Proc. Indian Acad. Sci. (Math. Sci.) Vol. 127, No. 3, June 2017, pp. 517–523.
DOI 10.1007/s12044-017-0343-6

On certain geodesic conjugacies of flat cylinders

C S ARAVINDA[1] and H A GURURAJA[2,*]

[1]TIFR Centre for Applicable Mathematics, P.O. Box 6503, Sharadanagar, GKVK Post, Bangalore 560 065, India
[2]Indian Institute of Science Education and Research, C/o Sree Rama Engineering College (Transit Campus), Rami Reddy Nagar, Karakambadi Road, Mangalam (P.O.), Tirupati 517 507, India
[*]Corresponding author.
E-mail: aravinda@math.tifrbng.res.in; gururaja@iisertirupati.ac.in

MS received 29 July 2016; revised 21 August 2016

Abstract. We prove C^0-conjugacy rigidity of any flat cylinder among two different classes of metrics on the cylinder, namely among the class of rotationally symmetric metrics and among the class of metrics without conjugate points.

Keywords. Geodesic flow; flat cylinder; geodesic conjugacy; rotationally symmetry metrics; closed geodesics.

2000 Mathematics Subject Classification. Primary: 43A85; Secondary: 22E30.

1. Introduction

Two complete Riemannian manifolds (M, g) and (N, h) are said to be *C^r-geodesically conjugate* if there exists a C^r diffeomorphism $\mathcal{F} : UM \to UN$ between the unit tangent bundles UM and UN of M and N such that $\mathcal{F} \circ g^t_M = g^t_N \circ \mathcal{F}$ for all $t \in \mathbb{R}$, where g^t_M and g^t_N denote the geodesic flows of M and N, respectively. A C^0-geodesic conjugacy is one in which $\mathcal{F}$ is a homeomorphism. A complete Riemannian manifold (M, g) is called *C^r-conjugacy rigid* if whenever (M, g) is C^r geodesically conjugate to a complete Riemannian manifold (N, h), then (M, g) must be isometric to (N, h).

It is of interest to know which manifolds are conjugacy rigid. The case of closed surfaces is well understood. It was proved in [3, 4] and, independently, in [7] that any closed surface of nonpositive curvature is C^0-conjugacy rigid. In particular, closed hyperbolic space forms in dimension 2 are C^0-conjugacy rigid. The only spherical space forms in dimension 2 are the round sphere and the real projective plane. Of these, round sphere is not C^∞-conjugacy rigid since its geodesic flow is conjugate to that of any Zoll surface (see Sec. 4F of [2]) whereas the real projective plane is C^0-conjugacy rigid (see Appendix D of [2]).

Whereas, to the best of our knowledge, the case of complete noncompact surfaces does not seem to have been discussed much in the literature. Motivated by the conjugacy rigidity of flat tori, we address here the conjugacy rigidity question flat cylinders and prove the following two theorems.

Theorem 1. *Any flat cylinder is C^0-conjugacy rigid within the class of rotationally symmetric metrics on the cylinder.*

We give a proof of this theorem in §2. We would like to point out here that there *do exist* rotationally symmetric Zoll metrics on the sphere (see [2]). Therefore Theorem 1 supports our belief that a flat cylinder may be conjugacy rigid among all smooth metrics on the cylinder.

Theorem 2. *Suppose that the complete Riemannian surface $\mathcal{S}$ is C^0-geodesically conjugate to a flat cylinder. Assume further that $\mathcal{S}$ has no conjugate points. Then $\mathcal{S}$ must be isometric to the flat cylinder.*

Our proof of Theorem 2 is based on the following result of Bangert and Emmerich (Theorem 2′ of [1]):

Let g be a complete Riemannian metric without conjugate points on $\mathbb{R} \times S^1$, and let $C = (\mathbb{R} \times S^1, g)$ denote the Riemannian Cylinder. If both the ends of C 'open out less than linearly', then g is flat. Here, an end $\mathcal{E}$ is said to 'open out less than linearly' if there exists a sequence $\{p_n\}$ in C converging to $\mathcal{E}$ such that

$$\lim_{n\to\infty} \frac{l(p_n)}{d(p_0, p_n)} = 0,$$

where $l(p)$ denotes the length of the shortest non-contractible loop based at $p \in C$.

Theorem 2 will be proved in §3. We wish to remark here that it has been proved by Croke and Kleiner in Corollary 4.2 of [5] that if two closed manifolds are C^0-geodesically conjugate and if one of them has no conjugate points then the same must be true of the other manifold as well. We do not know if the assumption of no conjugate points in Theorem 2 is redundant.

2. Proof of Theorem 1

It is well known (Theorem 2.5.7 of [8]) that any complete flat cylinder must be isometric to an infinite right circular cylinder in the three dimensional Euclidean space. Moreover, the radius of this right circular cylinder determines the flat cylinder completely up to isometry. For future reference we note that, since the radius of the right circular cylinder is determined by the length spectrum (=circumference) of the cylinder we conclude that any two complete flat cylinders with the same length spectra must be isometric.

In what follows, we denote by $\mathcal{C}$ the right circular cylinder of radius 1. And, $\mathcal{S}$ denotes the Riemannian cylinder $(\mathbb{R} \times S^1, g)$ where g is a rotationally symmetric metric $g = dr^2 + f^2(r)d\theta^2$ with $f : \mathbb{R} \to \mathbb{R}$ is a smooth positive function.

The rest of this section is devoted to the proof of Theorem 1. That is, if the surfaces $\mathcal{C}$ and $\mathcal{S}$ as above are C^0-geodesically conjugate then they must be isometric.

Let the geodesic flow of the surfaces $\mathcal{C}$ and $\mathcal{S}$ be conjugate via a homeomorphism $\mathcal{F} : U\mathcal{C} \to U\mathcal{S}$. It is easily seen that the conjugacy $\mathcal{F}$ takes a closed geodesic in $\mathcal{C}$ onto a closed geodesic in $\mathcal{S}$ of same length and conversely. Since all closed geodesics in $\mathcal{C}$ are of length 2π, the same must be true of any closed geodesic in $\mathcal{S}$. Note that the closed geodesics in $\mathcal{C}$ are all parallel, i.e., of the form r = constant. Also, it follows from the geodesic equation that a parallel $r = r_0$ of $\mathcal{S}$ is a closed geodesic if and only if $f'(r_0) = 0$.

PROPOSITION 2.1

Suppose all the closed geodesics in $\mathcal{S}$ are parallels. Then $\mathcal{S}$ must be isometric to $\mathcal{C}$.

Proof. It suffices to show that f must be a constant function. Since the length of any closed geodesic equals 2π, f will then be a constant function equal to 1.

By hypothesis, the set $\{r \in \mathbb{R} \mid f'(r) = 0\}$ is non-empty. Consider the set $\mathcal{A} = \{r \in \mathbb{R} \mid f'(r) \neq 0\}$. Clearly $\mathcal{A}$ is open and is a proper subset of $\mathbb{R}$. We show that $\mathcal{A}$ must be closed as well which implies that it must be empty.

Suppose $r_0 \in \mathbb{R}$ be such that $f'(r_0) = 0$. Then the parallel $r = r_0$ is a geodesic. Let v_0 be the unit tangent vector to this geodesic at the point $(r_0, 0)$. Set $u_0 = \mathcal{F}^{-1}(v_0)$ and let (s_0, θ_0) denote the base point of u_0 in $\mathcal{C}$. Now parametrize an open segment of the meridian $\theta = \theta_0$ through (s_0, θ_0) in $\mathcal{C}$ as $\alpha(s) = (s, \theta_0)$ and consider the curve $\overline{\alpha}(s) = (\alpha(s), \partial_{\theta_{\alpha(s)}})$ in the unit tangent bundle $U\mathcal{C}$. Observe that the image of $\overline{\alpha}$ under $\mathcal{F}$ gives a curve in $U\mathcal{S}$ such that the geodesic through each vector $\mathcal{F}(\overline{\alpha}(s))$ is closed by construction, and hence, is a parallel by assumption. Also note that for $s \neq s'$, the geodesics through $\mathcal{F}(\overline{\alpha}(s))$ and $\mathcal{F}(\overline{\alpha}(s'))$ are different. Therefore exactly one of the two curve segments $\mathcal{F}(\overline{\alpha}(s))$ $(s > s_0)$ or $\mathcal{F}(\overline{\alpha}(s))$ $(s < s_0)$ will have all base points which lie above the parallel $r = r_0$ and the other will have all its base points below the parallel $r = r_0$. Moreover, these base points must lie on different parallels. By continuity of $\mathcal{F} \circ \overline{\alpha}$ we conclude that the above parallel geodesics fill out a neighborhood of $(r_0, 0)$ in $\mathcal{S}$. We conclude that $f'(r) = 0$ for all r close to r_0. This proves that $\mathbb{R} \setminus \mathcal{A}$ must be open. □

We call a closed geodesic *slant* if it is not a parallel geodesic. We have the following proposition.

PROPOSITION 2.2

Let γ be a slant geodesic in $\mathcal{S}$ which is contained between and tangent to the parallels $r = a$ and $r = b$, $a < b$. Then

(*i*) $f(r) \geq f(a) = f(b)$ *for* $r \in [a, b]$;
(*ii*) f *admits a unique critical point on* $[a, b]$ (*which corresponds to max of* f *on* $[a, b]$) *at some* $r_0 \in (a, b)$ *and* $f(r_0) = 1$;
(*iii*) $f' > 0$ *on* $[a, r_0)$ *and* $f' < 0$ *on* $(r_0, b]$.

Proof. Suppose that $\gamma(t) = (r(t), \theta(t))$. Let $\phi(\gamma'(t))$ denote the angle that the tangent vector $\gamma'(t)$ makes with the parallel. Then Clairaut's relation asserts that $f(r(t)) \cos(\phi(\gamma'(t)) = C_\gamma$, where C_γ is a constant that depends on the geodesic. By reparametrizing γ if necessary we may assume that C_γ is positive.

(i) When $r(t) = a$ or b, $\phi(\gamma'(t)) = 0$ so that by Clairaut relation $f(a) = C_\gamma = f(b)$. Again from Clairaut's relation it is clear that $f(r(t)) \geq C_\gamma$ for all t which implies that $f(r) \geq C_\gamma$ for $r \in [a, b]$.
(ii) Since the parallels $r = a$ and $r = b$ are not geodesics $f'(a) \neq 0$ and $f'(b) \neq 0$. By part (i), we conclude that $f'(a) > 0$ and $f'(b) < 0$. Thus the maximum of f on $[a, b]$ must be attained at an interior point $r_0 \in (a, b)$. Since $f'(r_0) = 0$, the parallel $r = r_0$ is a closed geodesic of length 2π so that $f(r_0) = 1$.

If possible, let $r_1 \in (a, b)$ be another critical point of f. For the sake of convenience, assume that $r_0 < r_1$. Then the parallel $r = r_1$ is a closed geodesic of length 2π. Thus $f(r_1) = 1$. Therefore any critical point of f corresponds to the maximum. We claim that $f(r) = 1$ for all $r \in [r_0, r_1]$. Otherwise the minimum of f on $[r_0, r_1]$ (which must be less than 1) must be attained at an interior point, say, $r' \in (r_0, r_1)$. Since r' is a critical point of f we conclude by the above argument that $f(r') = 1$, a contradiction. Hence the part of $\mathcal{S}$ corresponding to the interval $[r_0, r_1]$ must be a (part of flat) cylinder.

Now let $A \in [a, b]$ be the smallest critical point of f and B the largest. By the above argument, $\mathcal{S}$ restricted to $[A, B]$ must be a flat cylinder. Observe that f increases strictly in $[a, A]$ and decreases strictly in $[B, b]$.

We will call any point $\gamma(t)$ on the geodesic γ for which $r(t) = b$ as a *highest point* of γ. It may happen that the slant geodesic γ may have more than one highest point. Of course, we can define highest point of any slant geodesic in a similar way. *Lowest point* of any slant geodesic will also be defined similarly.

Now using a method similar to that employed in the proof of Proposition 2.1, we can construct slant geodesics starting arbitrarily near $(B, 0)$ and above the parallel $r = B$ with initial velocity arbitrarily close to $\partial_{\theta_{(B,0)}}$. Observe that we do have such geodesics starting *above* the parallel $r = B$ because each vector $\mathcal{F}(\bar{\alpha}(s))$ gives rise to a *distinct* closed geodesic of $\mathcal{S}$. Thus, in the notation of Proposition 2.1, one of the curve segments $\mathcal{F}(\bar{\alpha}(s))$ $(s > s_0)$ or $\mathcal{F}(\bar{\alpha}(s))$ $(s < s_0)$ must have base points which lie above the parallel $r = B$. We now claim that all these geodesics must pass through the entire flat cylindrical part and go beyond the parallel $r = A$. For if any such slant geodesic γ_1 lies entirely above the parallel $r = B$, then Clairaut's relation gives $f(r(\gamma_1(t))) \geq f(r(\gamma_1(t_0)))$, where $\gamma_1(t_0)$ is a lowest point of γ_1. But this is impossible because f is strictly decreasing on the interval $[B, b]$. Using (i) we see that any such geodesic must pass through the cylindrical part and go beyond the parallel $r = A$. Moreover, the tangent vectors at the highest points of these geodesics are arbitrarily close to the tangent vector to the parallel geodesic $r = B$ at $(B, 0)$ because all are parallel vectors whose base points are arbitrarily close to $(B, 0)$ by construction. These slant geodesics are nothing but helices as long as they remain within the flat cylinderical part. However it can be easily checked that any helix (in any right circular cylinder) which makes a sufficiently small angle with the parallels (for any helix this angle is independent of parallels) must have length bigger than 2π. Therefore if a slant geodesic enters the cylinderical part at a small enough angle then its total length (which is bigger than the length of its helical part) must be greater than 2π, which is clearly impossible. Hence we can not have any cylindrical part in the first place. Therefore there exists exactly one critical point of f in $[a, b]$. This proves (ii).

(iii) This follows from combining parts (i) and (ii) above. □

The parallel geodesic corresponding to the unique critical point of f in $[a, b]$ will be called the *equator* of the slant geodesic γ.

Lemma 2.3. *Let v_n be a sequence in $U\mathcal{S}$ that converges to v. Assume that each geodesic γ_n of $\mathcal{S}$ defined by $\gamma_n(0) = \pi(v_n)$ and $\gamma'(0) = v_n$ is closed. Then the geodesic γ with $\gamma(0) = \pi(v)$ and $\gamma'(0) = v$ must also be closed.*

Proof. Since v_n gives a closed geodesic of $\mathcal{S}$, the vector $\mathcal{F}^{-1}(v_n)$ must be a parallel vector of $\mathcal{C}$ (i.e., a vector tangent to some parallel of $\mathcal{C}$). The sequence of vectors $\mathcal{F}^{-1}(v_n)$ in $U\mathcal{C}$

therefore consists only of parallel vectors that converges to the vector $\mathcal{F}^{-1}(v)$. However this can happen only if the vector $\mathcal{F}^{-1}(v)$ is itself parallel. Thus it gives rise to a closed geodesic of $\mathcal{C}$ whose image under $\mathcal{F}$ is a closed geodesic of $\mathcal{S}$ which has v as tangent vector at some point on it. □

Given two slant geodesics γ_1 and γ_2 with the same equator, we say γ_1 is *slanter* than γ_2 if the parallel containing the highest point(s) of γ_1 lies above the parallel containing the highest point(s) of γ_2.

Lemma 2.4. *Given a slant geodesic γ of $\mathcal{S}$, we can find another slant geodesic which is slanter than γ.*

Proof. Let v denote the tangent vector at the highest point of γ. Note that v must be a parallel vector. Let $u = \mathcal{F}^{-1}(v)$. Choose a neighborhood V of v in $U\mathcal{S}$ and let $U = \mathcal{F}^{-1}(V)$ be the corresponding neighborhood of u in $U\mathcal{C}$. Let $\mathcal{V} = \{w \in V : r(\pi(w)) \geq r(\pi(v))\}$. Observe that $\mathcal{V}$ is a subset of $U\mathcal{S}$ which has nonempty interior and contain v as a boundary point. Therefore the set $\mathcal{U} = \mathcal{F}^{-1}(\mathcal{V})$ is a subset of $U\mathcal{C}$ with nonempty interior which contains u as a boundary point. We can choose an *interior* parallel vector u_1 of $\mathcal{U}$ which is arbitrarily close to u. Then the vector $v_1 = \mathcal{F}(u_1)$ gives a closed geodesic in $\mathcal{S}$ and, since the base point of v_1 lies above the highest point of γ, this closed geodesic must be a slant geodesic which is slanter than γ. Observe that the constructed slant geodesic must have the same equator as γ because, with notation as in Proposition 2.2, there are no critical points of f in a neighborhood of the interval $[a, b]$ other than r_0. □

PROPOSITION 2.5

$\mathcal{S}$ does not admit slant geodesics.

Proof. The proof is by contradiction. We fix a slant geodesic in $\mathcal{S}$ and consider the family of all slant geodesics of $\mathcal{S}$ which have the same equator. First observe that at the highest point of a slant geodesic the velocity vector must be a parallel vector. Now, there are two possibilities: Either the highest point of these slant geodesics with the same equator (which can always be assumed to lie on the meridian $\theta = 0$ by rotation) converges to an end $\mathcal{E}_1$ of $\mathcal{S}$ or, by the completeness of $\mathcal{S}$, it must converge to some point $p \in \mathcal{S}$. In the latter case, the velocity vectors at the highest points of these geodesics converge to some vector v. By Lemma 2.3, we get a slant geodesic with v as a tangent vector. But then Lemma 2.4 says that we can construct a slant geodesic with the same equator whose highest point lies above p, which is impossible. Thus the highest points must converge to the end $\mathcal{E}_1$. In this case the corresponding lowest points must converge to the other end $\mathcal{E}_2$ of $\mathcal{S}$. Therefore, if γ is a slant geodesic with highest point $\gamma(t_1)$ and lowest point $\gamma(t_0)$ then

$$l(\gamma) \geq \int_{t_0}^{t_1} |\gamma'(t)| \mathrm{d}t \geq \int_{t_0}^{t_1} r'(t) \mathrm{d}t = r(t_1) - r(t_0) > 2\pi,$$

if γ is sufficiently slanter. This, however, contradicts the fact that the length of any slant geodesic is 2π. □

Theorem 1 now follows by combining Proposition 2.1 and Proposition 2.5.

3. Proof of Theorem 2

In this section, we give a proof of Theorem 2. Throughout this section, we assume that $\mathcal{C}$ is a flat cylinder and $\mathcal{S}$ is a complete Riemannian surface without conjugate points. Moreover we assume that $\mathcal{F} : U\mathcal{C} \to U\mathcal{S}$ is a C^0-geodesic conjugacy.

We begin by proving that $\mathcal{S}$ must be diffeomorphic to a cylinder.

Claim 1. $\pi_1(\mathcal{S}) = \mathbb{Z}$.

Consider the canonical projection $\pi : U\mathcal{S} \to \mathcal{S}$. The induced homomorphism $\pi_* : \pi_1(U\mathcal{S}, (p_0, v_0)) \to \pi_1(S, p_0)$ is surjective so that

$$\pi_1(S, p_0) = \pi_1(U\mathcal{S}, (p_0, v_0))/\mathrm{Ker}\, \pi_*.$$

Moreover $\pi_1(U\mathcal{S}, (p_0, v_0)) = \mathbb{Z} \oplus \mathbb{Z}$ since $U\mathcal{S}$ is homeomorphic to $U\mathcal{C} = S^1 \times \mathbb{R}^1 \times S^1$ by assumption. We may assume that the second factor comes from the fibre. Also $\mathrm{Ker}\, \pi_*$ is generated by the fibre $\pi^{-1}(p_0)$ and hence is isomorphic to $\mathbb{Z}$: clearly any homotopy class of loops in the fibre projects onto the point curve p_0 and hence is contained in $\mathrm{Ker}\, \pi_*$. Conversely, if a loop $(\gamma(x), v(\gamma(x)))$ in $\pi_1(U\mathcal{S}, (p_0, v_0))$ projects onto the null homotopic loop γ in $\mathcal{S}$ then take a homotopy $F(x, t)$ (on the surface $\mathcal{S}$) of γ with the point curve p_0. For each $x \in [0, 1]$ parallel translate the unit vector $v(F(x, 0))$ along the path $t \to F(x, t)$ (which can be assumed to be piecewise smooth) to p_0. This shows that the original loop is homotopic to a loop based at p_0 in the fibre $\pi^{-1}(p_0)$. Therefore $\pi_1(\mathcal{S}) = \mathbb{Z}$.

Claim 2. $\mathcal{S}$ is diffeomorphic to $\mathcal{C}$.

By Claim 1, we know that $\pi_1(\mathcal{S}) = \mathbb{Z}$. Assume, for the moment, that $\mathcal{S}$ is orientable. Put a complex structure on $\mathcal{S}$. It is a well known fact (see p. 216, p. 223 of [6]) that a Riemann surface with fundamental group $\mathbb{Z}$ must be diffeomorphic to the cylinder.

If $\mathcal{S}$ is nonorientable with fundamental group $\mathbb{Z}$, let $\hat{\mathcal{S}}$ denote the orientable double cover of $\mathcal{S}$. Since the image of the fundamental group of $\hat{\mathcal{S}}$ under the induced homomorphism associated to the covering map is a index 2 subgroup, the fundamental group of $\hat{\mathcal{S}}$ is infinite cyclic as well and hence, by the above, it must be diffeomorphic to a cylinder. Thus $\mathcal{S}$ must be diffeomorphic to the Mobius band. Since the unit tangent bundles of the cylinder and the Mobius band are not homeomorphic we have a contradiction. This proves Claim 2.

We now proceed to show that the ends of the cylinder $\mathcal{S}$ open out less than linearly. In view of the result of Bangert and Emmerich quoted in Section 1 (Theorem 2$'$ of [1]), this will prove that $\mathcal{S}$ must be flat.

Let $\bar{\alpha}(s)$ be the curve in the unit tangent bundle $U\mathcal{C}$ as in the proof of Proposition 2.1 of §2. As $s \to \pm\infty$, this curve approaches the two different ends of $U\mathcal{C}$. Since $\mathcal{F}$ is a homeomorphism, the image of this curve under $\mathcal{F}$ must satisfy the same property on $U\mathcal{S}$. That is, we obtain sequences of points $\{p_n\}$ and $\{q_n\}$ on $\mathcal{S}$ tending to the two different ends of $\mathcal{S}$ such that there exist closed geodesics of length 2π passing through the points p_n, and closed geodesics of length 2π passing through the points q_n. Thus we see that $l(p_n) \le 2\pi$ and $l(q_n) \le 2\pi$, and therefore,

$$\lim_{n\to\infty} \frac{l(p_n)}{d(p_0, p_n)} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{l(q_n)}{d(p_0, q_n)} = 0,$$

where $p_0 \in \mathcal{S}$ is a fixed point. This proves that the cylinder $\mathcal{S}$ is flat. Now, any two flat cylinders with the same length spectrum must be isometric. This completes the proof of the theorem.

Acknowledgement

The authors thank the anonymous referee for comments that have helped enhance clarity and readability of the paper.

References

[1] Bangert V and Emmerich P, Area growth and the rigidity of surfaces without conjugate points, *J. Diff. Geom.* **94** (2013) 367–385

[2] Besse A, Manifolds all of whose geodesics are closed, Ergebnisse Grenzgeb Math., No. 93 (1978) (Berlin: Springer)

[3] Croke C, Rigidity for surfaces of nonpositive curvature, *Comm. Math. Helv.* **65** (1990) 150–169

[4] Croke C, Fathi A and Feldman J, The marked length-spectrum of a surface of nonpositive curvature, *Topology* **31** (1992) 847–855

[5] Croke C and Kleiner B, Conjugacy and rigidity for manifolds with a parallel vector field, *J. Diff. Geom.* **39** (1994) 659–680

[6] Napier T and Ramachandran M, An introduction to Riemann surfaces (2011) (Birkhauser)

[7] Otal J-P, Le spectre marqué des longueurs des surfaces à courbure négative, *Ann. Math.* **131** (1990) 151–162

[8] Wolf J, Spaces of constant curvature, sixth edition AMS (2011) (Chelsea Publishing)

Communicating Editor: Mj Mahan