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Lie n-derivations on $\mathcal{J}$-subspace lattice algebras

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Abstract. Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X over the real or complex field $\mathbb{F}$ with $\dim X \geq 3$ and let $n \geq 2$ be an integer. Suppose that $\dim K \neq 2$ for every $K \in \mathcal{J}(\mathcal{L})$ and $L : \operatorname{Alg}\mathcal{L} \to \operatorname{Alg}\mathcal{L}$ is a linear map. It is shown that L satisfies $\sum_{i=1}^{n} p_n(A_1, \ldots, A_{i-1}, L(A_i), A_{i+1}, \ldots, A_n) = 0$ whenever $p_n(A_1, A_2, \ldots, A_n) = 0$ for $A_1, A_2, \ldots, A_n \in \operatorname{Alg}\mathcal{L}$ if and only if for each $K \in \mathcal{J}(\mathcal{L})$, there exists a bounded linear operator $T_K \in \mathcal{B}(K)$, a scalar λ_K and a linear functional $h_K : \operatorname{Alg}\mathcal{L} \to \mathbb{F}$ such that $L(A)x = (T_K A - AT_K + \lambda_K A + h_K(A)I)x$ for all $x \in K$ and all $A \in \operatorname{Alg}\mathcal{L}$. Based on this result, a complete characterization of linear n-Lie derivations on $\operatorname{Alg}\mathcal{L}$ is obtained.

Keywords. $\mathcal{J}$-subspace lattice algebras; Lie derivations; Lie n-derivations; derivations.

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1. Introduction

Let $\mathcal{A}$ be an algebra. Recall that a linear map $L : \mathcal{A} \to \mathcal{A}$ is a derivation if $L(AB) = L(A)B + AL(B)$ for all $A, B \in \mathcal{A}$; is a Lie derivation if $L([A, B]) = [L(A), B] + [A, L(B)]$ for all $A, B \in \mathcal{A}$, where $[A, B] = AB - BA$; is a Lie triple derivation if $L([[A, B], C]) = [[L(A), B], C] + [[A, L(B)], C] + [[A, B], L(C)]$ for all $A, B, C \in \mathcal{A}$. It is obvious that each derivation is a Lie derivation and each Lie derivation is a Lie triple derivation, but the converse is not true generally. The questions of characterizing Lie (triple) derivations and revealing the relationship between Lie (triple) derivations and derivations have attracted much attention, see [3, 4, 7, 12, 14, 15, 17] and the references therein.

In [1], Abdullaev introduced the concept of Lie n-derivations. Define the sequence of polynomials: $p_1(x) = x$ and

$$p_n(x_1, x_2, \ldots, x_n) = [p_{n-1}(x_1, x_2, \ldots, x_{n-1}), x_n]$$

for all integers $n \geq 2$. Here, $p_n(x_1, x_2, \ldots, x_n)$ is also called the $(n-1)$-th commutator. Thus, $p_2(x_1, x_2) = [x_1, x_2]$, $p_3(x_1, x_2, x_3) = [[x_1, x_2], x_3]$,$\ldots$ Let $n \geq 2$ be an integer. A linear map $L : \mathcal{A} \to \mathcal{A}$ is called a *Lie n-derivation* if

$$L(p_n(x_1, x_2, \ldots, x_n)) = \sum_{i=1}^{n} p_n(x_1, \ldots, x_{i-1}, L(x_i), x_{i+1}, \ldots, x_n)$$

for all $x_1, x_2, \dots, x_n \in \mathcal{A}$. Particularly, a Lie 2-derivation is a Lie derivation and a Lie 3-derivation is a Lie triple derivation. Abdullaev [1] discussed the form of Lie n-derivations on a certain von Neumann algebra. After that, the problem of how to characterize the structure of Lie n-derivations has been investigated for various rings and operator algebras (see [2, 6, 16, 18] and the references therein).

In the present paper, we will consider the characterization of linear Lie n-derivations on an important kind of subspace lattice algebras: $\mathcal{J}$-subspace lattice algebras.

Let X be a Banach space. A family $\mathcal{L}$ of subspaces of X is called a subspace lattice on X if it contains $\{0\}$ and X, and is closed under the operations of closed linear span $\vee$ and intersection $\wedge$ in the sense that $\vee_{\gamma\in\Gamma}L_\gamma \in \mathcal{L}$ and $\wedge_{\gamma\in\Gamma}L_\gamma \in \mathcal{L}$ for every family $\{L_\gamma : \gamma \in \Gamma\}$ of elements in $\mathcal{L}$. For a subspace lattice $\mathcal{L}$ on X, the associated subspace lattice algebra $\operatorname{Alg}\mathcal{L}$ is the set of operators on X leaving every subspace in $\mathcal{L}$ invariant. Given a subspace lattice $\mathcal{L}$ on X, put

$$\mathcal{J}(\mathcal{L}) = \{K \in \mathcal{L} : K \neq \{0\} \text{ and } K_- \neq X\},$$

where $K_- = \vee\{L \in \mathcal{L} : K \nsubseteq L\}$. Call $\mathcal{L}$ a $\mathcal{J}$-subspace lattice (simply, JSL) on X if it satisfies the following conditions:

(1) $\vee\{K : K \in \mathcal{J}(\mathcal{L})\} = X$;
(2) $\wedge\{K_- : K \in \mathcal{J}(\mathcal{L})\} = \{0\}$;
(3) $K \vee K_- = X, \forall K \in \mathcal{J}(\mathcal{L})$;
(4) $K \wedge K_- = \{0\}, \forall K \in \mathcal{J}(\mathcal{L})$.

If $\mathcal{L}$ is a JSL, the associated subspace lattice algebra $\operatorname{Alg}\mathcal{L}$ is called a *$\mathcal{J}$-subspace lattice algebra*, or briefly, JSL algebra. Note that a JSL algebra may not be prime. It also should be mentioned that both atomic Boolean subspace lattices and pentagon subspace lattices are $\mathcal{J}$-subspace lattices [9]. For $L \in \mathcal{L}$, denote $L_-^\perp = (L_-)^\perp$, where $L^\perp$ denotes the annihilator of L. Denote by $\langle\mathcal{J}(\mathcal{L})\rangle$ and $\langle\mathcal{J}(\mathcal{L})_-^\perp\rangle$ the (not necessarily closed) linear span of $\cup\{K : K \in \mathcal{J}(\mathcal{L})\}$ and the linear span of $\cup\{K_-^\perp : K \in \mathcal{J}(\mathcal{L})\}$, respectively. For $x \in X$ and $f \in X^*$, $x \otimes f$ stands for the operator on X with rank not greater than one defined by $(x \otimes f)y = f(y)x$. Sometimes we use $\langle x, f\rangle$ to present the value $f(x)$ of f at x. For $K \in \mathcal{J}(\mathcal{L})$, let $\mathcal{F}_{\mathcal{L}}(K)$ denote the subspace spanned by all rank one operators $x \otimes f$ with $x \in K$ and $f \in K_-^\perp$. For more properties of JSL algebras, see [8–11].

We first give a local characterization of linear Lie n-derivations on JSL algebras, which is the first main result in this paper.

Theorem 1.1. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X over the real or complex field $\mathbb{F}$ with* $\dim X \geq 3$ *and let $n \geq 2$ be an integer. Suppose that* $\dim K \neq 2$ *for every $K \in \mathcal{J}(\mathcal{L})$ and $L : \operatorname{Alg}\mathcal{L} \to \operatorname{Alg}\mathcal{L}$ is a linear map. Then the following statements are equivalent.*

(1) *L satisfies $\sum_{i=1}^{n} p_n(A_1, \dots, A_{i-1}, L(A_i), A_{i+1}, \dots, A_n) = 0$ for every $A_1, \dots, A_n \in \operatorname{Alg}\mathcal{L}$ with $p_n(A_1, \dots, A_n) = 0$.*
(2) *For each $K \in \mathcal{J}(\mathcal{L})$, there exists an operator $T_K \in \mathcal{B}(K)$, a scalar λ_K and a linear functional $h_K : \operatorname{Alg}\mathcal{L} \to \mathbb{F}$ such that $L(A)x = (T_K A - AT_K + \lambda_K A + h_K(A)I)x$ for all $x \in K$ and all $A \in \operatorname{Alg}\mathcal{L}$.*

Recall that every Lie derivation on $M_n(\mathbb{F})$ $(n \geq 1)$ is a sum of a derivation and a multiple of I by a linear functional vanishing at the commutators [3]. This fact, together with Theorem 1.1 and its proof given in Section 2 (including the case $\dim K = 2$ for some $K \in \mathcal{J}(\mathcal{L})$), one can immediately get a characterization of linear Lie n-derivations on JSL algebras, which is the second main result in the paper.

Theorem 1.2. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X over the real or complex field $\mathbb{F}$ and let $n \geq 2$ be an integer. Then a linear map $L : \operatorname{Alg}\mathcal{L} \to \operatorname{Alg}\mathcal{L}$ is a Lie n-derivation if and only if for each $K \in \mathcal{J}(\mathcal{L})$, there exists an operator $T_K \in \mathcal{B}(K)$, a scalar λ_K and a linear functional $h_K : \operatorname{Alg}\mathcal{L} \to \mathbb{F}$ satisfying $h_K(p_n(A_1, \ldots, A_n)) = 0$ for all $A_1, \ldots, A_n \in \operatorname{Alg}\mathcal{L}$ such that $L(A)x = (T_K A - AT_K + \lambda_K A + h_K(A)I)x$ for all $x \in K$ and all $A \in \operatorname{Alg}\mathcal{L}$.*

Remark 1. It is obvious that every Lie n-derivation of JSL algebras satisfies the statement (1) in Theorem 1.1, but the converse does not hold in general. Hence, our Theorem 1.1 is a generalization of Theorem 4.1 of [12] and Theorem 10 of [17], where linear Lie derivations and linear Lie triple derivations on $\mathcal{J}$-subspace lattice algebras are characterized, respectively.

2. Proof of Theorem 1.1

In this section, we will give a proof of Theorem 1.1.

We first give several lemmas, which are needed to prove our main result.

Lemma 2.1 [11]. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X. Then $x \otimes f \in \operatorname{Alg}\mathcal{L}$ if and only if there exists a subspace $K \in \mathcal{J}(\mathcal{L})$ such that $x \in K$ and $f \in K_-^{\perp}$.*

Lemma 2.2 [9]. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X. The following statements hold.*

(1) *For any $K, L \in \mathcal{J}(\mathcal{L})$, $K \neq L$ implies that $K \subseteq L_-$;*
(2) *For any $K, L \in \mathcal{J}(\mathcal{L})$, $K \neq L$ implies that $K \cap L = \{0\}$;*
(3) *Let $K \in \mathcal{J}(\mathcal{L})$. Then, for any nonzero vector $x \in K$, there exists $f \in K_-^{\perp}$ such that $f(x) = 1$; dually, for any nonzero functional $f \in K_-^{\perp}$, there exists $x \in K$ such that $f(x) = 1$.*

Lemma 2.3 (*Lemma* 3.6 *of* [13]). *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X and let $A, B \in \mathcal{F}_{\mathcal{L}}(K)$ for some $K \in \mathcal{J}(\mathcal{L})$. Then there is an idempotent $P \in \mathcal{F}_{\mathcal{L}}(K)$ such that $A = PAP$ and $B = PBP$.*

Let X be a Banach space over the real or complex field $\mathbb{F}$ with $\dim X > 2$. Denote by $\mathcal{B}(X)$ and $\mathcal{F}(X)$ the algebra of all bounded linear operators on X and the subalgebra of all finite rank operators in $\mathcal{B}(X)$, respectively. Recall that a subalgebra $\mathcal{A}$ of $\mathcal{B}(X)$ is called a *standard subalgebra* if $\mathcal{A}$ contains $\mathcal{F}(X)$ and the unit operator I.

The following lemma gives a characterization of linear Lie n-derivations on $\mathcal{F}(X)$ by local action.

Lemma 2.4. *Let X be a Banach space over the real or complex field $\mathbb{F}$ with* $\dim X \geq 3$ *and let $\mathcal{A}$ be a standard subalgebra of $\mathcal{B}(X)$. Suppose that $L : \mathcal{F}(X) \to \mathcal{A}$ is a linear map and $n \geq 2$ is an integer. Then L satisfies*

$$\sum_{i=1}^{n} p_n(A_1, \ldots, A_{i-1}, L(A_i), A_{i+1}, \ldots, A_n) = 0$$

whenever $p_n(A_1, \ldots, A_n) = 0$ for $A_1, \ldots, A_n \in \mathcal{F}(X)$ if and only if $L(A) = TA - AT + \lambda A + h(A)I$ for all $A \in \mathcal{F}(X)$, where $\lambda \in \mathbb{F}$, $T \in \mathcal{B}(X)$ and $h : \mathcal{F}(X) \to \mathbb{F}$ is a linear functional.

Proof. The 'if' part is easily checked. For the 'only if' part, assume that L satisfies

$$\sum_{i=1}^{n} p_n(A_1, \ldots, A_{i-1}, L(A_i), A_{i+1}, \ldots, A_n) = 0 \tag{2.1}$$

for all $A_1, A_2, \ldots, A_n \in \mathcal{F}(X)$ with $p_n(A_1, A_2, \ldots, A_n) = 0$. For any $A \in \mathcal{F}(X)$, letting $A_1 = A$ and $A_2 = A^2$ in eq. (2.1), we get $p_n(L(A), A^2, A_3, \ldots, A_n) + p_n(A, L(A^2), A_3, \ldots, A_n) = 0$, which implies that

$$p_{n-1}(L(A)A^2 - A^2L(A) + AL(A^2) - L(A^2)A, A_3, A_4, \ldots, A_n) = 0 \tag{2.2}$$

holds for all $A, A_3, \ldots, A_n \in \mathcal{F}(X)$. Write

$$\begin{aligned} Z = {}& L(A)A^2 - A^2L(A) + AL(A^2) - L(A^2)A \\ = {}& [A, L(A^2) - L(A)A - AL(A)] \in \mathcal{F}(X). \end{aligned}$$

Then eq. (2.2) becomes $p_{n-1}(Z, A_3, A_4, \ldots, A_n) = 0$, that is,

$$p_{n-2}(Z, A_3, A_4, \ldots, A_{n-1})A_n = A_n p_{n-2}(Z, A_3, A_4, \ldots, A_{n-1})$$

for all $A_3, \ldots, A_n \in \mathcal{F}(X)$. Since $\mathcal{F}(X)$ is strong dense in $\mathcal{B}(X)$, the above equation implies that

$$p_{n-2}(Z, A_3, A_4, \ldots, A_{n-1}) \in \mathcal{Z}(\mathcal{B}(X)) = \mathbb{F}I.$$

Note that I can not be the commutator. So $p_{n-2}(Z, A_3, A_4, \ldots, A_{n-1}) = 0$ holds for all $A_3, \ldots, A_{n-1} \in \mathcal{F}(X)$. Repeating the above process, the arbitrariness of $A_3, \ldots, A_{n-1}$ implies $Z = 0$, that is,

$$[A, L(A^2) - L(A)A - AL(A)] = 0 \quad \text{for all } A \in \mathcal{F}(X).$$

It is clear that both $\mathcal{F}(X)$ and $\mathcal{A}$ are prime algebras with characteristic 0 and $\deg(\mathcal{F}(X)), \deg(\mathcal{A}) \geq 3$. Thus, by a similar argument to that of Theorem 4 of [3], there exists a scalar λ, a linear derivation $\tau : \mathcal{F}(X) \to \mathcal{A}$ and a linear functional $h : \mathcal{F}(X) \to \mathbb{F}$ such that

$$L(A) = \lambda A + \tau(A) + h(A)I \quad \text{for all } A \in \mathcal{F}(X).$$

It is well-known that derivations on $\mathcal{F}(X)$ are inner (for example, see [5]). So there exists an operator $T \in \mathcal{B}(X)$ such that $\tau(A) = TA - AT$ for all $A \in \mathcal{F}(X)$, completing the proof of the lemma. □

Now, we are in a position to give a proof of Theorem 1.1.

Proof of Theorem 1.1. A straightforward verification shows that (2) $\Rightarrow$ (1) is true. For (1) $\Rightarrow$ (2), we will prove it by checking several claims.

Claim 1. For any $K \in \mathcal{J}(\mathcal{L})$, there exist a bounded linear operator $T_K \in \mathcal{B}(K)$, a scalar λ_K and a linear functional $h_K : \mathcal{F}_{\mathcal{L}}(K) \to \mathbb{F}$ such that

$$L(A)|_K = (T_K A - A T_K + \lambda_K A + h_K(A) P_K)|_K$$

for all $A \in \mathcal{F}_{\mathcal{L}}(K)$, where $P_K \in \mathcal{F}_{\mathcal{L}}(K)$ is an idempotent independent of A satisfying $P_K A = A P_K = A$.

For any $K \in \mathcal{J}(\mathcal{L})$, if $\dim K = 1$, then $\dim K_-^{\perp} = 1$. So $\dim \mathcal{F}_{\mathcal{L}}(K) = 1$. It is easy to check that, for any $A \in \mathcal{F}_{\mathcal{L}}(K)$, we have $L(A) = \lambda_K A$ for some $\lambda_K \in \mathbb{F}$. So Claim 1 holds.

Now assume that $\dim K > 2$. For any $A \in \mathcal{F}_{\mathcal{L}}(K)$, by Lemma 2.3, there is an idempotent operator $P = \sum_{i=1}^{n} y_i \otimes g_i \in \mathcal{F}_{\mathcal{L}}(K)$ such that $A = PAP$, where $y_i \in K$ and $g_i \in K_-^{\perp}$. We may require that both $\{y_1, y_2, \dots, y_n\}$ and $\{g_1, g_2, \dots, g_n\}$ are linearly independent sets. Since $P^2 = P$, it can be easily checked that $g_i(y_j) = \delta_{ij}$ for $1 \le i, j \le n$. Define the set

$$\mathcal{D}_P = \left\{ C \in \mathcal{F}_{\mathcal{L}}(K) : C = \sum_{i,j=1}^{n} \lambda_{ij} y_i \otimes g_j, \lambda_{ij} \in \mathbb{F} \right\}.$$

It is clear that $\mathcal{D}_P$ is a subalgebra of $\mathcal{F}_{\mathcal{L}}(K)$ and $\mathcal{D}_P$ is isomorphic to $M_n(\mathbb{F})$ via $C \mapsto [\lambda_{ij}]_{n\times n}$. Since $\dim K > 2$, we can choose P so that $\dim \operatorname{ran}(P) > 2$. For any $A, B \in \mathcal{F}_{\mathcal{L}}(K)$, by Lemma 2.3, there exists some $P \in \mathcal{F}_{\mathcal{L}}(K)$ such that $A = PAP$ and $B = PBP$. So $\mathcal{F}_{\mathcal{L}}(K)$ is a local matrix algebra. Note that $K \wedge K_- = \{0\}$ and $K \vee K_-^{\perp} = X$. We may regard $K_-^{\perp} \subseteq K^*$, the dual space K^* of K, since for any $f \in K_-^{\perp}$, $f|_K \in K^*$. Hence $\mathcal{F}_{\mathcal{L}}(K)$ is isomorphic to $\mathcal{F}(K) \subset \mathcal{B}(K)$, the algebra of all bounded finite rank operators from K into K. Thus, by Lemma 2.4, the claim is also true in this case.

Claim 2. For each $K \in \mathcal{J}(\mathcal{L})$, there exist an operator $T_K \in \mathcal{B}(K)$ and a scalar λ_K such that $L(A)x = (T_K A - A T_K + \lambda_K A + h_K(A) I)x$ holds for all invertible operator $A \in \operatorname{Alg}\mathcal{L}$, for all $x \in K$ and for some scalar $h_K(A)$ depending on A.

Assume that $A \in \operatorname{Alg}\mathcal{L}$ is invertible. For any $K \in \mathcal{J}(\mathcal{L})$ and for any nonzero vector $x \in K$, we have $0 \neq Ax \in K$. By Lemma 2.2, there exists $f \in K_-^{\perp}$ such that $\langle Ax, f\rangle = 1$. Moreover, Lemma 2.1 implies $x \otimes f, Ax \otimes f, Ax \otimes A^* f \in \operatorname{Alg}\mathcal{L}$. In the sequel, we will complete the proof of the Claim by considering two cases.

Case 1. $\langle x, f\rangle = \beta \neq 0$. Since $p_n(A - Ax \otimes A^* f, x \otimes f, x \otimes f, \dots, x \otimes f) = 0$, we have $L(p_n(A - Ax \otimes A^* f, x \otimes f, x \otimes f, \dots, x \otimes f)) = 0$, that is,

$$\begin{aligned} &p_n(L(A) - L(Ax \otimes A^* f), x \otimes f, x \otimes f, \dots, x \otimes f) \\ &\quad + p_n(A - Ax \otimes A^* f, L(x \otimes f), x \otimes f, \dots, x \otimes f) = 0. \end{aligned}$$

By Claim 1, we get

$$\begin{aligned} &p_n(L(A) - T_K(Ax \otimes A^* f) + (Ax \otimes A^* f)T_K \\ &\quad - \lambda_K(Ax \otimes A^* f) - h_K(Ax \otimes A^* f)P_K, x \otimes f, x \otimes f, \dots, x \otimes f) \\ &\quad + p_n(A - Ax \otimes A^* f, T_K(x \otimes f) - (x \otimes f)T_K + \lambda_K(x \otimes f) \\ &\quad + h_K(x \otimes f)P_K, x \otimes f, \dots, x \otimes f) = 0. \end{aligned} \tag{2.3}$$

Note that $P_K x\otimes f = x\otimes f P_K$ and $(x\otimes f)(AP_K - P_K A)x = 0$. Acting at x in eq. (2.3), and noting that $\langle Ax, f\rangle = 1$, a direct calculation can be obtained such that, if $n = 2k+1$ (k is any positive integer) is odd, then

$$\begin{aligned}
0 &= p_n(L(A) - T_K(Ax\otimes A^*f) + (Ax\otimes A^*f)T_K - \lambda_K(Ax\otimes A^*f) \\
&\quad - h_K(Ax\otimes A^*f)P_K, x\otimes f, x\otimes f, \ldots, x\otimes f)x \\
&\quad + p_n(A - Ax\otimes A^*f, T_K(x\otimes f) - (x\otimes f)T_K + \lambda_K(x\otimes f) \\
&\quad + h_K(x\otimes f)P_K, x\otimes f, \ldots, x\otimes f)x \\
&= (\beta^{2k-1}(L(A)x\otimes f - T_K Ax\otimes f + \langle AT_K x, f\rangle Ax\otimes f - \lambda_K Ax\otimes f) \\
&\quad - 2\beta^{2k-2}(\langle L(A)x, f\rangle - \lambda_K P_K)x\otimes f + \beta^{2k-1}(x\otimes f L(A) \\
&\quad - \langle T_K Ax, f\rangle x\otimes A^*f + x\otimes T_K^* A^* f - \lambda_K x\otimes A^*f))x \\
&\quad + \beta^{2k-1}(AT_K x\otimes f - \langle AT_K x, f\rangle Ax\otimes f - x\otimes A^* T_K^* f + \lambda_K x\otimes A^* f \\
&\quad + \langle T_K Ax, f\rangle x\otimes A^* f - \lambda_K x\otimes A^* f)x \\
&= \beta^{2k}(L(A) - T_K A + AT_K - \lambda_K A)x - \beta^{2k-1}(\langle L(A)x, f\rangle \\
&\quad - \langle T_K Ax, f\rangle + \langle AT_K x, f\rangle - \lambda_K)x.
\end{aligned}$$

If $n = 2k$ is even, then we have

$$\begin{aligned}
0 &= p_n(L(A) - T_K(Ax\otimes A^*f) + (Ax\otimes A^*f)T_K - \lambda_K(Ax\otimes A^*f) \\
&\quad - h_K(Ax\otimes A^*f)P_K, x\otimes f, x\otimes f, \ldots, x\otimes f)x \\
&\quad + p_n(A - Ax\otimes A^*f, T_K(x\otimes f) - (x\otimes f)T_K + \lambda_K(x\otimes f) \\
&\quad + h_K(x\otimes f)P_K, x\otimes f, \ldots, x\otimes f)x \\
&= (\beta^{2k-2}(L(A)x\otimes f - T_K Ax\otimes f + \langle AT_K x, f\rangle Ax\otimes f - \lambda_K Ax\otimes f) \\
&\quad - \beta^{2k-2}(x\otimes f L(A) - \langle T_K Ax, f\rangle x\otimes A^* f + x\otimes T_K^* A^* f \\
&\quad - \lambda_K x\otimes A^* f))x + \beta^{2k-2}(AT_K x\otimes f \\
&\quad - \langle AT_K x, f\rangle Ax\otimes f + x\otimes A^* T_K^* f - \lambda_K x\otimes A^* f \\
&\quad - \langle T_K Ax, f\rangle x\otimes A^* f + \lambda_K x\otimes A^* f)x \\
&= \beta^{2k-1}(L(A) - T_K A + AT_K - \lambda_K A)x \\
&\quad - \beta^{2k-2}(\langle L(A)x, f\rangle - \langle T_K Ax, f\rangle + \langle AT_K x, f\rangle - \lambda_K)x.
\end{aligned}$$

So the above two equations imply that, for any $n \geq 2$, we have

$$\begin{aligned}
(L(A) - T_K A + AT_K - \lambda_K A)x = \beta^{-1}(&\langle L(A)x, f\rangle - \langle T_K Ax, f\rangle \\
&+ \langle AT_K x, f\rangle - \lambda_K)x,
\end{aligned}$$

that is, $(L(A) - T_K A + AT_K - \lambda_K A)x$ is linearly dependent on x for every $x\in K$. It follows that there exists a scalar $h_K(A)$ depending on A such that $L(A) - T_K A + AT_K - \lambda_K A = h_K(A)I_K$ on K for all invertible $A\in \mathrm{Alg}\,\mathcal{L}$.

Case 2. $\langle x, f\rangle = 0$. In this case, since $p_n(A - Ax\otimes A^*f, x\otimes f, Ax\otimes f, Ax\otimes f, \ldots, Ax\otimes f) = 0$, we get $L(p_n(A - Ax\otimes A^*f, x\otimes f, Ax\otimes f, Ax\otimes f, \ldots, Ax\otimes f)) = 0$, that is,

$$\begin{aligned}
&p_n(L(A) - L(Ax\otimes A^*f), x\otimes f, Ax\otimes f, \ldots, Ax\otimes f) \\
&\quad + p_n(A - Ax\otimes A^*f, L(x\otimes f), Ax\otimes f, \ldots, Ax\otimes f) = 0.
\end{aligned}$$

This and Claim 1 imply that

$$\begin{aligned}&p_n(L(A) - T_K(Ax \otimes A^* f) + (Ax \otimes A^* f)T_K - \lambda_K(Ax \otimes A^* f)\\&\quad - h_K(Ax \otimes A^* f)P_K, x \otimes f, Ax \otimes f, \ldots, Ax \otimes f)\\&\quad + p_n(A - Ax \otimes A^* f, T_K(x \otimes f) - (x \otimes f)T_K\\&\quad + \lambda_K(x \otimes f) + h_K(x \otimes f)P_K, Ax \otimes f, \ldots, Ax \otimes f) = 0.\end{aligned}$$

Acting at Ax in the above equation, we can obtain that, if $n \geq 3$, then

$$\begin{aligned}0 &= p_n(L(A) - T_K(Ax \otimes A^* f) + (Ax \otimes A^* f)T_K - \lambda_K(Ax \otimes A^* f)\\&\quad - h_K(Ax \otimes A^* f)P_K, x \otimes f, Ax \otimes f, \ldots, Ax \otimes f)Ax\\&\quad + p_n(A - Ax \otimes A^* f, T_K(x \otimes f) - (x \otimes f)T_K\\&\quad + \lambda_K(x \otimes f) + h_K(x \otimes f)P_K, Ax \otimes f, \ldots, Ax \otimes f)Ax\\&= (L(A)x \otimes f - T_K Ax \otimes f - \langle L(A)Ax, f\rangle x \otimes f\\&\quad + \langle A^2 x, f\rangle\langle T_K Ax, f\rangle x \otimes f - \langle AT_K Ax, f\rangle x \otimes f\\&\quad + \lambda_K\langle A^2 x, f\rangle x \otimes f - \langle L(A)x, f\rangle Ax \otimes f\\&\quad + \langle T_K Ax, f\rangle Ax \otimes f)Ax + (AT_K x \otimes f - \langle AT_K x, f\rangle Ax \otimes f\\&\quad + \langle T_K A^2 x, f\rangle x \otimes f - \langle A^2 x, f\rangle\langle T_K Ax, f\rangle x \otimes f)Ax\\&= (L(A) - T_K A - \langle L(A)x, f\rangle A + \langle T_K Ax, f\rangle A)x\\&\quad + (-\langle L(A)Ax, f\rangle + \langle A^2 x, f\rangle\langle T_K Ax, f\rangle - \langle AT_K Ax, f\rangle\\&\quad + \lambda_K\langle A^2 x, f\rangle)x + (AT_K - \langle AT_K x, f\rangle A)x\\&\quad + (\langle T_K A^2 x, f\rangle - \langle A^2 x, f\rangle\langle T_K Ax, f\rangle)x\\&= (L(A) - T_K A + AT_K)x - (\langle AT_K x, f\rangle\\&\quad + \langle L(A)x, f\rangle - \langle T_K Ax, f\rangle)Ax - (\langle L(A)Ax, f\rangle + \langle AT_K Ax, f\rangle\\&\quad - \lambda_K\langle A^2 x, f\rangle - \langle T_K A^2 x, f\rangle)x.\end{aligned} \tag{2.4}$$

If $n = 2$, then

$$\begin{aligned}0 &= p_n(L(A) - T_K(Ax \otimes A^* f) + (Ax \otimes A^* f)T_K - \lambda_K(Ax \otimes A^* f)\\&\quad - h_K(Ax \otimes A^* f)P_K, x \otimes f, Ax \otimes f, \ldots, Ax \otimes f)Ax\\&\quad + p_n(A - Ax \otimes A^* f, T_K(x \otimes f) - (x \otimes f)T_K\\&\quad + \lambda_K(x \otimes f) + h_K(x \otimes f)P_K, Ax \otimes f, \ldots, Ax \otimes f)Ax\\&= (L(A)x \otimes f - T_K Ax \otimes f + \langle AT_K x, f\rangle Ax \otimes f - \lambda_K Ax \otimes f\\&\quad - x \otimes f L(A) + \langle T_K Ax, f\rangle x \otimes A^* f - x \otimes f T_K A + \lambda_K x \otimes f A)Ax\\&\quad + (AT_K x \otimes f - \langle AT_K x, f\rangle Ax \otimes f\\&\quad + x \otimes A^* T_K^* f - \langle T_K Ax, f\rangle x \otimes A^* f)Ax\\&= (L(A) - T_K A + \langle AT_K x, f\rangle A - \lambda_K A)x\\&\quad - (\langle L(A)Ax, f\rangle - \langle A^2 x, f\rangle\langle T_K Ax, f\rangle\\&\quad + \langle AT_K Ax, f\rangle - \lambda_K\langle A^2 x, f\rangle)x + (AT_K x - \langle AT_K x, f\rangle)Ax\\&\quad + (\langle T_K A^2 x, f\rangle - \langle A^2 x, f\rangle\langle T_K Ax, f\rangle)x\\&= (L(A) - T_K A + AT_K - \lambda_K A)x\\&\quad - (\langle L(A)Ax, f\rangle + \langle AT_K Ax, f\rangle - \langle T_K A^2 x, f\rangle - \lambda_K\langle A^2 x, f\rangle)x.\end{aligned} \tag{2.5}$$

Next, if $n = 2$, eq. (2.5) implies that

$$\begin{aligned}(L(A) - T_K A + AT_K - \lambda_K A)x = (&\langle L(A)Ax, f\rangle + \langle AT_K Ax, f\rangle\\&- \langle T_K A^2 x, f\rangle - \lambda_K\langle A^2 x, f\rangle)x\end{aligned} \tag{2.6}$$

holds for all invertible operators $A \in \operatorname{Alg}\mathcal{L}$. Note that $2A$ is also invertible in $\operatorname{Alg}\mathcal{L}$. Replacing A by $2A$ in eq. (2.6), we get

$$\begin{aligned}(L(A) - T_K A + AT_K - \lambda_K A)x = 2(&\langle L(A)Ax, f\rangle \\ &+ \langle AT_K Ax, f\rangle - \langle T_K A^2x, f\rangle \\ &- \lambda_K\langle A^2x, f\rangle)x.\end{aligned} \tag{2.7}$$

Comparing eq. (2.6) and eq. (2.7) we obtain $(L(A) - T_K A + AT_K - \lambda_K A)x = 0$, that is, $L(A)x = (T_K A - AT_K + \lambda_K A)x$ for all invertible $A \in \operatorname{Alg}\mathcal{L}$ and all $x \in K$.

If $n \geq 3$, eq. (2.4) implies that

$$\begin{aligned}(L(A) - T_K A + AT_K)x = (&\langle AT_K x, f\rangle + \langle L(A)x, f\rangle \\ &- \langle T_K Ax, f\rangle)Ax + (\langle L(A)Ax, f\rangle \\ &+ \langle AT_K Ax, f\rangle - \lambda_K\langle A^2x, f\rangle - \langle T_K A^2x, f\rangle)x\end{aligned}$$

holds for all invertible operators $A \in \operatorname{Alg}\mathcal{L}$. Similar to the discussion of equations (2.6)–(2.7), one can achieve that $L(A)x = (T_K A - AT_K)x$ holds for all invertible $A \in \operatorname{Alg}\mathcal{L}$ and all $x \in K$.

Now, combining Case 1 and Case 2, the claim is true.

Claim 3. For each $K \in \mathcal{J}(\mathcal{L})$, there exist an operator $T_K \in \mathcal{B}(K)$, a scalar λ_K and a linear functional $h_K : \operatorname{Alg}\mathcal{L} \to \mathbb{F}$ such that $L(A)x = (T_K A - AT_K + \lambda_K A + h_K(A)I)x$ for all $x \in K$ and all $A \in \operatorname{Alg}\mathcal{L}$. Therefore, Theorem 1.1 holds.

For any $A \in \operatorname{Alg}\mathcal{L}$, take a scalar c such that $|c| > \|A\|$. Then $cI - A$ is invertible with its inverse still in $\operatorname{Alg}\mathcal{L}$. By Claim 2, we have

$$\begin{aligned}L(cI - A)|_K &= T_K(cI - A)|_K - (cI - A)T_K + \lambda_K(cI - A)|_K \\ &\quad + h_K(cI - A)I|_K \\ &= -T_K A|_K + AT_K + c\lambda_K I|_K - \lambda_K A|_K + h_K(cI - A)I|_K.\end{aligned}$$

On the other hand,

$$L(cI - A)|_K = L(cI)|_K - L(A)|_K = c\lambda_K I|_K + h_K(cI)I|_K - L(A)|_K.$$

Combining the above two equations, we get

$$\begin{aligned}L(A)|_K &= T_K A|_K - AT_K + \lambda_K A|_K + (h_K(cI) - h_K(cI - A))I|_K \\ &= T_K A|_K - AT_K + \lambda_K A|_K + h_K(A)I|_K,\end{aligned}$$

where $h_K(A) = h_K(cI) - h_K(cI - A)$. Thus there exists a functional h_K of $\operatorname{Alg}\mathcal{L}$ such that

$$L(A)x = (T_K A - AT_K + \lambda_K A + h_K(A)I)x$$

for all $A \in \operatorname{Alg}\mathcal{L}$ and all $x \in K$. Since L is linear, we see that h_K is also linear.

The proof of Theorem 1.1 is completed. □

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