPOSITION 3.7

Let F be a non-Archimedean local field with q as the cardinality of the residue field of F. Let χ_1 be a character of $F^\times$ of conductor $a(\chi_1) > 1$. Let χ_2 be a character of $F^\times$ of conductor $a(\chi_2) = 1$. Let ψ is an additive character of F of conductor -1, then

$$J_1(\chi_1, \chi_2, a(\chi_1)) = q^{a(\chi_1)-1} \cdot \chi_2(y) \cdot G(\chi_2^{-1}, \psi), \tag{3.20}$$

where $y = y(\chi_1, \psi) \in F^\times$ such that $\chi_1(1 + x) = \psi(yx)$ for all $x \in F$ with valuation $\nu_F(x) \geqslant \frac{a(\chi_1)}{2}$.

Proof. For the assumptions, from eq. (3.8) we have

$$\epsilon(\chi_1\chi_2, \psi) = \frac{q^{a(\chi_1)-\frac{1}{2}}\epsilon(\chi_1, \psi)\epsilon(\chi_2, \psi)}{J_1(\chi_1, \chi_2, a(\chi_1))}. \tag{3.21}$$

Since $a(\chi_1) > 1 = a(\chi_2)$, hence $a(\chi_1) \geqslant 2 \cdot a(\chi_2)$. From the Deligne's formula (2.13) we have

$$\epsilon(\chi_1\chi_2, \psi) = \chi_2^{-1}(y) \cdot \epsilon(\chi_1, \psi), \tag{3.22}$$

where $y = y(\chi_1, \psi) \in F^\times$ such that $\chi_1(1 + x) = \psi(yx)$ for all $x \in F$ with valuation $\nu_F(x) \geqslant \frac{a(\chi_1)}{2}$.

Comparing equations (3.21) and (3.22) we have

$$J_1(\chi_1, \chi_2, a(\chi_1)) = q^{a(\chi_1)-\frac{1}{2}} \cdot \chi_2(y) \cdot \epsilon(\chi_2, \psi). \tag{3.23}$$

Again by the given conditions $a(\chi_2) = 1$ and $n(\psi) = -1$, hence $c = \pi_F^{a(\chi_2)+n(\psi)} = \pi_F^{1-1} = 1$. Thus we can write

$$\epsilon(\chi_2, \psi) = q^{-\frac{1}{2}} \sum_{x \in U_F/U_F^1} \chi_2^{-1}(x)\psi(x) = q^{-\frac{1}{2}} \cdot G(\chi_2^{-1}, \psi), \tag{3.24}$$

since $\psi|_{O_F}$ is an additive character of O_F/P_F.

By using eq. (3.24), from eq. (3.23) we obtain

$$J_1(\chi_1, \chi_2, a(\chi_1)) = q^{a(\chi_1)-1} \cdot \chi_2(y) \cdot G(\chi_2^{-1}, \psi).$$

□

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