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Analysing the Wu metric on a class of eggs in $\mathbb{C}^n$ – II

G P BALAKUMAR[1,*] and PRACHI MAHAJAN[2]

[1]Indian Statistical Institute, Chennai 600 113, India
[2]Department of Mathematics, Indian Institute of Technology – Bombay, Mumbai 400 076, India
[*]Corresponding author.
E-mail: gpbalakumar@isichennai.res.in; prachi.mjn@iitb.ac.in

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Abstract. We study the Wu metric for the non-convex domains of the form $E_{2m} = \{z \in \mathbb{C}^n : |z_1|^{2m} + |z_2|^2 + \cdots + |z_{n-1}|^2 + |z_n|^2 < 1\}$, where $0 < m < 1/2$. We give explicit expressions for the Kobayashi metric and the Wu metric on such pseudo-eggs E_{2m}. We verify that the Wu metric is a continuous Hermitian metric on E_{2m}, real analytic everywhere except along the complex hypersurface $Z = \{(0, z_2, \ldots, z_n) \in E_{2m}\}$. We also show that the holomorphic curvature of the Wu metric for this non-compact family of pseudoconvex domains is bounded above in the sense of currents by a negative constant independent of m. This verifies a conjecture of S. Kobayashi and H. Wu for such E_{2m}.

Keywords. Wu metric; Kobayashi metric; negative holomorphic curvature.

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1. Introduction

We continue our study of the Wu metric from [1] by focussing on the following class of non-convex pseudo-egg domains

$$E_{2m} = \{z \in \mathbb{C}^n : |z_1|^{2m} + |z_2|^2 + \ldots + |z_{n-1}|^2 + |z_n|^2 < 1\}, \tag{1.1}$$

for $0 < m < 1/2$. Such a pseudo-egg cannot be biholomorphic to any convex domain. This follows by comparing the Kobayashi indicatrices, which must be linearly equivalent if the domains are biholomorphic to each other. Linear maps preserve convexity and Kobayashi indicatrix of a convex domain is convex. On the other hand, the indicatrix at the origin for a pseudo-egg (being a copy of E_{2m}) is non-convex. This also means that the Kobayashi metric on E_{2m} fails to satisfy the triangle inequality on the tangent space at the origin. While the Wu metric is indeed a norm (on each tangent space), it is not clear if the Wu metric enjoys better regularity than the Kobayashi metric. In general, the Wu metric may fail to be upper semicontinuous [4] notwithstanding the fact that the Kobayashi metric is always upper semicontinuous. In fact, in the case of C^2-smooth convex eggs, i.e. when $m > 1$, the Wu metric is only C^1-smooth [1] while the Kobayashi metric is C^2-smooth. For pseudo-eggs as in (1.1), first note that ∂E_{2m} is not even C^1-smooth.

Specifically, the non-smooth points of the boundary are given by $\{z \in \partial E_{2m} : z_1 = 0\}$, which is the boundary of the set $Z = \{(0, z_2, \dots, z_n) \in E_{2m}\}$; the remaining piece $\partial E_{2m} \setminus Z$ is a smooth strongly pseudoconvex hypersurface. The form and regularity of the Kobayashi metric for non-convex ellipsoids in dimensions any bigger than two, have not really been dealt with. We do this first for the pseudo-eggs (1.1) and then derive the following result about their Wu metric.

Theorem 1.1. *For $0 < m < 1/2$, the Wu metric on E_{2m} is a continuous Hermitian metric which is real analytic on E_{2m} except along the thin set Z. It is nowhere-Kähler. Furthermore, its holomorphic curvature (the term 'holomorphic curvature' stands precisely for the holomorphic sectional curvature and is said to be strongly negative, if it is bounded above by a negative constant) is non-constant and is bounded above by a negative constant independent of m, in the sense of currents.*

This extends the work of Cheung and Kim [2, 3] and verifies the following conjecture of Kobayashi [5, 8] for such pseudo-eggs.

Conjecture K–W: On every Kobayashi complete hyperbolic complex manifold, there exists a C^k-smooth (for some $k \geq 0$) complete Hermitian metric with its holomorphic curvature bounded above by a negative constant in the sense of currents.

2. The Wu metric on E_{2m} for $0 < m < 1/2$

In order to analyse the Wu metric on E_{2m}, it is natural to first compute the Kobayashi metric on E_{2m}. Note that E_{2m} is a *balanced* pseudoconvex domain in $\mathbb{C}^n$ and hence the Kobayashi metric at the origin, $K(0, v) = q_{E_{2m}}(v)$, where $q_{E_{2m}}$ denotes the *Minkowski functional* of E_{2m}. Let $\langle \cdot, \cdot \rangle$ denote the standard Hermitian inner product in $\mathbb{C}^{n-1}$, and write $z \in \mathbb{C}^n$ as $z = (z_1, \hat{z})$, where $\hat{z} = (z_2, \dots, z_n)$. For $(p_1, \dots, p_n) \in E_{2m}$ with $p_1 \neq 0$ and Ψ an automorphism of $\mathbb{B}^{n-1}$ that takes $\hat{p}$ to the origin,

$$\Phi(z_1, \dots, z_n) = \left(\frac{|p_1|}{p_1} \frac{(1 - |\hat{p}|^2)^{1/2m}}{(1 - \langle \hat{z}, \hat{p} \rangle)^{1/m}} z_1, \Psi(\hat{z}) \right) \tag{2.1}$$

is an automorphism of E_{2m}. Furthermore,

$$\Phi\left((p_1, \dots, p_n)\right) = \left(\frac{|p_1|}{(1 - |p_1|^2)^{1/2m}}, \hat{0} \right).$$

It follows that the study of the Kobayashi and Wu metrics can be focussed on points $(p, \hat{0})$ for $0 \leq p < 1$, as in the case of convex eggs, in view of automorphisms of E_{2m} of the form (2.1). Note that E_{2m} is the orbit of $S \cup \{0\}$ under the real analytic action of its automorphism group, where

$$S = \{p = (p_1, \hat{0}) \in \mathbb{C}^n : 0 < p_1 < 1\}.$$

To state our result on the explicit expression of the the Kobayashi metric for E_{2m} at the point $(p, \hat{0}) \in S$, we introduce some notations. Let

$$w = \frac{p^2 \left(|v_2|^2 + \cdots + |v_n|^2\right)}{m^2 |v_1|^2},$$

$$t = \frac{2m^2 w}{1 + 2m(m-1)w + (1 + 4m(m-1)w)^{1/2}}$$

and α be the unique solution in the interval (0, 1) of the equation

$$\alpha^{2m} - t\alpha^{2m-2} - (1-t)p^{2m} = 0.$$

Theorem 2.1. *For $0 < m < 1/2$, the Kobayashi metric on E_{2m} at the point $(p, \hat{0}) \in S$ is given by*

$$4K((p,\hat{0}),v) = \begin{cases} K_1((p,\hat{0}),v) \ \textit{for } w \le 1, \\ K_2((p,\hat{0}),v) \ \textit{for } w \ge \frac{1}{4m(1-m)}, \\ \min\{K_1((p,\hat{0}),v), K_2((p,\hat{0}),v)\} \ \textit{for } 1 < w < \frac{1}{4m(1-m)}, \end{cases}$$

where

$$K_1((p,\hat{0}),v) = \frac{m\alpha(1-t)|v_1|}{p(1-\alpha^2)\,(m(1-t)+t)}$$

and

$$K_2((p,\hat{0}),v) = \left(\frac{m^2 p^{2m-2}|v_1|^2}{(1-p^{2m})^2} + \frac{|v_2|^2}{1-p^{2m}} + \cdots + \frac{|v_n|^2}{1-p^{2m}}\right)^{1/2}.$$

Moreover, there is a $1 < w_0 < 1/4m(1-m)$ such that

$$\begin{aligned} 3K((p,\hat{0}),v) &= K_1((p,\hat{0}),v) \qquad \textit{for } \ w \le w_0, \\ K((p,\hat{0}),v) &= K_2((p,\hat{0}),v) \qquad \textit{for } \ w \ge w_0. \end{aligned}$$

Furthermore, $w_0 = \frac{t_0}{(m+(1-m)t_0)^2}$, where $t_0 = \frac{x_0^{2m}-p^{2m}}{x_0^{2m-2}-p^{2m}}$ and x_0 is the solution of the equation

$$\begin{aligned} &-(1-m)^2 x^{4m} + (-1-2m+2m^2+p^{2m})x^{4m-2} - m^2 x^{4m-4} \\ &\quad + (1-(2m-1)p^{2m})x^{2m} + (1+(2m-1)p^{2m})x^{2m-2} - p^{2m} = 0. \end{aligned}$$

The above result extends the computation done for such pseudo-egg domains in $\mathbb{C}^2$ (Theorem 4 of [7]); proof relies on the description of complex geodesics with respect to the Kobayashi metric due to Pflug and Zwonek (see Proposition 2 of [7]). Proof of Theorem 2.1 follows using the techniques of [7].

PROPOSITION 2.2

In terms of the Euclidean coordinates on the tangent bundle $E_{2m} \times \mathbb{C}^n$, for every $(p,\hat{0}) \in S$, the unit sphere of the Wu metric in $T_{(p,\hat{0})}E_{2m}$ is given by

$$r_1|v_1|^2 + r_2(|v_2|^2 + \cdots + |v_n|^2) = 1,$$

where r_1 and r_2 are positive real-valued continuous functions of p.

It turns out that, to determine the best fitting ellipsoid at $(p, \hat{0})$, it suffices to find $r_1, r_2 > 0$ such that the set

$$\{(v_1, \ldots, v_n) : v_1 \geq 0, \ldots, v_n \geq 0, r_1 v_1^2 + r_2(v_2^2 + \cdots + v_n^2) = 1\}$$

encloses the smallest volume with the coordinate axes and contains the set

$$\{(v_1, \ldots, v_n) : v_1 \geq 0, \ldots, v_n \geq 0, K((p, \hat{0}), v) \leq 1\}. \tag{2.2}$$

Recall from Theorem 2.1 that there is a $1 < w_0 < \frac{1}{4m(1-m)}$ such that

$$4K\left((p, \hat{0}), v\right) = \begin{cases} K_1((p, \hat{0}), v) & \text{for } w \leq w_0, \\ K_2((p, \hat{0}), v) & \text{for } w \geq w_0. \end{cases}$$

It follows that the boundary of the set described by equation (2.2) is the union of two curves, determined by whether $w < w_0$ or $w \geq w_0$. Henceforth, the portion of the curve $K^2((p, \hat{0}), v) = 1$ for $w \geq w_0$ and $w < w_0$ will be referred to as the lower K-curve and the upper K-curve respectively. In terms of the square transformation

$$x = v_2^2 + \cdots + v_n^2$$

and

$$y = v_1^2,$$

the lower K-curve is described by the straight line segment

$$m^2 p^{2m-2}(1 - p^{2m})^{-2} y + (1 - p^{2m})^{-1} x = 1.$$

The upper K-curve is given by the parametric equations, as follows:

$$x(\alpha) = (v_2(\alpha))^2 + \ldots + (v_n(\alpha))^2 = \frac{\alpha^{4m-2} + p^{4m} - p^{2m}\alpha^{2m-2} - p^{2m}\alpha^{2m}}{\alpha^{4m-2}}$$

and

$$y(\alpha) = (v_1(\alpha))^2 = \left(\frac{p(m\alpha^{2m-2} - (m-1)\alpha^{2m} - p^{2m})}{m\alpha^{2m-1}}\right)^2,$$

with α varying between p and x_0, where x_0 is as in Theorem 2.1. The above two equations can be written as a single equation $y(\alpha) = (f(\sqrt{x(\alpha)}))^2$. The parametric form for the upper K-curve as above, shows that f is real analytic. It is straightforward to verify that f is strictly square convex for $p < \alpha < x_0$ (see [2]). This fact together with Proposition 2.2 renders the following result.

PROPOSITION 2.3

The Wu metric on E_{2m} for $m < 1/2$ at the point $(p, \hat{0})$, $0 < p < 1$ is given by

$$h_{E_{2m}}(p, \hat{0}) = \frac{1}{(1-p^2)^2} dz_1 \otimes d\bar{z}_1 + \frac{1}{(1-p^{2m})} dz_2 \otimes d\bar{z}_2 + \cdots + \frac{1}{(1-p^{2m})} dz_n \otimes d\bar{z}_n.$$

Further, using the invariance of the Wu metric under automorphisms of E_{2m} (as described in (2.1)), we arrive at the following result.

Theorem 2.4. *For $0 < m < 1/2$, the Wu metric on E_{2m} at the point $(z_1, \ldots, z_n)$ is given by*

$$\sum_{i,j=1}^{n} h_{i\bar{j}}(z_1, \ldots, z_n)\, dz_i \otimes d\bar{z}_j,$$

where

$$h_{1\bar{1}}(z_1, \ldots, z_n) = \frac{(1-|\hat{z}|^2)^{1/m}}{((1-|\hat{z}|^2)^{1/m} - |z_1|^2)^2},$$

$$h_{1\bar{j}}(z_1, \ldots, z_n) = \frac{(1-|\hat{z}|^2)^{-1+1/m}\bar{z}_1 z_j}{m((1-|\hat{z}|^2)^{1/m} - |z_1|^2)^2} \quad \text{for } 2 \le j \le n,$$

$$h_{i\bar{1}}(z_1, \ldots, z_n) = \overline{h_{1\bar{i}}}(z_1, z_2, \ldots, z_n) \quad \text{for } 2 \le i \le n,$$

$$h_{j\bar{j}}(z_1, \ldots, z_n) = \left(\frac{(1-|\hat{z}|^2)^{-2+1/m}|z_1|^2|z_j|^2}{m^2((1-|\hat{z}|^2)^{1/m} - |z_1|^2)^2} + \frac{1-|\hat{z}|^2+|z_j|^2}{(1-|\hat{z}|^2)(1-|\hat{z}|^2-|z_1|^{2m})} \right) \quad \text{for } 2 \le j \le n,$$

and

$$h_{i\bar{j}}(z_1, \ldots, z_n) = \left(\frac{(1-|\hat{z}|^2)^{-2+1/m}|z_1|^2\bar{z}_i z_j}{m^2((1-|\hat{z}|^2)^{1/m} - |z_1|^2)^2} + \frac{\bar{z}_i z_j}{(1-|\hat{z}|^2)(1-|\hat{z}|^2-|z_1|^{2m})} \right) \quad \text{for } 2 \le i, j \le n \text{ and } i \ne j.$$

It follows that the Wu metric is real analytic on the set $\{(z_1, \ldots, z_n) \in E_{2m} : z_1 \ne 0\}$. Moreover, the Wu metric is continuous at points $(0, z_2, \ldots, z_n)$ of E_{2m}. Indeed, the continuity and completeness of the Wu metric on E_{2m} follows from Proposition 4 of [4]. Completeness of the Wu metric here relies on Kobayashi completeness of the domains E_{2m}, which is guaranteed by [6]. Furthermore, it is straightforward to verify that

$$\frac{\partial h_{12}}{\partial z_2}(z_1, \hat{0}) \ne \frac{\partial h_{22}}{\partial z_1}(z_1, \hat{0})$$

at points of S. This shows that the Wu metric is not Kähler. Since $E_{2m} \setminus Z$ is the orbit of S under the action of the automorphism group of E_{2m}, it follows that the Wu metric is nowhere Kähler on $E_{2m} \setminus Z$. Notice that $E_{2m} \setminus Z$ is an open dense subset of E_{2m} and hence, the Wu metric is nowhere Kähler on E_{2m}, for $0 < m < 1/2$.

3. Negative holomorphic curvature

Let $G = \sum_{i,j=1}^{n} g_{i\bar{j}} dz_i \otimes d\bar{z}_j$ be a C^2-smooth Hermitian metric on a complex manifold X. Then the holomorphic curvature of G along the direction of $\xi = (\xi_1, \ldots, \xi_n) \in T_p X$ at $p \in X$ is given by

$$\frac{\sum R_{i\bar{j}k\bar{l}}(p)\xi_i\bar{\xi}_j\xi_k\bar{\xi}_l}{\sum g_{i\bar{j}}(p) g_{k\bar{l}}(p)\xi_i\bar{\xi}_j\xi_k\bar{\xi}_l},$$

where $R_{i\bar{j}k\bar{l}}$ are the components of the curvature tensor given by

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \sum_{\alpha,\beta} g^{\alpha\bar{\beta}} \frac{\partial g_{i\bar{\beta}}}{\partial z_k} \frac{\partial g_{\alpha\bar{j}}}{\partial \bar{z}_l},$$

and $(g^{\alpha\bar{\beta}})$ denotes the inverse of the matrix $(g_{\alpha\bar{\beta}})$.

In case G is only continuous (and not C^2-smooth), then the holomorphic curvature is defined in a distributional sense, as a *current* of type (1, 1) (cf. [8]) and is said to be bounded above by a negative constant c if every embedded Riemann surface S in X with $G\big|_S = g_0\, d\xi \otimes d\bar{\xi}$ satisfies

$$\Delta_\xi \log g_0 = \frac{\partial^2 \log g_0}{\partial \xi \partial \bar{\xi}} > -c g_0\, \partial\xi \wedge \partial\bar{\xi},$$

in the sense of currents.

PROPOSITION 3.1

For $0 < m < 1/2$, the holomorphic curvature of the Wu metric on E_{2m} is bounded above by $-1/2$ at every point where the Wu metric is smooth.

A direct computation using Theorem 2.4 shows that

$$R_{1\bar{1}1\bar{1}}((p,\hat{0})) = -\frac{2}{(1-p^2)^4},$$

$$R_{1\bar{1}j\bar{j}}((p,\hat{0})) = -\frac{1+p^2}{m(1-p^2)^3} + \frac{p^2(1-p^{2m})}{m^2(1-p^2)^4} \quad \text{for } 2 \le j \le n,$$

$$R_{1\bar{i}i\bar{1}}((p,\hat{0})) = R_{i\bar{1}1\bar{i}}((p,\hat{0})) = \frac{-1-p^2}{m(1-p^2)^3} + \frac{p^{2m}}{(1-p^2)^2(1-p^{2m})} \quad \text{for } 2 \le i \le n$$

$$R_{i\bar{i}1\bar{1}}((p,\hat{0})) = -\frac{m^2 p^{2m-2}}{(1-p^{2m})^3} \quad \text{for } 2 \le i \le n,$$

$$R_{i\bar{i}j\bar{j}}((p,\hat{0})) = -\frac{1}{(1-p^{2m})^2} \quad \text{for } 2 \le i, j \le n \text{ and } i \ne j,$$

$$R_{i\bar{j}j\bar{i}}((p,\hat{0})) = -\frac{p^2}{m^2(1-p^2)^2} - \frac{1}{1-p^{2m}} \quad \text{for } 2 \le i, j \le n \text{ and } i \ne j,$$

$$R_{j\bar{j}j\bar{j}}((p,\hat{0})) = -\frac{p^2}{m^2(1-p^2)^2} - \frac{1}{1-p^{2m}} - \frac{1}{(1-p^{2m})^2} \quad \text{for } 2 \le j \le n,$$

and all other curvature components vanish. Now, arguments similar to those employed in [2] using the above computations yield that

$$\sum R_{i\bar{j}k\bar{l}}(p)\xi_i\bar{\xi}_j\xi_k\bar{\xi}_l < -\frac{1}{2}\sum g_{i\bar{j}}(p) g_{k\bar{l}}(p)\xi_i\bar{\xi}_j\xi_k\bar{\xi}_l.$$

It remains to establish the uniform negativity of curvature on the thin set Z.

PROPOSITION 3.2

There is a negative constant c (independent of m), such that the holomorphic curvature of the Wu metric at points of $Z \subset E_{2m}$, is bounded above by c in the sense of currents.

To prove this proposition, one cannot rely solely on the arguments used in the case of convex eggs (i.e., those presented in Appendix B of [2]), since the Wu metric is not even C^1-smooth on Z. However, the pseudo-eggs have the following property – for $m < 1/2$, $E_{2m} \subset \mathbb{B}^n$, where $\mathbb{B}^n$ denotes the unit ball in $\mathbb{C}^n$. Hence, one can pull back the standard Poincare–Bergman metric g on $\mathbb{B}^n$ by the inclusion mapping $i : E_{2m} \to \mathbb{B}^n$, to get the metric i^*g on E_{2m} and compare it with the Wu metric $h_{E_{2m}}$. It follows that $i^*g \leq \sqrt{n} h_{E_{2m}}$. Moreover, in this case, the best fitting ellipsoid at the origin is $\mathbb{B}^n$. Furthermore, since the Kobayashi-indicatrix at the origin is (a copy of) E_{2m}, it follows that i^*g coincides with $h_{E_{2m}}$ at the origin. Now, let M be any embedded Riemann surface with complex coordinate s and passing through the origin at $s = 0$. To establish the strong negativity of the holomorphic curvature near the origin, in the sense of currents, compare the restriction $G = i^*g_{|M}$ with $H = h_{E_{2m}}|_M$. Real analyticity of G ensures that its logarithmic average on small 'discs' about $s = 0$ in M equals $\Delta \log G(s)$ at $s = 0$ – this can be verified by a Taylor expansion of G in $s, \bar{s}$. The decreasing property of holomorphic curvature together with the facts that G is a conformal metric on M of constant negative holomorphic curvature and $H \geq 1/\sqrt{n} G$, establishes the strong negativity of the holomorphic curvature current of the Wu metric in a neighbourhood of the origin as in [3]. Notice that the bound on the curvature that we get here, is a constant that depends on the dimension n. But this constant does not depend on m. As Z is the orbit of the origin under the action of $\mathrm{Aut}(E_{2m})$, this analysis carries forth to hold throughout Z.

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