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Some inequalities for the Bell numbers

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Abstract. In this paper, we present derivatives of the generating functions for the Bell numbers by induction and by the Faà di Bruno formula, recover an explicit formula in terms of the Stirling numbers of the second kind, find the (logarithmically) absolute and complete monotonicity of the generating functions, and construct some inequalities for the Bell numbers. From these inequalities, we derive the logarithmic convexity of the sequence of the Bell numbers.

Keywords. Bell number determinant; product; inequality; generating function; derivative; absolutely monotonic function; completely monotonic function; logarithmically absolutely monotonic function; logarithmically completely monotonic function; Stirling number of the second kind; induction; Faà di Bruno formula; logarithmic convexity.

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1. Introduction

In combinatorics, the Bell numbers, usually denoted by B_n for $n \in \{0\} \cup \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers, count the number of ways a set with n elements can be partitioned into disjoint and nonempty subsets. These numbers have been studied by mathematicians since the 19th century, and their roots go back to medieval Japan, but they are named after Eric Temple Bell, who wrote about them in the 1930s. The first few Bell numbers B_n are

$$\begin{aligned} &B_0 = 1, \quad B_1 = 1, \quad B_2 = 2, \quad B_3 = 5, \quad B_4 = 15, \\ &B_5 = 52, \quad B_6 = 203, \quad B_7 = 877, \quad B_8 = 4140, \quad B_9 = 21147. \end{aligned}$$

All of the Bell numbers B_n can be generated by

$$e^{e^x} = e \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k \tag{1.1}$$

or, equivalently, by

$$e^{e^{-x}} = e\sum_{k=0}^{\infty}(-1)^k B_k \frac{x^k}{k!}.$$

We call $e^{e^{\pm x}}$ the generating functions of the Bell numbers B_k. For more information on the Bell numbers B_k, please refer to [2,5,9,16] and references therein.

In this paper, we present derivatives of the generating functions $e^{e^{\pm x}}$ for the Bell numbers B_k by induction and by the Faà di Bruno formula, recover an explicit formula in terms of the Stirling numbers of the second kind $S(n,k)$, find the (logarithmically) absolute and complete monotonicity of the generating functions $e^{e^{\pm x}}$, and construct some inequalities for the Bell numbers B_k. From these inequalities, we derive the logarithmic convexity of the sequence of the Bell numbers B_k.

2. Derivatives of the generating function $e^{e^{\pm x}}$

In this section, we elementarily present derivatives of the generating function $e^{e^{\pm x}}$ by induction and by the famous Faà di Bruno formula. Although these results are not new and elementary, for completeness of this paper and for utilization in this paper later, we would like to state them in detail.

Theorem 2.1. *For $n \in \mathbb{N}$, the n-th derivative of the function e^{e^x} can be computed by*

$$\frac{\mathrm{d}^n e^{e^x}}{\mathrm{d}x^n} = e^{e^x}\sum_{k=1}^{n} S(n,k)e^{kx}, \tag{2.1}$$

where

$$S(n,k) = \frac{1}{k!}\sum_{\ell=1}^{k}(-1)^{k-\ell}\binom{k}{\ell}\ell^n$$

for $n \ge k \ge 1$ are the Stirling numbers of the second kind.

First proof. A straightforward computation yields

$$\begin{aligned}
&\frac{\mathrm{d}e^{e^x}}{\mathrm{d}x} = e^{e^x+x}, \quad \frac{\mathrm{d}^2 e^{e^x}}{\mathrm{d}x^2} = e^{e^x+x}\big(e^x+1\big),\\
&\frac{\mathrm{d}^3 e^{e^x}}{\mathrm{d}x^3} = e^{e^x+x}\big(e^{2x}+3e^x+1\big),\\
&\frac{\mathrm{d}^4 e^{e^x}}{\mathrm{d}x^4} = e^{e^x+x}\big(e^{3x}+6e^{2x}+7e^x+1\big),\\
&\frac{\mathrm{d}^5 e^{e^x}}{\mathrm{d}x^5} = e^{e^x+x}\big(e^{4x}+10e^{3x}+25e^{2x}+15e^x+1\big),\\
&\frac{\mathrm{d}^6 e^{e^x}}{\mathrm{d}x^6} = e^{e^x+x}\big(e^{5x}+15e^{4x}+65e^{3x}+90e^{2x}+31e^x+1\big).
\end{aligned}$$

This helps us to conjecture the formula

$$\frac{\mathrm{d}^n e^{e^x}}{\mathrm{d}x^n} = e^{e^x+x}\sum_{k=0}^{n-1} a_{n,k}e^{kx}, \quad n \in \mathbb{N}.$$

Based on this conjectured formula, a direct calculation gives

$$\begin{aligned}
\frac{\mathrm{d}^{n+1} e^{e^x}}{\mathrm{d}x^{n+1}} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^n e^{e^x}}{\mathrm{d}x^n}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{e^x+x}\sum_{k=0}^{n-1} a_{n,k}e^{kx}\right) \\
&= e^{e^x+x}(e^x+1)\sum_{k=0}^{n-1} a_{n,k}e^{kx} + e^{e^x+x}\sum_{k=0}^{n-1} k a_{n,k}e^{kx} \\
&= e^{e^x+x}\left[\sum_{k=0}^{n-1} a_{n,k}e^{(k+1)x} + \sum_{k=0}^{n-1}(k+1)a_{n,k}e^{kx}\right] \\
&= e^{e^x+x}\left[\sum_{k=1}^{n} a_{n,k-1}e^{kx} + \sum_{k=0}^{n-1}(k+1)a_{n,k}e^{kx}\right] \\
&= e^{e^x+x}\left[a_{n,n-1}e^{nx} + \sum_{k=1}^{n-1}[a_{n,k-1}+(k+1)a_{n,k}]e^{kx} + a_{n,0}\right] \\
&= e^{e^x+x}\sum_{k=0}^{n} a_{n+1,k}e^{kx}.
\end{aligned}$$

Equating the last two lines in the above equation yields

$$a_{n,0} = a_{n+1,0}, \quad a_{n,n-1} = a_{n+1,n}, \quad n \ge 1 \tag{2.2}$$

and

$$a_{n+1,k} = a_{n,k-1} + (k+1)a_{n,k}, \quad 1 \le k \le n-1. \tag{2.3}$$

The first derivative $\left(e^{e^x}\right)' = e^{e^x+x}$ means

$$a_{1,0} = 1. \tag{2.4}$$

Combining this with the two recursions in (2.2) leads to

$$a_{n,0} = a_{n+1,n} = 1, \quad n \ge 1. \tag{2.5}$$

Letting $k = 1$ in (2.3) and considering (2.4) and (2.5) give

$$a_{n+1,1} = a_{n,0} + 2a_{n,1} = 1 + 2a_{n,1}, \quad n \ge 2.$$

Using $a_{2,1} = 1$ and recurring reveal that

$$a_{n,1} = \frac{1}{2}\left(2^n - 2\right), \quad n \ge 2. \tag{2.6}$$

Taking $k=2$ in (2.3) and using (2.6) gives

$$a_{n+1,2}=a_{n,1}+3a_{n,2}=2^{n-1}-1+3a_{n,2}, \quad n\ge 3.$$

From $a_{3,2}=1$, the above recursion figures out that

$$a_{n,2}=\frac{1}{3!}\bigl(3^n-3\times 2^n+3\bigr), \quad n\ge 3. \tag{2.7}$$

Taking $k=3$ in (2.3) and using (2.7) results in

$$a_{n+1,3}=a_{n,2}+4a_{n,3}=\frac{1}{6}\bigl(-3\times 2^n+3^n+3\bigr)+4a_{n,3}, \quad n\ge 4.$$

From $a_{4,3}=1$, the above recursion means that

$$a_{n,3}=\frac{1}{4!}\bigl(4^n-4\times 3^n+6\times 2^n-4\bigr), \quad n\ge 4.$$

Similarly, we can deduce

$$\begin{aligned}
a_{n,4}&=\frac{1}{5!}\bigl(5^n-5\times 4^n+10\times 3^n-10\times 2^n+5\bigr), \quad n\ge 5,\\
a_{n,5}&=\frac{1}{6!}\bigl(6^n-6\times 5^n+15\times 4^n-20\times 3^n+15\times 2^n-6\bigr), \quad n\ge 6,\\
a_{n,6}&=\frac{1}{7!}\bigl(7^n-7\times 6^n+21\times 5^n-35\times 4^n+35\times 3^n-21\times 2^n+7\bigr),\\
&\quad n\ge 7.
\end{aligned}$$

Inductively, we can conclude that

$$\begin{aligned}
a_{n,k}&=\frac{1}{(k+1)!}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k+1}{\ell}(k-\ell+1)^n\\
&=\frac{1}{(k+1)!}\sum_{\ell=1}^{k+1}(-1)^{k+1-\ell}\binom{k+1}{\ell}\ell^n\\
&=S(n,k+1)
\end{aligned}$$

for $n\ge k\ge 1$. The proof of Theorem 2.1 is complete. □

Second proof. It is easy to verify that, when $n=1,2$, the formula (2.1) is valid.

A straightforward computation gives

$$\begin{aligned}
\frac{\mathrm{d}^{n+1}e^{e^x-1}}{\mathrm{d}x^{n+1}}&=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^n e^{e^x-1}}{\mathrm{d}x^n}\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{e^x-1}\sum_{k=1}^{n}S(n,k)e^{kx}\right]\\
&=e^{e^x+x-1}\sum_{k=1}^{n}S(n,k)e^{kx}+e^{e^x-1}\sum_{k=1}^{n}S(n,k)ke^{kx}
\end{aligned}$$

$$
\begin{aligned}
&= e^{e^x-1}\left[\sum_{k=1}^{n} S(n,k)e^{(k+1)x} + \sum_{k=1}^{n} S(n,k)ke^{kx}\right] \\
&= e^{e^x-1}\left[\sum_{k=2}^{n+1} S(n,k-1)e^{kx} + \sum_{k=1}^{n} S(n,k)ke^{kx}\right] \\
&= e^{e^x-1}\left[S(n,n)e^{(n+1)x} + \sum_{k=2}^{n} S(n,k-1)e^{kx} + S(n,1)e^{x}\right. \\
&\quad \left. + \sum_{k=2}^{n} S(n,k)ke^{kx}\right] \\
&= e^{e^x-1}\left[S(n,n)e^{(n+1)x} + \sum_{k=2}^{n}[S(n,k-1)+kS(n,k)]e^{kx} + S(n,1)e^{x}\right] \\
&= e^{e^x-1}\left[S(n+1,n+1)e^{(n+1)x} + \sum_{k=2}^{n} S(n+1,k)e^{kx} + S(n+1,1)e^{x}\right] \\
&= e^{e^x-1}\sum_{k=1}^{n+1} S(n+1,k)e^{kx},
\end{aligned}
$$

where the classical recurrence

$$
S(n+1,k) = S(n,k-1) + kS(n,k), \quad 1 \le k \le n, \tag{2.8}
$$

listed in [1, p. 825], was used in the above argument. By induction, Theorem 2.1 is proved. □

Third proof. In combinatorics, the Bell polynomials of the second kind, or say, the partial Bell polynomials, denoted by $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$, are defined by

$$
B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1\le i\le n,\ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n} i\ell_i=n\\ \sum_{i=1}^{n}\ell_i=k}} \frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}
$$

for $n \ge k \ge 0$, see [7, p. 134, Theorem A], and satisfy

$$
B_{n,k}\left(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}\right) = a^k b^n B_{n,k}(x_1, x_2, \dots, x_{n-k+1})
$$

and

$$
B_{n,k}(1, 1, \dots, 1) = S(n,k),
$$

see [7, p. 135], where a and b are any complex numbers. The well-known Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind $B_{n,k}$ by

$$
\frac{\mathrm{d}^n}{\mathrm{d}x^n} f\circ g(x) = \sum_{k=0}^{n} f^{(k)}(g(x))B_{n,k}(g'(x), g''(x), \dots, g^{(n-k+1)}(x)), \tag{2.9}
$$

see [7, p. 139, Theorem C]. Applying $f(u)=e^u$ and $u=g(x)=e^x$ to (2.9) yields

$$\begin{aligned}\frac{\mathrm{d}^n e^{e^x}}{\mathrm{d}x^n}&=e^{e^x}\sum_{k=1}^{n}B_{n,k}\big(e^x,e^x,\dots,e^x\big)\\&=e^{e^x}\sum_{k=1}^{n}e^{kx}B_{n,k}(1,1,\dots,1)\\&=e^{e^x}\sum_{k=1}^{n}e^{kx}S(n,k).\end{aligned}$$

The proof of Theorem 2.1 is thus complete. □

Theorem 2.2. *For $n\in\mathbb{N}$, the n-th derivative of the function $e^{e^{-x}}$ can be computed by*

$$\frac{\mathrm{d}^n e^{e^{-x}}}{\mathrm{d}x^n}=(-1)^n e^{e^{-x}}\sum_{k=1}^{n}S(n,k)e^{-kx}. \tag{2.10}$$

First proof. This can be deduced inductively as the argument in the first two proofs of Theorem 2.1. We omit the details for lack of space. □

Second proof. In [7, p. 133], it was listed that

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty}x_m\frac{t^m}{m!}\right)^k=\sum_{n=k}^{\infty}B_{n,k}(x_1,x_2,\dots,x_{n-k+1})\frac{t^n}{n!} \tag{2.11}$$

for $k\ge 0$. Letting $x_1=\alpha\in\mathbb{C}$ and $x_{m+1}=0$ for $m\in\mathbb{N}$ in (2.11) gives

$$\alpha^k\frac{t^k}{k!}=\sum_{n=k}^{\infty}B_{n,k}(\alpha,0,\dots,0)\frac{t^n}{n!}$$

which implies that

$$B_{n,n}(\alpha)=\alpha^n \quad\text{and}\quad B_{n+k,k}(\alpha,0,\dots,0)=0 \tag{2.12}$$

for $k,n\in\{0\}\cup\mathbb{N}$. Applying $f(u)=e^{e^u}$ and $g(x)=-x$ in (2.9) and taking $\alpha=-1$ in (2.12) leads to (2.10) once again. The proof of Theorem 2.2 is complete. □

Remark 2.1. The recurrence relation (2.3) implies that

$$S(n+1,k+1)=S(n,k)+(k+1)S(n,k+1),\quad 1\le k\le n-1.$$

This is a recovery of the classical recurrence relation (2.8).

Remark 2.2. In mathematics, it is not easy to compute the Bell polynomials of the second kind $B_{n,k}$. In [10,11], some explicit formulas for special values of the Bell polynomials of the second kind $B_{n,k}$, similar to (2.12), were established and collected.

3. An explicit formula of B_n in terms of $S(n,k)$

In this section, with the help of Theorems 2.1 and 2.2, we will easily recover an explicit formula for the Bell numbers B_n in terms of the Stirling numbers $S(n,k)$, although this formula existed for several centuries.

Theorem 3.1. *For $n \in \mathbb{N}$, the Bell numbers B_n can be computed in terms of the Stirling numbers of the second kind $S(n,k)$ for $n \ge k \ge 1$ by*

$$B_n = \sum_{k=1}^{n} S(n,k). \tag{3.1}$$

Proof. Differentiating n times on both sides of (1.1) gives

$$\frac{\mathrm{d}^n e^{e^x-1}}{\mathrm{d}x^n} = \sum_{k=n}^{\infty} \frac{B_k}{(k-n)!} x^{k-n}.$$

Combining this with (2.1) leads to

$$\sum_{k=n}^{\infty} \frac{B_k}{(k-n)!} x^{k-n} = e^{e^x-1} \sum_{k=1}^{n} S(n,k) e^{kx}.$$

Taking the limit $x \to 0$ on both sides of the above equation gives (3.1).

Alternatively, the numbers eB_k are coefficients of the power series expansion of the function e^{e^x} at $x=0$, or say, the numbers $(-1)^k eB_k$ are coefficients of the power series expansion of the function $e^{e^{-x}}$ at $x=0$, consequently, by Theorems 2.1 and 2.2,

$$eB_k = \lim_{x\to 0} \frac{\mathrm{d}^k e^{e^x}}{\mathrm{d}x^k} = \lim_{x\to 0} e^{e^x} \sum_{\ell=1}^{k} S(k,\ell) e^{\ell x} = e \sum_{\ell=1}^{k} S(k,\ell)$$

and

$$\begin{aligned}(-1)^k eB_k &= \lim_{x\to 0} \frac{\mathrm{d}^k e^{e^{-x}}}{\mathrm{d}x^k} = (-1)^k \lim_{x\to 0} e^{e^{-x}} \sum_{\ell=1}^{k} S(k,\ell) e^{-\ell x} \\ &= (-1)^k e \sum_{\ell=1}^{k} S(k,\ell).\end{aligned}$$

The formula (3.1) immediately follows. The proof of Theorem 3.1 is complete. □

4. Monotonicity of the generating functions $e^{e^{\pm x}}$

Recall from [17,24,28] that a function f is said to be absolutely monotonic on an interval I if it has derivatives of all orders and $f^{(k-1)}(t) \ge 0$ for $t \in I$ and $k \in \mathbb{N}$.

It was introduced in [8, p. 23, Definition 1] that a positive function f is said to be logarithmically absolutely monotonic on an interval I if it has derivatives of all orders and $[\ln f(t)]^{(k)} \geq 0$ for $t \in I$ and $k \in \mathbb{N}$.

An infinitely differentiable function f is said to be completely monotonic on an interval I if it satisfies $(-1)^k f^{(k)}(x) \geq 0$ on I for all $k \geq 0$. For more information, please refer to [14, Chapter XIII], [25, Chapter 1] and [28, Chapter IV].

An infinitely differentiable function f is said to be logarithmically completely monotonic on an interval I if $(-1)^k[\ln f(x)]^{(k)} \geq 0$ on I for all $k \geq 1$. For the history of this terminology, refer to [3,4,20], [21, Remark 8], [22, pp. 2315–2316], [23, Remark 4.7], and related texts in [25, p. 67] and references therein.

We are now in a position to find the (logarithmically) absolute and complete monotonicity of the generating functions $e^{e^{\pm x}}$ for the Bell numbers B_k. These results are seemingly simple, even almost trivial, however, they are our basis of constructing inequalities for the Bell numbers B_k in next section.

Theorem 4.1. *The generating function e^{e^x} is absolutely monotonic on $\mathbb{R}$. More strongly, it is a logarithmically absolutely monotonic function on $\mathbb{R}$.*

Proof. By definition, the proof of the logarithmically absolute monotonicity is straightforward.

By definition, the absolute monotonicity of e^{e^x} follows from the formula (2.1) and the positivity of the Stirling numbers of the second kind $S(n, k)$.

The absolute monotonicity of e^{e^x} can also follow from [8, p. 23, Theorem 1] which reads that a logarithmically absolutely monotonic function on an interval I is also absolutely monotonic on I, but not conversely. □

Theorem 4.2. *The generating function $e^{e^{-x}}$ is completely monotonic on $\mathbb{R}$. More strongly, it is a logarithmically completely monotonic function on $\mathbb{R}$.*

Proof. By definition, the proof of the logarithmically complete monotonicity is straightforward.

By definition, the complete monotonicity of $e^{e^{-x}}$ follows from the formula (2.10) and the positivity of the Stirling numbers of the second kind $S(n, k)$.

The complete monotonicity of $e^{e^{-x}}$ can also follow from [8, p. 23, Theorem 4], see also [4,6,19,20], which reads that a logarithmically completely monotonic function on an interval I is also completely monotonic on I, but not conversely. □

Remark 4.1. It is easy to see that a function $f(x)$ is (logarithmically) absolutely monotonic on an interval I if and only if $f(-x)$ is a (logarithmically) completely monotonic function on the interval $-I$. Hence, Theorems 4.1 and 4.2 are equivalent to each other.

5. Inequalities for the Bell numbers B_k

In light of the absolute and complete monotonicity of the generating functions $e^{e^{\pm x}}$ in Theorems 4.1 and 4.2 and with the aid of properties of absolutely and completely monotonic functions, we now start to construct some inequalities for the Bell numbers B_n. From these inequalities, we can derive the logarithmic convexity of the sequence of the Bell numbers B_k. These inequalities are our main results in this paper.

Theorem 5.1. *Let $m \ge 1$ be a positive integer and let $|a_{ij}|_m$ denote a determinant of order m with elements a_{ij}.*

(1) *If a_i for $1 \le i \le m$ are non-negative integers, then*

$$|B_{a_i+a_j}|_m \ge 0 \tag{5.1}$$

and

$$\left|(-1)^{a_i+a_j} B_{a_i+a_j}\right|_m \ge 0. \tag{5.2}$$

(2) *If $a = (a_1, a_2, \dots, a_n)$ and $b = (b_1, b_2, \dots, b_n)$ are non-increasing n-tuples of non-negative integers such that $\sum_{i=1}^{k} a_i \ge \sum_{i=1}^{k} b_i$ for $1 \le k \le n-1$ and $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$, then*

$$\prod_{i=1}^{n} B_{a_i} \ge \prod_{i=1}^{n} B_{b_i}. \tag{5.3}$$

Proof. In [13] and [14, p. 367], it was obtained that if f is completely monotonic on $[0, \infty)$, then

$$|f^{(a_i+a_j)}(x)|_m \ge 0 \tag{5.4}$$

and

$$|(-1)^{a_i+a_j} f^{(a_i+a_j)}(x)|_m \ge 0. \tag{5.5}$$

Applying $f(x)$ to the function $e^{e^{-x}}$ in (5.4) and (5.5) and taking the limit $x \to 0^+$ give

$$\lim_{x\to 0^+} |(e^{e^{-x}})^{(a_i+a_j)}|_m = |(-1)^{a_i+a_j} e B_{a_i+a_j}|_m \ge 0$$

and

$$\lim_{x\to 0^+} |(-1)^{a_i+a_j} (e^{e^{-x}})^{(a_i+a_j)}|_m = |(-1)^{a_i+a_j} (-1)^{a_i+a_j} e B_{a_i+a_j}|_m \ge 0.$$

The determinant inequalities (5.1) and (5.2) follow.

In [14, p. 367, Theorem 2], it was stated that if f is a completely monotonic function on $[0, \infty)$, then

$$\prod_{i=1}^{n} [(-1)^{a_i} f^{(a_i)}(x)] \ge \prod_{i=1}^{n} [(-1)^{b_i} f^{(b_i)}(x)]. \tag{5.6}$$

Applying $f(x)$ to the function $e^{e^{-x}}$ in (5.6) and taking the limit $x \to 0^+$ give

$$\lim_{x\to 0^+}\prod_{i=1}^{n}[(-1)^{a_i}(e^{e^{-x}})^{(a_i)}]=\prod_{i=1}^{n}(eB_{a_i})$$
$$\ge \lim_{x\to 0^+}\prod_{i=1}^{n}[(-1)^{b_i}l(e^{e^{-x}}r)^{(b_i)}]=\prod_{i=1}^{n}(eB_{b_i}).$$

The product inequality (5.3) follows. The proof of Theorem 5.1 is complete. □

COROLLARY 5.1.1

The sequence B_k *for* $k \in \{0\}\cup\mathbb{N}$ *is logarithmically convex.*

Proof. In [14, p. 369] and [15, p. 429, Remark], it was stated that if $f(t)$ is a completely monotonic function such that $f^{(k)}(t)\ne 0$ for $k\ge 0$, then the sequence

$$\ln[(-1)^{k-1}f^{(k-1)}(t)],\quad k\ge 1 \tag{5.7}$$

is convex. Applying this result to the function $e^{e^{-x}}$ figures out that the sequence

$$\ln[(-1)^{k-1}(e^{e^{-x}})^{(k-1)}]\to 1+\ln B_{k-1},\quad x\to 0^+$$

for $k\ge 1$ is convex. Hence, the sequence $\{B_n\}_{n\ge 0}$ is logarithmically convex.

Alternatively, letting

$$\ell\ge 1,\quad n=2,\quad a_1=\ell+2,\quad a_2=\ell\quad\text{and}\quad b_1=b_2=\ell+1$$

in the inequality (5.3) leads to $B_\ell B_{\ell+2}\ge B_{\ell+1}^2$ which means that the sequence B_k for $k\in\mathbb{N}$ is logarithmically convex. The proof of Corollary 5.1.1 is complete. □

COROLLARY 5.1.2

If $\ell\ge 0$ *and* $n\ge k\ge 0$, *then*

$$B_{n+\ell}^{k}B_{\ell}^{n-k}\ge B_{k+\ell}^{n}.$$

Proof. This follows by taking

$$a=(\overbrace{n+\ell,\dots,n+\ell}^{k},\overbrace{\ell,\dots,\ell}^{n-k})\quad\text{and}\quad b=(k+\ell,k+\ell,\dots,k+\ell)$$

in the inequality (5.3). The proof of Corollary 5.1.2 is complete. □

Theorem 5.2. *If* $\ell\ge 0$, $n\ge k\ge m$, $2k\ge n$ *and* $2m\ge n$, *then*

$$B_{k+\ell}B_{n-k+\ell}\ge B_{m+\ell}B_{n-m+\ell}. \tag{5.8}$$

Proof. In [26, p. 397, Theorem D], it was found that if $f(x)$ is completely monotonic on $(0,\infty)$ and if $n \ge k \ge m$, $k \ge n-k$ and $m \ge n-m$, then

$$(-1)^n f^{(k)}(x) f^{(n-k)}(x) \ge (-1)^n f^{(m)}(x) f^{(n-m)}(x).$$

Replacing $f(x)$ by the function $(-1)^\ell (e^{e^{-x}})^{(\ell)}$ in the above inequality leads to

$$(-1)^n (e^{e^{-x}})^{(k+\ell)} (e^{e^{-x}})^{(n-k+\ell)} \ge (-1)^n (e^{e^{-x}})^{(m+\ell)} (e^{e^{-x}})^{(n-m+\ell)}.$$

Further taking $x \to 0^+$ finds the inequality (5.8). The proof of Theorem 5.2 is complete. □

Theorem 5.3. *For $\ell \ge 0$ and $m, n \in \mathbb{N}$, let*

$$\begin{aligned}
\mathcal{G}_{\ell,m,n} &= B_{\ell+2m+n} B_\ell^2 - B_{\ell+m+n} B_{\ell+m} B_\ell - B_{\ell+n} B_{\ell+2m} B_\ell + B_{\ell+n} B_{\ell+m}^2,\\
\mathcal{H}_{\ell,m,n} &= B_{\ell+2m+n} B_\ell^2 - 2B_{\ell+m+n} B_{\ell+m} B_\ell + B_{\ell+n} B_{\ell+m}^2,\\
\mathcal{I}_{\ell,m,n} &= B_{\ell+2m+n} B_\ell^2 - 2B_{\ell+n} B_{\ell+2m} B_\ell + B_{\ell+n} B_{\ell+m}^2.
\end{aligned}$$

Then

$$\begin{aligned}
&\mathcal{G}_{\ell,m,n} \ge 0, \quad \mathcal{H}_{\ell,m,n} \ge 0,\\
&\mathcal{H}_{\ell,m,n} \lesseqgtr \mathcal{G}_{\ell,m,n} \quad \text{when } m \lesseqgtr n,\\
&\mathcal{I}_{\ell,m,n} \ge \mathcal{G}_{\ell,m,n} \ge 0 \quad \text{when } n \ge m.
\end{aligned} \tag{5.9}$$

Proof. In [27, Theorem 1 and Remark 2], it was obtained that if f is completely monotonic on $(0,\infty)$ and

$$\begin{aligned}
G_{m,n} &= (-1)^n \{ f^{(n+2m)} f^2 - f^{(n+m)} f^{(m)} f - f^{(n)} f^{(2m)} f + f^{(n)} [f^{(m)}]^2 \},\\
H_{m,n} &= (-1)^n \{ f^{(n+2m)} f^2 - 2 f^{(n+m)} f^{(m)} f + f^{(n)} [f^{(m)}]^2 \},\\
I_{m,n} &= (-1)^n \{ f^{(n+2m)} f^2 - 2 f^{(n)} f^{(2m)} f + f^{(n)} [f^{(m)}]^2 \}
\end{aligned}$$

for $n, m \in \mathbb{N}$, then

$$\begin{aligned}
&G_{m,n} \ge 0, \quad H_{m,n} \ge 0,\\
&H_{m,n} \lesseqgtr G_{m,n} \quad \text{when } m \lesseqgtr n,\\
&I_{m,n} \ge G_{m,n} \ge 0 \quad \text{when } n \ge m.
\end{aligned} \tag{5.10}$$

Replacing $f(t)$ by $(-1)^\ell (e^{e^{-t}})^{(\ell)}$ in $G_{m,n}$ $H_{m,n}$ and $I_{m,n}$, and simplifying gives

$$\begin{aligned}
G_{m,n} = (-1)^{\ell+n} \big\{ &(e^{e^{-t}})^{(\ell+2m+n)} [(e^{e^{-t}})^{(\ell)}]^2 - (e^{e^{-t}})^{(\ell+m+n)} (e^{e^{-t}})^{(\ell+m)} (e^{e^{-t}})^{(\ell)}\\
&- (e^{e^{-t}})^{(\ell+n)} (e^{e^{-t}})^{(\ell+2m)} (e^{e^{-t}})^{(\ell)} + (e^{e^{-t}})^{(\ell+n)} [(e^{e^{-t}})^{(\ell+m)}]^2 \big\},
\end{aligned}$$

$$\begin{aligned}H_{m,n} &= (-1)^{\ell+n}\{(e^{e^{-t}})^{(\ell+2m+n)}[(e^{e^{-t}})^{(\ell)}]^2\\&\quad -2(e^{e^{-t}})^{(\ell+m+n)}(e^{e^{-t}})^{(\ell+m)}(e^{e^{-t}})^{(\ell)}+(e^{e^{-t}})^{(\ell+n)}[(e^{e^{-t}})^{(\ell+m)}]^2\},\\ I_{m,n} &= (-1)^{\ell+n}\{(e^{e^{-t}})^{(\ell+2m+n)}[(e^{e^{-t}})^{(\ell)}]^2-2(e^{e^{-t}})^{(\ell+n)}(e^{e^{-t}})^{(\ell+2m)}(e^{e^{-t}})^{(\ell)}\\&\quad +(e^{e^{-t}})^{(\ell+n)}[(e^{e^{-t}})^{(\ell+m)}]^2\}.\end{aligned}$$

Further taking $t \to 0^+$ gives

$$\lim_{t\to 0^+} G_{m,n}=e^3\mathcal{G}_{\ell,m,n}, \quad \lim_{t\to 0^+} H_{m,n}=e^3\mathcal{H}_{\ell,m,n} \quad \text{and} \quad \lim_{t\to 0^+} I_{m,n}=e^3\mathcal{I}_{\ell,m,n}.$$

Substituting these quantities into (5.10) and simplifying bring about inequalities in (5.9). The proof of Theorem 5.3 is complete. □

Theorem 5.4. *For $k \ge 0$ and $n \in \mathbb{N}$, we have*

$$\left(\prod_{\ell=0}^{n} B_{k+2\ell}\right)^{1/(n+1)} \ge \left(\prod_{\ell=0}^{n-1} B_{k+2\ell+1}\right)^{1/n}. \tag{5.11}$$

Proof. If f is a completely monotonic function on $(0,\infty)$, then, by the convexity of the sequence (5.7) and Nanson's inequality listed in [12, p. 205, 3.2.27],

$$\left[\prod_{\ell=0}^{n}(-1)^{k+2\ell+1}f^{(k+2\ell+1)}(t)\right]^{1/(n+1)} \ge \left[\prod_{\ell=1}^{n}(-1)^{k+2\ell}f^{(k+2\ell)}(t)\right]^{1/n}, \quad k\ge 0.$$

Replacing $f(t)$ by $e^{e^{-t}}$ in the above inequality gives

$$\left[\prod_{\ell=0}^{n}(-1)^{k+2\ell+1}\left(e^{e^{-t}}\right)^{(k+2\ell+1)}\right]^{1/(n+1)} \ge \left[\prod_{\ell=1}^{n}(-1)^{k+2\ell}\left(e^{e^{-t}}\right)^{(k+2\ell)}\right]^{1/n}, \quad k\ge 0.$$

Letting $t \to 0^+$ in the above inequality leads to (5.11). The proof of Theorem 5.4 is complete. □

Remark 5.1. This paper is a slightly corrected and revised version of the preprint [18].

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