Proc. Indian Acad. Sci. (Math. Sci.) Vol. 127, No. 4, September 2017, pp. 565–584.
DOI 10.1007/s12044-017-0353-4

Contributions to a conjecture of Mueller and Schmidt on Thue inequalities

N SARADHA* and DIVYUM SHARMA

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400 005, India
*Corresponding author.
E-mail: saradha@math.tifr.res.in; divyum@math.tifr.res.in

MS received 11 April 2016; revised 2 September 2016; published online 3 August 2017

Abstract. Let $F(X,Y) = \sum_{i=0}^{s} a_i X^{r_i} Y^{r-r_i} \in \mathbb{Z}[X,Y]$ be a form of degree $r = r_s \geq 3$, irreducible over $\mathbb{Q}$ and having at most $s+1$ non-zero coefficients. Mueller and Schmidt showed that the number of solutions of the Thue inequality

$$|F(X,Y)| \leq h$$

is $\ll s^2 h^{2/r}(1+\log h^{1/r})$. They *conjectured* that s^2 may be replaced by s. Let

$$\Psi = \max_{0\leq i\leq s} \max\left(\sum_{w=0}^{i-1} \frac{1}{r_i - r_w}, \sum_{w=i+1}^{s} \frac{1}{r_w - r_i}\right).$$

Then we show that s^2 may be replaced by $\max(s\log^3 s, se^{\Psi})$. We also show that if $|a_0| = |a_s|$ and $|a_i| \leq |a_0|$ for $1 \leq i \leq s-1$, then s^2 may be replaced by $s\log^{3/2} s$. In particular, this is true if $a_i \in \{-1, 1\}$.

Keywords. Thue equations; Thue inequalities; Archimedean Newton polygon.

2010 Mathematics Subject Classification. Primary: 11J25; 11D59; Secondary: 11D45.

1. Introduction

Let $F(X,Y)$ be a form of degree $r \geq 3$ with integer coefficients, irreducible over $\mathbb{Q}$ and having at most $s+1$ non-zero coefficients. Write

$$F(X,Y) = \sum_{i=0}^{s} a_i X^{r_i} Y^{r-r_i} \tag{1}$$

with $0 = r_0 < r_1 < \cdots < r_s = r$. Let Δ, H and M denote the discriminant, height and Mahler height of $F(X,1)$ respectively. For $h \geq 1$, consider the Thue inequality

$$|F(X,Y)| \leq h. \tag{2}$$

Let $N_F(h)$ denote the number of integer solutions (x,y) of (2). Schmidt [12] proved that

$$N_F(h) \ll \sqrt{rs}\, h^{2/r}(1 + \log h^{1/r}). \tag{3}$$

His result depended on the analysis of location of roots of $F(X, 1)$. Bombieri modified a conjecture of Siegel on the inequality (2) as

$$N_F(h) \leq C(s, h),$$

where $C(s, h)$ depends only on s and h (see [11, p. 208]). This was shown to be true in the case $s = 1$ by Hyyrö [7], Evertse [3] and Mueller [9]. The case $s \geq 2$ was considered by Mueller and Schmidt in [10] and [11]. They proved that

$$N_F(h) \ll s^2 C_1(r, h), \tag{4}$$

where $C_1(r, h) = h^{2/r}(1 + \log h^{1/r})$. In all the estimates for $N_F(h)$, the factor $h^{2/r}$ in $C_1(r, h)$ is unavoidable. The logarithmic factor in $C_1(r, h)$ was improved by Thunder [13,14] when h is large. We refer to [4] and [6] for more results on Thue equations and Thue inequalities. It was *conjectured* in [11] that it may be possible to replace the factor s^2 in (4) by s. In Theorem 1.1 below, we show that when the non-zero terms of F are sufficiently far apart, then s^2 can be improved. In Theorem 1.5, we improve (4) for forms with restricted coefficients.

Theorem 1.1. *Let $F(X, Y)$ be given by* (1). *Put*

$$\Psi = \max_{0 \leq i \leq s} \max\left(\sum_{w=0}^{i-1} \frac{1}{r_i - r_w}, \sum_{w=i+1}^{s} \frac{1}{r_w - r_i}\right) \tag{5}$$

and

$$\Phi = \max(\Psi, 3 \log\log s).$$

Then we have

$$N_F(h) \ll s e^{\Phi} C_1(r, h). \tag{6}$$

In (5), *an empty sum is taken to be equal to zero.*

Remark 1.2. Suppose $r \leq 4se^{2\Phi}$. Then (3) implies (6). Thus, for proving Theorem 1.1, we may assume that

$$r > 4se^{2\Phi}. \tag{7}$$

Further, we may take $s \gg 1$, as otherwise inequality (4) is sufficient.

Throughout the paper, $c_1, c_2, \ldots$ denote positive absolute constants. We shall illustrate Theorem 1.1 with some examples below.

Remark 1.3. Since $|r_i - r_w| \geq |i - w|$, for all $i, w \in \{0, \ldots, s\}$, it follows that

$$\Psi \leq \max_{0 \leq i \leq s} \max \left(\sum_{n=1}^{i} \frac{1}{n}, \sum_{n=1}^{s-i} \frac{1}{n} \right) \leq \log s + c_1,$$

(see [1, p. 55]). Thus we get (4).

Remark 1.4. We give some instances when s^2 in (4) can be improved.

(i) Suppose that there is a $c_2 > 1$ such that $|r_i - r_w| \geq c_2 |i - w|$ for all $i, w \in \{0, \ldots, s\}$. Then $\Psi \leq \frac{1}{c_2} \log s + c_3$. Hence

$$N_F(h) \ll s^{1+\frac{1}{c_2}} C_1(r, h).$$

(ii) Suppose $|r_i - r_w| \geq \frac{1}{3} |i - w| \log |i - w|$ for all $i, w \in \{0, \ldots, s\}$ with $i \neq w$. Then

$$\Psi \leq 1 + \max_{0 \leq i \leq s} \max \left(\sum_{n=2}^{i} \frac{3}{n \log n}, \sum_{n=2}^{s-i} \frac{3}{n \log n} \right) \leq 3 \log \log s + c_4,$$

(see [1, p. 70]).

In a different direction, we impose restrictions on the coefficients of F and improve (4).

Theorem 1.5. *Suppose that the coefficients of $F(X, Y)$ satisfy*

$$\left| \frac{a_0}{a_s} \right|^{1/r} \leq \left| \frac{a_0}{a_i} \right|^{1/r_i} \quad \text{for } i = 1, \ldots, s-1. \tag{8}$$

If $r \geq s \log^3 s$, then

$$N_F(h) \ll s (\log s)\, h^{2/r}.$$

Remark 1.6. If $r < s \log^3 s$, then by (3) we have

$$N_F(h) \ll s \log^{3/2} s\; C_1(r, h).$$

As an immediate consequence of Theorem 1.5 and Remark 1.6, we get

COROLLARY 1.7

Suppose that $|a_0| = |a_s| = H$, where H is the height of F. Then

$$N_F(h) \ll s \log^{3/2} s\; C_1(r, h).$$

In particular, the above estimate is valid if the coefficients a_i assume only the values ± 1.

We shall derive Theorem 1.1 as a consequence of the following result.

PROPOSITION 1.8

Let $F(X, Y)$ be given by (1). Then

$$N_F(h) \ll s\left(\frac{\log s}{\Phi} + e^{\Phi + c_5(\log^3 s)e^{-\Phi}}\right) C_1(r, h).$$

In the proofs of the above results, we follow the method of Mueller and Schmidt with necessary modifications.

2. Preliminaries

Let X_1 and X_2 be positive numbers. Divide the solutions (x, y) of (2) into three sets according as

$$\max(|x|, |y|) > X_1; \max(|x|, |y|) \leq X_1$$

and

$$\min(|x|, |y|) \geq X_2; \min(|x|, |y|) < X_2.$$

Denote the number of *primitive* solutions, i.e. solutions (x, y) with $\gcd(x, y) = 1$, in these sets by $P_{\ell ar}(X_1)$, $P_{med}(X_1, X_2)$ and $P_{sma}(X_2)$ respectively. If $X_2 > X_1$, put $P_{med}(X_1, X_2) = 0$. Choose numbers a, b with $0 < a < b < 1$. Define

$$t = \sqrt{2/(r + a^2)},\ \lambda = 2/((1 - b)t),$$

$$\delta = \frac{(r + b^2)t^2 - 2}{r - 1},\ A = \frac{1}{a^2}\left(\log M + \frac{r}{2}\right).$$

Further, we put

$$B_1 = \frac{2^r r^{r/2} M^r h}{\sqrt{|\Delta|}},$$

$$Y_E = (2B\sqrt{|\Delta|})^{1/(r-\lambda)}(4e^A)^{\lambda/(r-\lambda)},$$

$$Y_G = (2B_1)^{\frac{1}{r-2} + \frac{1}{r^2}} \text{ and } Y_W = R_1^{1/(r-\lambda)} Y_E \tag{9}$$

where

$$R_1 = e^{800 \log^3 r}.$$

Since $|\Delta| \leq r^r M^{2r-2}$ ([8, Theorem 1]), we have $B_1 > 1$. It was shown by Bombieri and Schmidt [2] that $P_{\ell ar}(Y_E) \ll r$. Further, Győry [5] showed that $P_{\ell ar}(Y_G) \leq 25r$. (In fact, he proved that $P_{\ell ar}(Y_G) \leq 5r$ if r is sufficiently large.) Since Y_G is much smaller than Y_E, the latter result is better than the former. Further, Schmidt [12] showed that

$$P_{\ell ar}(Y_E) \ll \sqrt{rs}$$

while Mueller and Schmidt obtained that

$$P_{\ell ar}(Y_W) \ll s. \tag{10}$$

In the following lemma, we estimate the cardinality of a set whose elements satisfy certain gap conditions.

Lemma 2.1. *Let $n \geq 2$ and let $U = \{u_1, \dots, u_n\}$ be a set together with a map $T : U \to \mathbb{R}^*$ such that*

$$A_1 \leq T(u_1) \leq T(u_2) \leq \cdots \leq T(u_n)$$

and

$$T(u_i) \geq \beta\, T(u_{i-1})^{\gamma} \quad \text{for } 2 \leq i \leq n, \tag{11}$$

where $\beta > 0$, $\gamma \geq 2$. Let

$$\kappa = \begin{cases} 2 & \text{if } \beta > 1, \\ 1 & \text{if } \beta \leq 1. \end{cases}$$

Suppose that $T(u_n) \leq B_1$ and $A_1\beta^{1/(\kappa(\gamma-1))} > 1$. Then

$$n \leq 1 + \frac{1}{\log \gamma} \log\left(\frac{\log B_1}{\log A_1 + (\log \beta)/(\kappa(\gamma-1))}\right). \tag{12}$$

Proof. From (11), by induction, we get

$$\begin{aligned} T(u_n) &\geq \beta^{1+\gamma+\cdots+\gamma^{n-2}} T(u_1)^{\gamma^{n-1}} \\ &\geq (\beta^{1/(\kappa(\gamma-1))} T(u_1))^{\gamma^{n-1}}. \end{aligned} \tag{13}$$

Since $T(u_n) \leq B_1$, (13) implies that

$$1 < (\beta^{1/(\kappa(\gamma-1))} T(u_1))^{\gamma^{n-1}} \leq B_1.$$

Taking logarithm and using $T(u_1) \geq A_1$, we find

$$\gamma^{n-1} \leq \frac{\log B_1}{\log A_1 + (\log \beta)/(\kappa(\gamma-1))}.$$

Since $\gamma \geq 2$, $n \geq 2$, the right-hand side of the above inequality is >1. Taking logarithm once again, we get (12). □

Let S be any finite set of complex numbers and let ξ be a real number. Define the distance of ξ from S, denoted by $d(S, \xi)$, as

$$d(S, \xi) = \min_{\eta \in S} |\xi - \eta|.$$

Thus if $\xi \in S$, then $d(S, \xi) = 0$. In the ensuing discussions, we specialize S as the set of roots $\alpha_1, \ldots, \alpha_r$ of

$$f(Z) = F(Z, 1).$$

These discussions are valid if we replace S by S^*, which is the set of roots $\beta_1, \ldots, \beta_r$ of $F(1, Z)$. Note that $\{\beta_1, \ldots, \beta_r\} = \{\alpha_1^{-1}, \ldots, \alpha_r^{-1}\}$ and $F(1, Z)$ has the same discriminant, height and Mahler height as $F(Z, 1)$. In [11], the following result was shown.

Lemma 2.2 [11, Lemma 7]. *There is a set* $S_1 \subseteq S$ *with* $|S_1| \le 6s + 4$ *such that for any real* ξ, *we have*

$$d(S_1, \xi) \le R_1\, d(S, \xi)$$

where $R_1 = e^{800 \log^3 r}$.

In §§ 3, 4 and 5, we improve the mechanism for dealing with medium solutions developed by Mueller and Schmidt.

3. Archimedean Newton polygon and large derivatives

Let F be as in (1). The Archimedean Newton polygon of $f(z) = F(z, 1)$ is the lower boundary of the convex hull of the points

$$P_i = (r_i, -\log |a_i|),\ i = 0, \ldots, s.$$

We label the vertices of the Newton polygon as

$$P_0 = P_{i(0)}, P_{i(1)}, \ldots, P_{i(\ell)} = P_s,$$

where $0 = i(0) < i(1) < \cdots < i(\ell) = s$. For $k = 1, \ldots, \ell$, define $\sigma(i(k))$ as the slope of the line segment $P_{i(k-1)}, P_{i(k)}$. For $k = 0, \ldots, \ell - 1$, define $\sigma^+(i(k))$ as the slope of the line segment $P_{i(k)}, P_{i(k+1)}$. Let α be a root of f. We define $K(\alpha)$, $k(\alpha)$ as follows. If $\sigma^+(i(\ell - 1)) = \sigma(s) < \log |\alpha| + \Psi + \log 3$, then put $K(\alpha) = \ell$. If not, then define $K(\alpha)$ as the least integer K in $0 \le K \le \ell - 1$ with $\sigma^+(i(K)) \ge \log |\alpha| + \Psi + \log 3$. If $\sigma(i(1)) = \sigma^+(0) > \log |\alpha| - \Psi - \log 3$, then put $k(\alpha) = 0$. Otherwise, define $k(\alpha)$ as the largest integer k in $1 \le k \le \ell$ with $\sigma(i(k)) \le \log |\alpha| - \Psi - \log 3$. Clearly, $k(\alpha) \le K(\alpha)$. In [11], $k(\alpha)$ and $K(\alpha)$ are defined with Ψ replaced by $\log s$. As in [11, equation (6.3)], we have

$$k(\alpha) < K(\alpha)$$

for every root α of f.

Lemma 3.1. *The coefficients of* F *satisfy condition* (8) *if and only if the Newton polygon of* F *is a straight line joining* $(0, -\log |a_0|)$ *and* $(r, -\log |a_s|)$. *Further, in this case the height* H *of* F *is either* $|a_0|$ *or* $|a_s|$.

Proof. The Newton polygon is a straight line if and only if the slope of the line joining $(0, -\log |a_0|)$ and $(r, -\log |a_s|)$ does not exceed the slope of the line joining $(0, -\log |a_0|)$ and $(r_i, -\log |a_i|)$ for $i = 1, \ldots, s - 1$, i.e. for every i with $1 \le i \le s - 1$, we have

$$\frac{-\log |a_s| + \log |a_0|}{r} \le \frac{-\log |a_i| + \log |a_0|}{r_i}$$

or

$$\left|\frac{a_0}{a_s}\right|^{1/r} \le \left|\frac{a_0}{a_i}\right|^{1/r_i}$$

which is condition (8). This proves the first assertion. Condition (8) means that for $i = 1, \dots, s-1$, we have

$$|a_s||a_0|^{\frac{r}{r_i}-1} \ge |a_i|^{\frac{r}{r_i}}.$$

If $|a_0| \le |a_s|$, then the above inequality implies that

$$|a_s|^{\frac{r}{r_i}} \ge |a_i|^{\frac{r}{r_i}} \quad \text{for } i = 1, \dots, s-1.$$

Thus

$$|a_s| \ge |a_i| \quad \text{for } i = 0, \dots, s-1.$$

Hence $H = |a_s|$. Similarly, if $|a_s| \le |a_0|$, then we get

$$|a_0|^{\frac{r}{r_i}} \ge |a_i|^{\frac{r}{r_i}} \quad \text{for } i = 1, \dots, s-1.$$

Thus $H = |a_0|$. This proves the second assertion. □

Remark 3.2. By Lemma 3.1, when the coefficients of F satisfy (8), we have $\ell = 1$. Since $k(\alpha) < K(\alpha)$, we find in this case

$$k(\alpha) = 0 \text{ and } K(\alpha) = 1$$

for any $\alpha \in S$.

As a consequence of Lemma 3.1, we get the following result which is a special case of Lemma 1(i) of [11].

Lemma 3.3. *Suppose that the coefficients of F satisfy* (8). *Then for every root α of f, we have*

$$\frac{1}{2}e^{\sigma} < |\alpha| < 2e^{\sigma},$$

where σ denotes the slope of the line joining $(0, -\log|a_0|)$ *and* $(r, -\log|a_s|)$.

Proof. By Lemma 3.1, the coefficients of F satisfy (8) if and only if

$$\sigma \le \frac{-\log|a_i| + \log|a_0|}{r_i} \quad \text{for } i = 1, \dots, s-1.$$

This implies that

$$|a_s|e^{\sigma r} = |a_0| \ge |a_i|e^{\sigma r_i} \quad \text{for } i = 0, \dots, s-1.$$

When $z = e^{\sigma} w$ with $|w| \geq 2$, we have

$$\begin{aligned}|f(z)| &= |a_s e^{\sigma r} w^r + a_{s-1} e^{\sigma r_{s-1}} w^{r_{s-1}} + \cdots + a_0| \\ &\geq |a_s e^{\sigma r}||w^r| - |a_{s-1} e^{\sigma r_{s-1}}||w^{r_{s-1}}| - \cdots - |a_0| \\ &\geq |a_s e^{\sigma r}|(|w^r| - |w^{r_{s-1}}| - \cdots - 1) > 0.\end{aligned}$$

Therefore if α is a root of f, then

$$|\alpha| < 2e^{\sigma}.$$

Now consider the reciprocal polynomial $\hat{f}$ of f, i.e.

$$\hat{f}(z) = z^r f(1/z).$$

The Newton polygon of $\hat{f}$ is the single line with slope $-\sigma$. Hence every root $\hat{\alpha}$ of $\hat{f}$ satisfies

$$|\hat{\alpha}| < 2e^{-\sigma}.$$

Since the roots of $\hat{f}$ are the reciprocals of the roots of f, we obtain

$$|\alpha| > \frac{1}{2} e^{\sigma}$$

for every root α of f. This completes the proof of the lemma. □

We shall now use the Newton polygon to prove that for each root α of f, there exists u with $1 \leq u \leq r$ such that $|f^{(u)}(\alpha)|$ is large. This will enable us to obtain good rational approximations to α from the solutions of (2) (see Lemma 4.1). We introduce some notation. Let e, h be two non-negative integers. Let $(e)_h$ be given by

$$(e)_h = \begin{cases} 0 & \text{if } e = 0, \\ 1 & \text{if } h = 0, \\ e(e-1)\cdots(e-h+1) & \text{otherwise.} \end{cases}$$

For a positive integer t, define

$$\Delta_t^-(e) = \begin{pmatrix} (e)_0 \\ \vdots \\ (e)_t \end{pmatrix}.$$

Further, for $0 \leq u \leq t$, let

$$\Delta_{t,u}^-(e) = \begin{pmatrix} (e)_0 \\ \vdots \\ (e)_{u-1} \\ (e)_{u+1} \\ \vdots \\ (e)_t \end{pmatrix}.$$

If $\{a_1, \ldots, a_{t+1}\}$ is a set of positive integers, then

$$\det\left(\Delta_t^-(a_1), \ldots, \Delta_t^-(a_{t+1})\right) = \prod_{1\le i<j\le t+1} (a_j - a_i). \tag{14}$$

Let $\{b_1, \ldots, b_t\}$ be any set of positive integers. Put

$$E_u^{(t)} = E_u^{(t)}(b_1, \ldots, b_t) = (-1)^{t+u} \det\left(\Delta_{t,u}^-(b_1), \ldots, \Delta_{t,u}^-(b_t)\right).$$

Then for any positive integer e, we have

$$\sum_{u=0}^{t} (e)_u E_u^{(t)}(b_1, \ldots, b_t) = \det\left(\Delta_t^-(b_1), \ldots, \Delta_t^-(b_t), \Delta_t^-(e)\right).$$

We denote $\det\left(\Delta_t^-(b_1), \ldots, \Delta_t^-(b_t), \Delta_t^-(e)\right)$ by $D(b_1, \ldots, b_t, e)$. Note that $D(b_1, \ldots, b_t, e) = 0$ whenever $e = b_i$ for any i with $1 \le i \le t$.

Lemma 3.4. *Let $P(z) = p_1 z^{e_1} + \cdots + p_m z^{e_m}$, $e_1 < \cdots < e_m$, be a polynomial and let $\{b_1, \ldots, b_t\}$ be any set of positive integers. Then*

$$\sum_{u=0}^{t} E_u^{(t)} z^u P^{(u)}(z) = \sum_{i=1}^{m} p_i z^{e_i} D(b_1, \ldots, b_t, e_i),$$

where $E_u^{(t)} = E_u^{(t)}(b_1, \ldots, b_t)$.

Proof. Observe that for $0 \le u \le t$, we have

$$z^u P^{(u)}(z) = (e_1)_u\, p_1 z^{e_1} + \cdots + (e_m)_u\, p_m z^{e_m}.$$

Then

$$\begin{aligned}\sum_{u=0}^{t} E_u^{(t)} z^u P^{(u)}(z) &= p_1 z^{e_1} \sum_{u=0}^{t} (e_1)_u E_u^{(t)} + \cdots + p_m z^{e_m} \sum_{u=0}^{t} (e_m)_u E_u^{(t)} \\ &= p_1 z^{e_1} D(b_1, \ldots, b_t, e_1) + \cdots + p_m z^{e_m} D(b_1, \ldots, b_t, e_m).\end{aligned}$$

This proves the lemma. □

COROLLARY 3.5

Let α be a root of $f(z)$. Then

$$\sum_{u=1}^{i(K)} E_u^{(i(K))} \alpha^u f^{(u)}(\alpha) = a_{i(K)} D_{i(K)}^{(1)} \alpha^{r_{i(K)}} + \sum_{j>i(K)} a_j D_j^{(1)} \alpha^{r_j}, \tag{15}$$

where $K = K(\alpha)$, $E_u^{(i(K))} = E_u^{(i(K))}(r_0, \ldots, r_{i(K)-1})$, $D_j^{(1)} = D(r_0, \ldots, r_{i(K)-1}, r_j)$ and the sum with $j > i(K)$ is taken as zero if $K = \ell$. Also

$$\sum_{u=1}^{s-i(k)} E_u^{(s-i(k))} \alpha^u f^{(u)}(\alpha) = a_{i(k)} D_{i(k)}^{(2)} \alpha^{r_{i(k)}} + \sum_{j<i(k)} a_j D_j^{(2)} \alpha^{r_j}, \tag{16}$$

where $k = k(\alpha)$, $E_u^{(s-i(k))} = E_u^{(s-i(k))}(r_{i(k)+1}, \ldots, r_s)$, $D_j^{(2)} = D(r_{i(k)+1}, \ldots, r_s, r_j)$ *and the sum with* $j < i(k)$ *is taken as zero if* $k = 0$.

Proof. In Lemma 3.4, take $P(z) = f(z)$ with $(e_1, \ldots, e_m) = (r_0, \ldots, r_s)$ and $(p_1, \ldots, p_m) = (a_0, \ldots, a_s)$. Now (15) follows from the lemma by taking $z = \alpha$, $t = i(K)$, $(b_1, \ldots, b_t) = (r_0, \ldots, r_{i(K)-1})$ and using the fact that $D(r_0, \ldots, r_{i(K)-1}, r_j) = 0$ for $j < i(K)$. Similarly, we obtain (16) by taking $z = \alpha$, $t = s - i(k)$, $(b_1, \ldots, b_t) = (r_{i(k)+1}, \ldots, r_s)$ and using the fact that $D(r_{i(k)+1}, \ldots, r_s, r_j) = 0$ for $j > i(k)$. □

Note that the sums on the left-hand sides of (15) and (16) are non-empty since $0 \leq i(k) < i(K) \leq s$.

Lemma 3.6. *Let* (x, y) *be a solution of* (2) *with* $y \neq 0$. *Let* α *be a root of* f. *Then*

(*i*) *There exists* u *with* $1 \leq u \leq i(K)$ *such that*

$$|f^{(u)}(\alpha)| \geq \frac{1}{4s}(2s^2r)^{1-s}|a_{i(K)}||\alpha|^{r_{i(K)}-u}.$$

(*ii*) *There exists* v *with* $1 \leq v \leq s - i(k)$ *such that*

$$|f^{(v)}(\alpha)| \geq \frac{1}{4s}(2s^2r)^{1-s}|a_{i(k)}||\alpha|^{r_{i(k)}-v}.$$

Proof. Suppose that $K < \ell$. Let $j > i(K)$. Put

$$W_j^{(1)} = \left|\frac{a_j D_j^{(1)}\alpha^{r_j}}{a_{i(K)}D_{i(K)}^{(1)}\alpha^{r_{i(K)}}}\right|.$$

By (15), we have

$$\sum_{u=1}^{i(K)} E_u^{(i(K))}\alpha^u f^{(u)}(\alpha) = a_{i(K)}D_{i(K)}^{(1)}\alpha^{r_{i(K)}}\left(1 + \sum_{j>i(K)} \frac{a_j D_j^{(1)}\alpha^{r_j}}{a_{i(K)}D_{i(K)}^{(1)}\alpha^{r_{i(K)}}}\right). \quad (17)$$

From the definition of $D_j^{(1)}$ and (14), we obtain

$$\left|\frac{D_j^{(1)}}{D_{i(K)}^{(1)}}\right| = \prod_{w<i(K)}\left(\frac{r_j - r_w}{r_{i(K)} - r_w}\right) = \prod_{w<i(K)}\left(1 + \left(\frac{r_j - r_{i(K)}}{r_{i(K)} - r_w}\right)\right).$$

This implies

$$\log\left|\frac{D_j^{(1)}}{D_{i(K)}^{(1)}}\right| \leq (r_j - r_{i(K)})\sum_{w<i(K)}\frac{1}{r_{i(K)} - r_w}$$
$$\leq (r_j - r_{i(K)})\Psi.$$

Then

$$\begin{aligned}\log W_j^{(1)} &\le (r_j - r_{i(K)})\,(\log|\alpha| + \Psi) + \log|a_j| - \log|a_{i(K)}| \\ &= (r_j - r_{i(K)})\,(\log|\alpha| + \Psi - \sigma(j, i(K))),\end{aligned}$$

where $\sigma(j, i(K))$ denotes the slope of the line segment joining $P_{i(K)}$ and P_j. From the convexity of the Newton polygon and the definition of $K = K(\alpha)$, we get

$$\sigma(j, i(K)) \ge \sigma^+(i(K)) \ge \log|\alpha| + \Psi + \log 3.$$

Hence

$$\log W_j^{(1)} \le -(r_j - r_{i(K)}) \log 3,$$

implying that

$$W_j^{(1)} \le 3^{-(r_j - r_{i(K)})}.$$

Thus

$$\sum_{j > i(K)} W_j^{(1)} \le 3^{-1} + 3^{-2} + \cdots = \frac{1}{2}.$$

Using this in (17), we obtain

$$\left| \sum_{u=1}^{i(K)} E_u^{(i(K))} \alpha^u f^{(u)}(\alpha) \right| \ge \frac{1}{2} |a_{i(K)} D_{i(K)}^{(1)} \alpha^{r_{i(K)}}|. \tag{18}$$

It is easy to see that the above inequality holds also for $K = \ell$ since then the right hand side of (17) reduces to $|a_s D_s^{(1)} \alpha^r|$. Suppose that $k > 0$. As above, it can be shown that for $j < i(k)$,

$$\log \left| \frac{D_j^{(2)}}{D_{i(k)}^{(2)}} \right| \le (r_{i(k)} - r_j)\Psi$$

and hence

$$\left| \sum_{u=1}^{s-i(k)} E_u^{(s-i(k))} \alpha^u f^{(u)}(\alpha) \right| \ge \frac{1}{2} |a_{i(k)} D_{i(k)}^{(2)} \alpha^{r_{i(k)}}|. \tag{19}$$

This inequality is also true for $k = 0$. From [11, equations (6.12) and (6.13)] it follows that

$$|E_u^{(i(K))}| \le 2^s (s^2 r)^{s-1} |D_{i(K)}^{(1)}| \quad \text{for } 1 \le u \le i(K) \tag{20}$$

and

$$|E_u^{(s-i(k))}| \leq 2^s (s^2 r)^{s-1} |D_{i(k)}^{(2)}| \quad \text{for } 1 \leq u \leq s - i(k). \tag{21}$$

Substituting (20) in (18), we find that

$$|f^{(u)}(\alpha)| \geq \frac{1}{4s}(2s^2 r)^{1-s} |a_{i(K)}||\alpha|^{r_{i(K)}-u}$$

for some u with $1 \leq u \leq i(K)$. In a similar manner, (19) and (21) yield the second part of the lemma. □

4. Good rational approximations

The results in this section correspond to [11, Section 14], with minor changes. We include the details for the convenience of the reader. Throughout this section, assume that (x, y) is a solution of (2) with $y \neq 0$ and that α is a root of f with

$$d\left(S, \frac{x}{y}\right) = \left|\alpha - \frac{x}{y}\right|.$$

Let q be the smallest integer with

$$|a_q| = H = \max_{0 \leq j \leq s} |a_j|.$$

Note that $(q, -\log |a_q|)$ is a vertex of the Newton polygon of f. The following lemma from [11] gives a rational approximation to α in terms of the derivatives of f.

Lemma 4.1 [11, *Lemma* 10]. *If* $f^{(u)}(\alpha) \neq 0$ *for some* u *in* $1 \leq u \leq r$, *then*

$$d\left(S, \frac{x}{y}\right) \leq \frac{r}{2}\left(\frac{2^r h}{|f^{(u)}(\alpha) y^r|}\right)^{1/u}.$$

The following two results are a consequence of the above lemma.

Lemma 4.2. *If* $q < i(K)$, *where* $K = K(\alpha)$, *then*

$$d\left(S, \frac{x}{y}\right) \leq \frac{1}{H^{(1/u)-(1/r)}}\left(\frac{(rs)^{2s}(6e^{\Psi})^r h}{|y|^r}\right)^{1/u}, \tag{22}$$

where u *is chosen according to Lemma* 3.6(*i*).

Proof. Combining Lemmas 3.6(i) and 4.1, we obtain

$$\left|\alpha - \frac{x}{y}\right| \leq \left(\frac{2^r (rs)^{2s} h}{|a_{i(K)}||\alpha|^{r_{i(K)}-u}|y|^r}\right)^{1/u}. \tag{23}$$

Denote $|a_{i(K)}||\alpha|^{r_{i(K)}-u}$ by $\Delta(\alpha, u)$. Thus

$$\log \Delta(\alpha, u) = (r_{i(K)} - u) \log |\alpha| + \log |a_{i(K)}|.$$

From the definition of $K(\alpha)$ it follows that

$$\log |\alpha| \geq \sigma(i(K)) - \Psi - \log 3.$$

Since $q < i(K)$, we have $\sigma(i(K)) \geq \sigma(q, i(K))$ as the Newton polygon is convex. Thus

$$\begin{aligned} \log \Delta(\alpha, u) &\geq (r_{i(K)} - u)\,(\sigma(i(K)) - \Psi - \log 3) + \log |a_{i(K)}| \\ &\geq (r_{i(K)} - u)\sigma(q, i(K)) + \log |a_{i(K)}| - r \log(3e^{\Psi}) \\ &= (r_q - u)\sigma(q, i(K)) + \log |a_q| - r \log(3e^{\Psi}). \end{aligned}$$

Since P_q is one of the lowest vertices of the Newton polygon and $q < i(K)$, the slope $\sigma(q, i(K))$ is non-negative. Therefore if $r_q \geq u$, we have

$$\log \Delta(\alpha, u) \geq \log |a_q| - r \log(3e^{\Psi}) = \log H - r \log(3e^{\Psi}).$$

If $r_q < u$, we have $\sigma(q, i(K)) \leq \sigma(q, s)$. Hence

$$\begin{aligned} \log \Delta(\alpha, u) &\geq (r_q - u)\sigma(q, s) + \log |a_q| - r \log(3e^{\Psi}) \\ &= \left(1 - \frac{u - r_q}{r_s - r_q}\right) \log |a_q| + \frac{u - r_q}{r_s - r_q} \log |a_s| - r \log(3e^{\Psi}) \\ &\geq \left(1 - \frac{u - r_q}{r - r_q}\right) \log |a_q| - r \log(3e^{\Psi}) \\ &\geq \left(1 - \frac{u}{r}\right) \log H - r \log(3e^{\Psi}). \end{aligned}$$

Using this estimate in (23), we obtain the assertion of the lemma. □

Lemma 4.3. *If $x \neq 0$, $|y| \geq 2(rs)^{2s/r} h^{1/r}$ and $i(k) < q$, where $k = k(\alpha)$, then*

$$d\left(S^*, \frac{y}{x}\right) \leq \frac{1}{H^{(1/v)-(1/r)}} \left(\frac{(rs)^{2s}(12e^{\Psi})^r h}{|x|^r}\right)^{1/v}, \tag{24}$$

where v is chosen according to Lemma 3.6(ii) and $S^* = \{\alpha^{-1} | \alpha \in S\}$.

Proof. Combining Lemmas 3.6(ii) and 4.1, we obtain

$$\left|\alpha - \frac{x}{y}\right| \le \left(\frac{2^r (rs)^{2s} h}{|a_{i(k)}||\alpha|^{r_{i(k)}-v}|y|^r}\right)^{1/v}.$$

Denote $|a_{i(k)}||\alpha|^{r_{i(k)}-v}$ by $\Delta^*(\alpha, v)$. Then

$$\log(|\alpha|^v \Delta^*(\alpha, v)) = r_{i(k)} \log|\alpha| + \log|a_{i(k)}|.$$

If $k = 0$, we have $r_{i(0)} = 0$ and hence $\log(|\alpha|^v \Delta^*(\alpha, v)) \ge 0$. Now suppose that $k > 0$. From the definition of $k(\alpha)$, it follows that

$$\log|\alpha| \ge \sigma(i(k)) + \Psi + \log 3.$$

This implies that

$$\begin{aligned}\log(|\alpha|^v \Delta^*(\alpha, v)) &\ge r_{i(k)}\sigma(i(k)) + \log|a_{i(k)}| \\ &\ge r_{i(k)}\sigma(0, i(k)) + \log|a_{i(k)}| = \log|a_0| \ge 0.\end{aligned}$$

Therefore

$$\left|\alpha - \frac{x}{y}\right| \le |\alpha| \left(\frac{2^r (rs)^{2s} h}{|y|^r}\right)^{1/v} \le |\alpha|$$

by the assumption on y. Hence

$$|x| \le |2\alpha y|.$$

Using this we obtain

$$\begin{aligned}\left|\alpha^{-1} - \frac{y}{x}\right| = \left|\frac{y}{x\alpha}\right| \left|\alpha - \frac{x}{y}\right| &\le \left(\frac{2^r (rs)^{2s} h}{|a_{i(k)}||\alpha|^{r_{i(k)}}}\right)^{1/v} \frac{1}{|x||y|^{(r/v)-1}} \\ &\le \left(\frac{4^r (rs)^{2s} h}{\Gamma(\alpha, v)|x|^r}\right)^{1/v},\end{aligned}$$

where $\Gamma(\alpha, v) = |a_{i(k)}||\alpha|^{-(r-r_{i(k)}-v)}$. Note that $r - r_{i(k)} - v \ge 0$. It is enough to show that

$$\Gamma(\alpha, v) \ge (3e^{\Psi})^{-r} H^{1-(v/r)}.$$

Since $i(k) < q$, we have $\sigma(i(k), q) \ge \sigma^+(i(k)) > \log|\alpha| - \Psi - \log 3$. Thus

$$\begin{aligned}\log\Gamma(\alpha, v) &\ge (r - r_{i(k)} - v)(-\sigma^+(i(k)) - \log(3e^{\Psi})) + \log|a_{i(k)}| \\ &\ge -(r - r_{i(k)} - v)\sigma(i(k), q) + \log|a_{i(k)}| - r\log(3e^{\Psi}) \\ &= -(r - r_q - v)\sigma(i(k), q) + \log|a_q| - r\log(3e^{\Psi}).\end{aligned}$$

When $v \le r - r_q$, we have

$$\log \Gamma(\alpha, v) \ge \log |a_q| - r \log(3e^{\Psi}) = \log H - r \log(3e^{\Psi})$$

as $\sigma(i(k), q) \le 0$. When $v > r - r_q$, we get

$$\begin{aligned}\log \Gamma(\alpha, v) &\ge -(r - r_q - v)\sigma(0, q) + \log |a_q| - r \log(3e^{\Psi}) \\ &= (r - r_q - v)((\log |a_q| - \log |a_0|)/r_q) \\ &\quad + \log |a_q| - r \log(3e^{\Psi}) \\ &\ge ((r - v)/r_q) \log |a_q| - r \log(3e^{\Psi}) \\ &\ge (1 - (v/r)) \log H - r \log(3e^{\Psi}).\end{aligned}$$

This proves the claim and hence the assertion of the lemma. □

We combine Lemmas 4.2, 4.3 and 2.2 to get the following result, which is analogous to [11, Lemma 17].

Lemma 4.4. *There is a set S_1 of roots of $F(x, 1)$ and a set S_1^* of roots of $F(1, y)$, both with cardinalities $\le 6s + 4$, such that any solution (x, y) of (2) with*

$$\min(|x|, |y|) \ge 12e^{\Psi}(rs)^{2s/r} h^{1/r}$$

either has

$$\left|\alpha - \frac{x}{y}\right| \le \frac{R_1}{H^{\frac{1}{s} - \frac{1}{r}}} \left(\frac{(rs)^{2s}(12e^{\Psi})^r h}{|y|^r}\right)^{1/s} \tag{25}$$

for some $\alpha \in S_1$ or has

$$\left|\alpha^* - \frac{y}{x}\right| \le \frac{R_1}{H^{\frac{1}{s} - \frac{1}{r}}} \left(\frac{(rs)^{2s}(12e^{\Psi})^r h}{|x|^r}\right)^{1/s} \tag{26}$$

for some $\alpha^* \in S_1^*$. Here $R_1 = e^{800 \log^3 r}$.

Proof. Since $\min(|x|, |y|) \ge 12e^{\Psi}(rs)^{2s/r} h^{1/r}$, Lemmas 4.2 and 4.3 imply that there is either a root α of $F(Z, 1)$ with (22) or a root α^{-1} of $F(1, Z)$ with (24). Further, the right-hand sides of these inequalities increase with u and v respectively. Therefore we may replace u, v with s. This, together with Lemma 2.2, gives the assertion of the lemma. □

5. Estimation of medium solutions

Let Y_W be as given in (9). In this section, we shall estimate $P_{med}(Y_W, Z)$ for suitably chosen Z.

Lemma 5.1. *Let $F(X,Y)$ be given by* (1) *and $R_1 = 800\log^3 r$. Further let* (7) *hold.*

(i) *Let*

$$Y_S = ((12e^{\Psi})^r R_1^{2s} h)^{\frac{1}{r-2s}}.$$

Then

$$P_{med}(Y_W, Y_S) \ll \frac{s}{\Phi}(\log s + \log(1+\log h^{1/r})).$$

(ii) *Suppose that the coefficients of $F(X,Y)$ satisfy* (8) *and that $r \geq 4s$. Let*

$$Y_S' = (8^r R_1^s (s^2 r)^{3s} h)^{\frac{1}{r-2s}}.$$

Then

$$P_{med}(Y_W, Y_S') \ll s(\log s + \log(1+\log h^{1/r})).$$

Proof.

(i) By Lemma 4.4, it is enough to estimate the number of primitive pairs (x,y) satisfying (25) for some $\alpha \in S_1$ or (26) for some $\alpha^* \in S_1^*$ with

$$Y_S \leq y \leq Y_W.$$

We consider the case when (25) is satisfied. The other case is similar. Let $\alpha \in S_1$ and let $U = \{(x_1,y_1),\dots,(x_\nu,y_\nu)\}$ be the set of all solutions of (25) with $\gcd(x_i,y_i)=1$ and

$$Y_S \leq y_1 \leq \cdots \leq y_\nu \leq Y_W.$$

Suppose that $\nu \geq 2$. Then

$$\begin{aligned}\frac{1}{y_i y_{i+1}} &\leq \left|\frac{x_i}{y_i} - \frac{x_{i+1}}{y_{i+1}}\right| \leq \left|\alpha - \frac{x_i}{y_i}\right| + \left|\alpha - \frac{x_{i+1}}{y_{i+1}}\right| \\ &\leq \frac{K_1}{2y_i^{r/s}} + \frac{K_1}{2y_{i+1}^{r/s}} \leq \frac{K_1}{y_i^{r/s}},\end{aligned}$$

where

$$K_1 = 2R_1(rs)^2(12e^{\Psi})^{r/s}h^{1/s}H^{(1/r)-(1/s)}.$$

Thus we have

$$y_{i+1} \geq K_1^{-1} y_i^{(r/s)-1}.$$

In Lemma 2.1(i), take $T((x_i,y_i)) = y_i$, $\beta = \frac{1}{K_1}$, $\gamma = \frac{r-s}{s}$, $A_1 = Y_S$ and $B_1 = Y_W$. Note that $\gamma = (r/s)-1 > 4e^{2\Phi} - 1 \geq e^{\Phi} \geq 2$. Further, since $R_1 \geq 4(rs)^4$, we have

$$\frac{Y_S}{K_1^{\frac{1}{\kappa(\gamma-1)}}} \geq \frac{R_1^{\frac{s}{r-2s}} H^{\frac{r-s}{2r(r-2s)}}}{(rs)^{\frac{2s}{r-2s}} 2^{\frac{s}{r-2s}}} \geq R_1^{\frac{s}{2(r-2s)}} H^{\frac{r-s}{2r(r-2s)}}.$$

Using the inequality $M \le (r+1)H$ ([8, eq. (6)]), we get

$$\log Y_W \ll \sqrt{r} + \log H + \log h^{1/r}.$$

Now we apply Lemma 2.1(i) to obtain

$$\begin{aligned}\nu &\ll 1 + \frac{1}{\log \gamma} \log \left(\frac{2r(r-2s)\left(\sqrt{r} + \log H + \log h^{1/r}\right)}{(r-s)\log H + rs \log R_1} \right) \\ &\ll \frac{\log r + \log\left(1 + \log h^{1/r}\right)}{\log \gamma}.\end{aligned}$$

If $r \ll s^3$, we get

$$\nu \ll \frac{\log s + \log(1 + \log h^{1/r})}{\Phi}.$$

If $r \gg s^3$, then $\gamma \gg r^{2/3}$, which implies that

$$\nu \ll 1 + \frac{\log(1 + \log h^{1/r})}{\log r} \ll \frac{\log s + \log(1 + \log h^{1/r})}{\Phi}$$

as $\Phi \ll \log s$ (see Remark 1.3). Since this is true for each $\alpha \in S_1$, we obtain the assertion of part (i) of the lemma.
(ii) By Remark 3.2 and Lemmas 4.1, 3.6 and 2.2, there is a set $S_1 \subseteq S$ with $|S_1| \ll s$, such that for some $\alpha \in S_1$,

$$d\left(S_1, \frac{x}{y}\right) = \left|\alpha - \frac{x}{y}\right| \le \frac{rR_1}{2} \left(\frac{s(rs^2)^{s-1} 2^{r+s+1} h}{|y|^r |a_s| |\alpha|^{r-u}}\right)^{1/u}. \tag{27}$$

By Lemma 3.3, it follows that every root α of f satisfies

$$\frac{1}{2} e^{\sigma} < |\alpha| < 2e^{\sigma},$$

where σ is the slope of the line joining $(0, -\log|a_0|)$ and $(r, -\log|a_s|)$. This implies that

$$\log|\alpha| > \sigma - \log 2 = \frac{-\log|a_s| + \log|a_0|}{r} - \log 2.$$

Therefore

$$|a_s||\alpha|^{r-u} \ge \frac{|a_s|^{u/r} |a_0|^{(r-u)/r}}{2^{r-u}}.$$

By (27) and $|y| \ge Y_S'$, we get that

$$d\left(S_1, \frac{x}{y}\right) \le \frac{rR_1}{2} \left(\frac{s(rs^2)^{s-1} 2^{3r} h}{|y|^r |a_s|^{s/r} |a_0|^{(r-s)/r}} \right)^{1/s}.$$

As the height of F is either $|a_0|$ or $|a_s|$, we obtain that

$$\left|\alpha - \frac{x}{y}\right| \leq \frac{rR_1}{2H^{1/r}} \left(\frac{s(rs^2)^{s-1}2^{3r}h}{|y|^r}\right)^{1/s}. \tag{28}$$

Let $U = \{(x_1, y_1), \ldots, (x_\nu, y_\nu)\}$ be the set of all the solutions of (28) with $\gcd(x_i, y_i) = 1$ and

$$Y'_S \leq y_1 \leq \ldots \leq y_\nu \leq Y_W.$$

Put

$$K_2 = \frac{R_1 8^{r/s}(sh)^{1/s}(rs)^2}{H^{1/r}}.$$

Suppose that $\nu \geq 2$. As in the proof of (i), we get from (28) that

$$y_{i+1} \geq K_2^{-1} y_i^{(r/s)-1}.$$

In Lemma 2.1(i), take $T((x_i, y_i)) = y_i$, $\beta = \frac{1}{K_2}$, $\gamma = \frac{r-s}{s}$, $A_1 = Y'_S$ and $B_1 = Y_W$. Note that

$$Y'_S \geq K_2^{\frac{1}{\kappa(\gamma-1)}} H^{\frac{s}{2r(r-2s)}} (rs^3)^{\frac{s}{r-2s}} \text{ and } \gamma \geq 2.$$

Thus by Lemma 2.1(i), we obtain

$$\begin{aligned} \nu &\ll 1 + \frac{1}{\log \gamma} \log\left(\frac{2r(r-2s)(\sqrt{r} + \log H + \log h^{1/r})}{s \log H + 2rs \log(rs^3)}\right) \\ &\ll \frac{\log r + \log(1 + \log h^{1/r})}{\log \gamma}. \end{aligned}$$

Now we argue as in the proof of (i) to get the assertion. □

6. Estimation of small solutions

To estimate the number of small solutions, we use the following lemma from [11].

Lemma 6.1 [11, *Lemma* 18]. *Let $F(X, Y)$ be given by* (1) *and let $r \geq 4s$. Then for any $X_2 \geq 1$, we have*

$$P_{sma}(X_2) \ll (rs^2)^{2s/r} h^{2/r} + s\, X_2.$$

Lemma 6.2. *Let $F(X, Y)$ be given by* (1). *Take Y_S and Y'_S as in Lemma* 5.1. *Then*

(i)

$$P_{sma}(Y_S) \ll e^{c_7 (\log s) e^{-2\Phi}} h^{\frac{2}{r}} + s e^{\Phi + c_8 \left(\log^3 s\right) e^{-\Phi}} h^{\frac{1}{r-2s}}.$$

(ii)

$$P_{sma}(Y'_S) \ll sh^{2/r} \textit{ whenever } r \geq s\log^3 s.$$

Proof. (i) follows from (7) and Lemma 6.1 with X_2 as Y_S. Similarly, (ii) follows by taking X_2 as Y'_S in Lemma 6.1 and using $r \geq s\log^3 s$. □

7. Proofs of Proposition 1.8 and Theorems 1.1, 1.5

Proof of Proposition 1.8. Suppose that $r \geq 4s$. Let $P(h)$ denote the number of primitive solutions of (2). Then

$$P(h) = P_{\ell ar}(Y_W) + P_{med}(Y_W, Y_S) + P_{sma}(Y_S).$$

The upper bounds for the three quantities on the right-hand side are obtained from (10), Lemma 5.1(i) and Lemma 6.2(i). This together with $\Phi \ll \log s$ (see Remark 1.3) yields

$$P(h) \ll \rho h^{2/r},$$

where

$$\rho = \frac{s\log s}{\Phi} + s^{c_7 e^{-2\Phi}} + se^{\Phi + c_8(\log^3 s)e^{-\Phi}}.$$

Using a partial summation argument, it was shown in [11, p. 212] that

$$N_F(h) \ll P(h) + h^{1/r} r^{-1} \sum_{n=1}^{h-1} P(n) n^{-1-(1/r)}.$$

Substituting our estimate for $P(h)$, we obtain that

$$N_F(h) \ll \rho h^{2/r}.$$

When $r < 4s$, we use (3) to obtain

$$N_F(h) \ll s\, C_1(r, h).$$

This proves the proposition. □

Proof of Theorem 1.1. Observe that $\Phi \geq 3\log\log s$. Thus from Proposition 1.8, we get

$$N_F(h) \ll s\left(\log s + e^{\Phi}\right) C_1(r, h) \ll se^{\Phi} C_1(r, h).$$

□

Proof of Theorem 1.5. We have

$$P(h) = P_{\ell ar}(Y_W) + P_{med}\left(Y_W, Y'_S\right) + P_{sma}(Y'_S).$$

We use the respective upper bounds for the three quantities on the right hand side from (10), Lemma 5.1(ii) and Lemma 6.2(ii) and proceed as in the proof of Proposition 1.8 to give the assertion. □

References

[1] Apostol T M, Introduction to analytic number theory (1976) (New York: Springer)

[2] Bombieri E and Schmidt W M, On Thue's equation, *Invent. Math.* **88** (1987) 69–81

[3] Evertse J-H, On the equation $ax^n - by^n = c$, *Compos. Math.* **47** (1982) 288–315

[4] Evertse J-H and Győry K, Thue inequalities with a small number of solutions, in: The mathematical heritage of C. F. Gauss, George M Rassias (ed) (1991) (Singapore: World Scientific Publication Co.) pp. 204–224

[5] Győry K, Thue inequalities with a small number of primitive solutions, *Period. Math. Hungar.* **42** (2001) 199–209

[6] Győry K, On the number of primitive solutions of Thue equations and Thue inequalities, Paul Erdős and his mathematics I (2002) (Berlin: Springer) pp. 279–294

[7] Hyyrö S, Über die Gleichung $ax^n - by^n = c$ und das Catalansche Problem, *Ann. Acad. Sci. Fenn. Ser. AI* **355** (1964) 1–50

[8] Mahler K, An inequality for the discriminant of a polynomial, *Mich. Math. J.* **11** (1964) 257–262

[9] Mueller J, Counting solutions of $|ax^r - by^r| \leq h$, *Quart. J. Math. Oxf.* **38(2)** (1987) 503–513.

[10] Mueller J and Schmidt W M, Trinomial Thue equations and inequalities, *J. Reine Angew. Math.* **379** (1987) 76–99

[11] Mueller J and Schmidt W M, Thue's equation and a conjecture of Siegel, *Acta Math.* **160** (1988) 207–247

[12] Schmidt W M, Thue equations with few coefficients, *Trans. Am. Math. Soc.* **303** (1987) 241–255

[13] Thunder J L, The number of solutions to cubic Thue inequalities, *Acta Arith.* **66(3)** (1994) 237–243

[14] Thunder J L, On Thue inequalities and a conjecture of Schmidt, *J. Number Theory* **52** (1995) 319–328

Communicating Editor: S D Adhikari