Proc. Indian Acad. Sci. (Math. Sci.) Vol. 127, No. 3, June 2017, pp. 547–549.
DOI 10.1007/s12044-017-0344-5

Line bundles and flat connections

INDRANIL BISWAS[1,*] and GEORG SCHUMACHER[2]

[1]School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400 005, India
[2]Fachbereich Mathematik und Informatik, Philipps-Universität Marburg, Lahnberge, Hans-Meerwein-Strasse, D-35032 Marburg, Germany
[*]Corresponding author.
E-mail: indranil@math.tifr.res.in; schumac@mathematik.uni-marburg.de

MS received 27 October 2016

Abstract. We prove that there are cocompact lattices Γ in $\mathrm{SL}(2, \mathbb{C})$ with the property that there are holomorphic line bundles L on $\mathrm{SL}(2, \mathbb{C})/\Gamma$ with $c_1(L) = 0$ such that L does not admit any unitary flat connection.

Keywords. Calabi–Yau threefold; torsion; cocompact lattice; unitary representation.

2000 Mathematics Subject Classification. 81T30, 14D21, 53C07.

1. Stable bundles and unitary flat connections

1.1 *Admitting flat connections*

Let X be a compact connected complex manifold of complex dimension δ. Let ω be the $(1, 1)$-form on X associated to a Gauduchon metric on X, so $\partial\overline{\partial}\omega^{\delta-1} = 0$; such a metric on X exists [6]. The degree of a torsionfree coherent analytic sheaf F on X is defined as

$$\text{degree}(F) = \int_X c_1^h(\det F) \wedge \omega^{\delta-1},$$

where $\det F$ is the determinant line bundle on X associated to F (Ch. V, §6, p. 165 of [9]), and $c_1^h(\det F)$ is the first Chern form associated to a Hermitian structure h on $\det F$; it should be clarified that degree(F) is independent of the choice of h. A holomorphic vector bundle E on X is called *stable* if

$$\frac{\text{degree}(F)}{\text{rank}(F)} < \frac{\text{degree}(E)}{\text{rank}(E)}$$

for all coherent analytic subsheaves $F \subset E$ with $0 < \text{rank}(F) < \text{rank}(E)$ [11]. In particular, E is stable if it is a line bundle.

If (X, ω) is Kähler, meaning $d\omega = 0$, and E is a stable vector bundle on X with $c_i(E) = 0$ for all $i > 0$, then a theorem of Uhlenbeck and Yau implies that E admits a unitary flat connection [13]. More generally, if (X, ω) is astheno-Kähler [8], meaning $\partial\overline{\partial}\omega^{\delta-2} = 0$ with $\delta \geq 3$, and E is a stable vector bundle on X with $c_i(E) = 0$ for all $i > 0$, then E admits a unitary flat connection.

1.2 *Not admitting flat connections*

Now take X to be a Calabi–Eckmann manifold $X = S^{2m+1} \times S^{2n+1}$, $m, n > 0$, which is a holomorphic elliptic curve bundle over $\mathbb{CP}^m \times \mathbb{CP}^n$ [4]. Then we have $\dim H^1(X, \mathcal{O}_X) = 1$ (II.9 of [3], p. 232 of [7]). Therefore, using the long exact sequence of cohomologies

$$\begin{aligned} H^1(X, 2\pi\sqrt{-1} \cdot \mathbb{Z}) &\longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^*) \\ &\longrightarrow H^2(X, 2\pi\sqrt{-1} \cdot \mathbb{Z}) \end{aligned} \tag{1}$$

associated to the short exact sequence of sheaves

$$0 \longrightarrow 2\pi\sqrt{-1} \cdot \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 0,$$

we conclude that the space of all isomorphism classes of holomorphic line bundles L on X with $c_1(L) = 0$ is of complex dimension 1.

On the other hand, there is no nontrivial line bundle on X with a flat unitary connection because X is simply connected.

2. Trivial tangent bundle

Assume that the holomorphic tangent bundle TX of X is holomorphically trivial. This implies that there is a complex Lie group G and a co-compact lattice $\Gamma \subset G$, such that X is biholomorphic to G/Γ [14]. It is known that X admits a balanced metric ω, meaning $d\omega^{\delta-1} = 0$, where $\delta = \dim_{\mathbb{C}} X$ [2].

We will construct an example of a pair $(G/\Gamma, L)$, where L is a holomorphic line bundle on G/Γ such that $c_1(L) = 0$ but L does not admit a unitary flat connection.

Take $G = \mathrm{SL}(2, \mathbb{C})$. The Lie algebra of G is $\mathrm{sl}(2, \mathbb{C})$ (complex 2×2 matrices of trace zero). Let $\Gamma \subset G$ be a co-compact lattice such that

$$\operatorname{rank}(\Gamma/[\Gamma, \Gamma]) > 0\,; \tag{2}$$

we note that such subgroups exist (see p. 3393, Theorem 2.1 of [10]). Denote the quotient $\mathrm{SL}(2, \mathbb{C})/\Gamma$ by X. These are examples of non-Kähler Calabi–Yau manifolds; see [5, 12] for their relevance is string theory.

A theorem of Akhiezer says that

$$\dim H^1(X, \mathcal{O}_X) = H^1(X, \mathbb{C}) \tag{3}$$

(p. 608, Theorem 1 of [1]). Note that Γ is the fundamental group of X.

The (real) dimension of the space of all homomorphisms from Γ to U(1) is

$$\dim_{\mathbb{R}} H^1(X, \mathbb{R}) = \dim_{\mathbb{C}} H^1(X, \mathbb{C})$$

because any homomorphism $\pi_1(X) \longrightarrow \mathrm{U}(1)$ factors through $\pi_1(X)/[\pi_1(X), \pi_1(X)] = H_1(X, \mathbb{Z})$. On the other hand, from the exact sequence in (1) it follows that the (complex) dimension of the space of all isomorphism classes of holomorphic line bundles L on X such that $c_i(L) = 0$ is $\dim_{\mathbb{C}} H^1(X, \mathcal{O}_X)$. Now from (2) and (3),

$$\begin{aligned} \dim_{\mathbb{R}} H^1(X, \mathcal{O}_X) &= 2 \cdot \dim_{\mathbb{C}} H^1(X, \mathcal{O}_X) = 2 \cdot \dim_{\mathbb{C}} H^1(X, \mathbb{C}) \\ &> \dim_{\mathbb{C}} H^1(X, \mathbb{C}) = \dim_{\mathbb{R}} H^1(X, \mathbb{R})\,. \end{aligned}$$

Therefore, from dimension count, not all holomorphic line bundles L on X with $c_1(L) = 0$ can have a unitary flat connection. So we have proved the following:

Theorem 2.1. *There is holomorphic line bundle L on the compact complex parallelizable manifold $SL(2, \mathbb{C})/\Gamma$ such that $c_1(L) = 0$ but L does not admit any unitary flat connection.*

References

[1] Akhiezer D, Group actions on the Dolbeault cohomology of homogeneous manifolds, *Math. Zeit.* **226** (1997) 607–621

[2] Biswas I, Semistability of invariant bundles over G/Γ, II, *Comp. Ren. Math. Acad. Sci. Paris* **350** (2012) 277–280

[3] Borel A, A spectral sequence for complex analytic bundles, Appendix to Topological Methods in Algebraic Geometry (ed.) F Hirzebruch (1966) (New York: Springer-Verlag New York, Inc.)

[4] Calabi E and Eckmann B, A class of compact, complex manifolds which are not algebraic, *Ann. Math.* **58** (1953) 494–500

[5] Fei T and Yau S-T, Invariant solutions to the Strominger system on complex Lie groups and their quotients, *Comm. Math. Phys.* **338** (2015) 1183–1195

[6] Gauduchon P, La 1-forme de torsion d'une variété hermitienne compacte, *Math. Ann.* **267** (1984) 495–518

[7] Höfer T, Remarks on torus principal bundles, *J. Math. Kyoto Univ.* **33** (1993) 227–259

[8] Jost J and Yau S-T, A nonlinear elliptic system for maps from Hermitian to Riemannian manifolds and rigidity theorems in Hermitian geometry, *Acta Math.* **170** (1993) 221–254

[9] Kobayashi S, Differential geometry of complex vector bundles, Publications of the Math. Society of Japan 15 (1987) (Iwanami Shoten Publishers and Princeton University Press)

[10] Lackenby M, Some 3-manifolds and 3-orbifolds with large fundamental group, *Proc. Amer. Math. Soc.* **135** (2007) 3393–3402

[11] Li J and Yau S-T, Hermitian Yang-Mills connections on non-Kähler manifolds, Mathematical Aspects of String Theory (ed.) S-T Yau (1987) (World Sci Publ.) pp. 560–573

[12] Strominger A, Superstrings with torsion, *Nucl. Phys. B* **274** (1986) 253–284

[13] Uhlenbeck K and Yau S-T, On the existence of Hermitian–Yang–Mills connections in stable vector bundles, *Comm. Pure Appl. Math.* **39** (1986) 257–293

[14] Wang H-C, Complex parallisable manifolds, *Proc. Amer. Math. Soc.* **5** (1954) 771–776

COMMUNICATING EDITOR: Parameswaran Sankaran