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On the number of special numbers

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Abstract. For lack of a better word, a number is called *special* if it has mutually distinct exponents in its canonical prime factorizaton for all exponents. Let $V(x)$ be the number of special numbers $\leq x$. We will prove that there is a constant $c > 1$ such that $V(x) \sim \dfrac{cx}{\log x}$. We will make some remarks on determining the error term at the end. Using the explicit *abc* conjecture, we will study the existence of 23 consecutive special integers.

Keywords. Special numbers; squarefull numbers; Thue–Mahler equations; *abc* conjecture.

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1. Introduction

For lack of a better word, a natural number n is called *special* if in its prime power factorization

$$n = p_1^{\alpha_1} \dots p_k^{\alpha_k}, \tag{1}$$

we have all the non-zero α_i as distinct. This concept was introduced by Bernardo Recamán Santos (see [8]). He asked if there is a sequence of 23 consecutive special numbers. This seemingly simple question is still unsolved and it is curious that it is easy to see that there is no sequence of 24 consecutive special numbers. We will show this below. If one assumes the *abc* conjecture, then Pace Nielsen showed that there are only finitely many sequences of 23 consecutive special numbers [8]. Presumably there are none. We explore this problem using explicit forms of the *abc* conjecture.

It is well known that assuming the *abc* conjecture one can prove that some diophantine equations have only a finite number of solutions. Due to its ineffective nature, the *abc* conjecture does not give an explicit computable bound for the number of solutions or their size. On the other hand, assuming a weak effective version of the *abc* conjecture (as in [3]), we can find all solutions of some diophantine equations. Several authors have formulated explicit versions of the *abc* conjecture. Notable among these are the ones due to Baker [1] and Browkin [3] and [4]. We will use this to show:

Theorem 1.1. *Assuming the explicit abc conjecture, there are only finitely many sequences of* 23 *consecutive special numbers. If there are* 23 *consecutive special numbers* $x, x+1, \ldots, x+22$, *then*

$$x < 2e^{63727} 2^{3^{85} e^{254908} (63727)^5}.$$

The question of Recamán Santos raises the interesting question of the distribution of special numbers. Clearly every prime power is special, but are there significantly more? In this context, we show:

Theorem 1.2. *Let* $V(x)$ *be the number of special numbers* $\leq x$. *Then, there exists a constant* $c > 1$ *such that*

$$V(x) = \frac{cx}{\log x} + O\left(\frac{x}{\log^2 x}\right).$$

In fact, c is approximately 1.7.

Thus, the number of special numbers is comparable in size to the set of prime powers.

2. Preliminaries

Conjecture 2.1. Let us recall the *abc* conjecture of Oesterlé and Masser (1985). Fix $\epsilon > 0$. Then, there exists a constant $\kappa(\epsilon)$ depending only on ϵ such that

$$a + b = c, \tag{2}$$

where a, b and c are coprime positive integers. Then

$$c \leqslant \kappa(\epsilon) \left(\prod_{p|abc} p\right)^{1+\epsilon}.$$

As the referee remarks, the conjecture is inspired by the folklore conjecture that if D is a multiplicatively closed set generated by finitely many numbers (which we could take, without loss of any generality to be the first k primes), then the equation $a + b = c$ has only finitely many solutions in mutually coprime elements a, b, c of D.

For any positive integer $n > 1$, let $R = \mathrm{Rad}(n) = \prod_{p|n} p$ be the radical of n, and $\omega(n)$ be the number of distinct prime factors of n. Baker [1] made a more precise conjecture:

$$c < \frac{6}{5} N \frac{(\log N)^{\omega}}{\omega!}, \tag{3}$$

where $N = R(abc)$ and $\omega = \omega(N)$. A refined version of this conjecture due to Laishram and Shorey [9] yields the following:

Theorem 2.2. *Let a, b and c be pairwise coprime positive integers satisfying* (2) *and $R = \mathrm{Rad}(abc)$. Then, Baker's conjecture* (3) *implies*

$$c < R^{1+\frac{3}{4}}.$$

Further, under the same hypothesis (3), *for $0 < \epsilon \le \frac{3}{4}$, there exists ω_ϵ depending only on ϵ such that when $R = \mathrm{Rad}(abc) \ge R_\epsilon = \prod_{p \le p_{\omega_\epsilon}} p$, we have*

$$c < \kappa(\epsilon) R^{1+\epsilon},$$

where

$$\kappa(\epsilon) = \frac{6}{5\sqrt{2\pi \max(\omega, \omega_\epsilon)}} \le \frac{6}{5\sqrt{2\pi \omega_\epsilon}}$$

with $\omega = \omega(R)$. Here are some values of ϵ, ω_ϵ and R_ϵ:

ϵ	$\frac{3}{4}$	$\frac{7}{12}$	$\frac{6}{11}$	$\frac{1}{2}$	$\frac{34}{71}$	$\frac{5}{12}$	$\frac{1}{3}$
ω_ϵ	14	49	72	127	175	548	6460
R_ϵ	$e^{37.1101}$	$e^{204.75}$	$e^{335.71}$	$e^{679.585}$	$e^{1004.763}$	$e^{3894.57}$	e^{63727}

We will apply this theorem below with $\epsilon = 1/3$. We will also need to invoke some results on the Thue equation in order to use the explicit *abc*. These are equations of the form

$$F(x, y) = h, \tag{4}$$

where $F(x, y) \in \mathbb{Z}[X, Y]$ is an irreducible binary form of degree $n \ge 3$ and h is a non-zero rational integer. Bombieri and Schmidt [2] obtained bounds for the number of solutions to this equation. The most recent results on the bounds for the solutions of Thue equation were obtained by Bugeaud and Györy [5]. They improved the best known bounds for the solutions of Thue equations over $\mathbb{Z}$ in the following theorem.

Theorem 2.3. *All solutions x, y of equation* (4) *of degree n satisfy*

$$\max\{|x|, |y|\} < \exp\{c_4 H^{2n-2} (\log H)^{2n-1} \log h^*\},$$

where $h^* = \max(|h|, e)$, $c_4 = c_4(n) = 3^{n+9} n^{18(n+1)}$ and $H (\ge 3)$ is the maximum absolute value of the coefficients of F.

3. Consecutive special numbers

We now discuss the Recamán Santos problem. Let us first show that there is no sequence of consecutive special numbers of length 24.

Lemma 3.1. *There is no sequence of* 24 *consecutive special numbers.*

Proof. We begin with the elementary observation that each of four consecutive numbers cannot be a multiple of 2 or 3. The easy proof of this is left to the reader. Now suppose

$$n+1, n+2, \ldots, n+24$$

are all special. Four of these are divisible by 6, say $6k, 6(k+1), 6(k+2), 6(k+3)$. We claim all of $k, k+1, k+2, k+3$ are not coprime to 6. Indeed, if any one is coprime to 6, (say) m, then $6m$ is not special which gives a contradiction. So we may suppose each is divisible by either 2 or 3. But this is impossible by our elementary observation. This completes the proof. □

It seems difficult to reduce 24 to 23 in the previous lemma without using some advanced techniques. We now address this issue. We begin with the following elementary lemma.

Lemma 3.2. *Any squarefull number can be written as a^3b with b cubefree.*

Proof. Every squarefull number n can be written in the form of $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ for all p_i's are distinct primes and each exponent $\alpha_i \geq 2$, where $i = \{1, 2, \ldots, k\}$. Let p^α be one of these prime powers dividing n, where $\alpha = 3\beta + \delta \geq 2$ with $\delta = 0, 1, 2$. The result is now immediate. □

We follow the strategy of our earlier lemma. Suppose now that there are 23 consecutive special numbers. At least 11 of them are even. Among these at least five of them are divisible by 2 and not by 4. Let us write these five numbers as

$$2(m-4), 2(m-2), 2m, 2(m+2), 2(m+4),$$

where $m-4, m-2, m, m+2, m+4$ are all odd and squarefull. Now consider the *abc* equation

$$(m^2-4)+4 = m^2.$$

We apply Theorem 2.2 with $\epsilon = 1/3$:

$$m^2 \leq \frac{6}{5\sqrt{2\pi}\cdot 6460}(m^{1/2}2m^{1/2}m^{1/2})^{4/3}, \tag{5}$$

$$\leq \frac{6}{5\sqrt{2\pi}\cdot 6460}(2m^{3/2})^{4/3} \tag{6}$$

which leads to a contradiction if $\mathrm{Rad}(m(m-2)(m+2)) \geq e^{63727}$. Thus, we may assume that $\mathrm{Rad}(m(m-2)(m+2) < e^{63727}$. We now apply the theory of the Thue equation to obtain an effective bound on m. Indeed, by Lemma 3.2, we can write $m^2 = ba^3$ and $m^2 - 4 = cd^3$ with b, c cubefree. By the above, both b, c are bounded since they are cubefree and all their prime factors are less than e^{63727}. Now we have a finite number of Thue equations:

$$ba^3 - cd^3 = 4,$$

with b, c less than $e^{2(63727)}$. By Theorem 2.3, we have the following.

PROPOSITION 3.3

Any solution (a, d) of the Thue equation

$$ba^3 - cd^3 = 4$$

satisfies

$$|a|, |d| < 4^{3^{84}e^{254908}(63727)^5}.$$

Since b and c are effectively bounded by $e^{2(63727)}$, we find that m^2 is effectively bounded by

$$e^{2(63727)}4^{3^{85}e^{254908}(63727)^5}.$$

In other words, if there are 23 consecutive special numbers $x, x+1, ..., x+22$, then

$$x < 2e^{63727}2^{3^{85}e^{254908}(63727)^5}.$$

This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

Special numbers can be grouped into two disjoint groups V_0 and V_1 respectively, where in V_0, some α_i is 1 in (1) and in V_1, all $\alpha_i \geq 2$. The elements of V_1 are then squarefull numbers. In 1970, Golomb [7] proved an estimate for the number of squarefull numbers. If $V_1 = \#\{n \leq x : n \text{ squarefull for } \alpha_i \geq 2\}$, then by our earlier study of squarefull numbers we see that $V_1(x) = O(\sqrt{x})$. Thus, to find the asymptotic formula for $V(x)$, we need only study $V_0(x) = \#\{n \leq x | n = mp, (m, p) = 1, m \text{ squarefull and special}\}$.

Every number in V_0 has a unique representation as

$$n = pm,$$

where p is prime and $(m, p) = 1$ with m squarefull and special. Let y be a parameter (to be choosen later). We first consider the contribution of those squarefull $m \in V_0$ with $m > y$. Clearly, the number of such numbers is

$$\leq \sum_{\substack{m > y \\ m \text{ squarefull}}} \frac{x}{m}. \tag{7}$$

Since the number of squarefull numbers $\leq x$ is $\sim c_1\sqrt{x}$, it is not difficult to see by partial summation that

$$\sum_{\substack{m > y \\ m \text{ squarefull}}} \frac{1}{m} \ll \frac{1}{\sqrt{y}}. \tag{8}$$

This is an elementary exercise in partial summation (see [12]). Thus, the sum in (7) is $\ll \frac{x}{\sqrt{y}}$. We will choose $y = \log^4 x$ so that the contribution from $m > y$ is $O(x/\log^2 x)$. Henceforth, we assume $m < y$.

We now need to count the number of elements of V_0 which have a unique representation as $n = pm$, $(p, m) = 1$ and m special and squarefull. We consider first those n with $p < y$. Since $m < y$, the number of $n = pm$ with $p < y$ and $m < y$ is clearly $\leq y^2 = \log^8 x$ which is negligible compared to our main term. Thus we may assume $p > y$. Since now $p > y$ and $m < y$, the condition $(p, m) = 1$ is automatically satisfied.

Hence,

$$V_0(x) = \sum_{\substack{m < y \\ m \text{ sq-full and special}}} \frac{x/m}{\log(x/m)} \left(1 + O\left(\frac{1}{\log(x/m)}\right)\right)$$

by an application of the prime number theorem. Since $m < y = \log^4 x$, we see that

$$\begin{aligned} \frac{1}{\log(x/m)} &= \frac{1}{\log x}\left(1 + O\left(\frac{\log m}{\log x}\right)\right) \\ &= \frac{1}{\log x}\left(1 + O\left(\frac{\log m}{\log x}\right)\right) + O(y^2). \end{aligned}$$

Hence

$$V_0(x) = \frac{x}{\log x} \sum_{\substack{m < y \\ m \text{ sq-full and special}}} \frac{1}{m} + O\left(\frac{x}{\log^2 x}\right),$$

since

$$\sum_{m \text{ sqfull}} \frac{\log m}{m}$$

is convergent as well as the sum

$$\sum_{\substack{m = 1 \\ m \text{ sq-full and special}}}^{\infty} \frac{1}{m}.$$

We also include $m = 1$ in this sum. In fact,

$$\sum_{\substack{m < y \\ \text{m sq-full and special}}} \frac{1}{m} = \sum_{m=1}^{\infty}{}' \frac{1}{m} - \sum_{m>y} \frac{1}{m},$$

where the dash on the summation means that either $m = 1$ or m is squarefull, and by our remarks in (8), we see this is equal to

$$\sum_{m=1}^{\infty}{}' \frac{1}{m} + O\left(\frac{1}{\sqrt{y}}\right). \tag{9}$$

We remark that the error term that emerges from the proof above is $O(x/\log^2 x)$. This completes the proof.

It is possible to improve the error term and actually show that for any $k \geq 2$, we have constants $c_0, c_1, \ldots, c_k$ such that

$$V(x) = \sum_{j=0}^{k} \frac{c_j x}{\log^j x} + O\left(\frac{x}{\log^{k+1} x}\right).$$

This is easily done by inserting a stronger form of the prime number theorem into the proof above.

5. Concluding remarks

An effective version of the *abc* conjecture can be used to determine (conjecturally) that there cannot be 12 consecutive special numbers. Suppose not. Let

$$n + 1, n + 2, \ldots, n + 12$$

be all special. Six of these are even, say

$$2k, 2(k + 1), 2(k + 2), 2(k + 3), 2(k + 4), 2(k + 5).$$

Of these six, at least three are not divisible by 4 and so we may write these as

$$2j, 2j + 4, 2j + 8$$

with j odd and powerful as well as $j + 2$, $j + 4$. That is,

$$j, j + 2, j + 4 = l - 2, l, l + 2,$$

(say) are all powerful and the *abc* equation

$$(l^2 - 4) + 4 = l^2$$

implies

$$l^2 \ll (l^{1/2} l^{1/2} l^{1/2})^{1+\varepsilon}.$$

This implies that l is bounded.

Perhaps more is true. We may conjecture (based on numerical evidence) that there are no 6 consecutive special numbers. The only known five consecutive special integers are $\{1, 2, 3, 4, 5\}$, $\{16, 17, 18, 19, 20\}$, $\{241, 242, 243, 244, 245\}$, $\{2644, 2645, 2646, 2647, 2648\}$ and $\{4372, 4373, 4374, 4375, 4376\}$. Those are only ones with entries smaller than 10^6. There are a few more known up to $7 \cdot 10^8$. In case of squarefull (or powerful) numbers, as a summary of results on three consecutive powerful numbers, Erdös [6] conjectured that there does not exist three consecutive powerful numbers. Golomb [7] also considered this question, as did Mollin and Walsh [10]. See an application of the *abc* conjecture in page 135 of [11].

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