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Symplectic S_5 action on symplectic homotopy K3 surfaces

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Abstract. Let X be a symplectic homotopy K3 surface and $G = S_5$ act on X symplectically. In this paper, we give a weak classification of the G action on X by discussing the fixed-point set structure. Besides, we analyse the exoticness of smooth structures of X under the action of G.

Keywords. K3 surfaces; symplectic actions; exotic smooth structure.

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1. Introduction

Let X be a symplectic homotopy K3 surface, namely X is a closed, oriented smooth 4-manifold homeomorphic to the standard K3 surface and admits an orientation-compatible symplectic structure. A finite group G is called a symplectic K3 group if G can be realized as a subgroup of the symplectic automorphism group of a K3 surface. In [8, 9], they classified finite abelian groups of symplectic automorphisms and determined all the symplectic K3 groups. The following 11 groups are the maximal symplectic K3 groups:

$$L_2(7), A_6, S_5, M_{20}, F_{384}, A_{4,4}, T_{192}, H_{192}, N_{72}, M_9, T_{48}.$$

In [3], the relation between the smooth structure of the symplectic homotopy K3 surface and its symplectic symmetries were studied. In particular, they introduced a measurement of the exoticness of smooth structures of symplectic homotopy K3 surfaces. By this measurement, an effective action by various maximal symplectic K3 groups may force the corresponding homotopy K3 surface to be minimally exotic. Concretely, they proved the following theorem.

Theorem 1.1 **[3].** *Let G be one of the following maximal symplectic $K3$ groups*:

$$L_2(7), A_6, M_{20}, A_{4,4}, T_{192}, T_{48},$$

and let X be a symplectic homotopy $K3$ surface. If X adimts an effective G-action via symplectic symmetries, *then X must be minimally exotic*, *i.e.*, $r_X = 0$.

In this paper, we study the maximal symplectic K3 group S_5 action on symplectic homotopy K3 surface. We give a classification of symplectic S_5 action on symplectic homotopy K3 and try to study the exoticness of smooth structures of symplectic homotopy K3 surfaces under this symplectic S_5 action. Concretely, we get the following main theorem.

Theorem 1.2. *Let X be a symplectic homotopy $K3$ surface and $G = S_5$ act on X symplectically. If $b_2^+(X/G) = 3$, then X must be minimally exotic.*

We organize the rest of the paper as follows. In §2, we review some results about the symplectic group actions on symplectic homotopy K3 surfaces and some formulas used in our study. In §3, we prove the main theorems, Theorem 3.1 and Theorem 3.5.

2. Preliminaries

In this section, we collect some theorems used in our study. These results are mainly about the symplectic group actions on symplectic homotopy K3 surfaces. We also review the G-signature theorem and Lefschetz fixed point theorem.

Lemma 2.1 [3]. *Let G_0 be the maximal subgroup of G such that $b_2^+(X/\langle G_0\rangle) = 3$. Then G/G_0 is cyclic. Moreover, the commutator $[G, G]$ is contained in G_0.*

In [2], Bryan studied the fixed point set of spin involution on a rational cohomology K3. In [6], Kim studied the fixed point set of spin $\mathbb{Z}/4$ action on homotopy K3 surface. Additionally, applying the theory of J-holomorphic curve, Chen and Kwasik [3, 4] studied the fixed point set of order 2, 3, 4, 5 elements of a symplectic group action on a symplectic homotopy K3 surface under the condition $g \in G_0$, which are the following three lemmas.

Lemma 2.2 [3].

(1) *Let $g \in G$ be an involution. If $g \in G_0$, then* Fix(g) *consists of* 8 *isolated fixed points. If $g \in G\backslash G_0$, then* Fix(g) *is either empty or a disjoint union of embedded J-holomorphic curves Σ_j such that $c_1(K_X) \cdot \Sigma_j = 0$ for each j.*
(2) *Let $g \in G_0$ be an element of order* 4. *Then* Fix(g) *consists of* 4 *isolated fixed points, all with a local representation contained in $SL_2(\mathbb{C})$.*

Lemma 2.3 [3, 4]. *Let $g \in G_0$ be an element of order* 3. *Then* Fix(g) *may be divided into subsets of the following four types.*

(*I*) *One fixed point with local representation in $SL_2(\mathbb{C})$.*
(*II*) *Three fixed points, all with local representation of type (k, k) for some $k \neq 0 \bmod 3$.*
(*III*) *One fixed point of local representation type (k, k), $k \neq 0 \bmod 3$, and one fixed spherical component of self-intersection*-2.
(*IV*) *One fixed toroidal component of self-intersection* 0.

Lemma 2.4 [3]. *Suppose $H \subset G_0$ is a subgroup isomorphic to either A_5 or A_6. Let $g \in H$ be an element of odd order. Then g is either pseudofree or has only toroidal fixed components. Moreover, g has* 4 *isolated fixed points if $|g| = 5$, and g has either* 6 *isolated fixed points of type* (*I*) *or* 12 *isolated fixed points of type* (*II*) *when $|g| = 3$.*

Lemma 2.5 [3]. *Suppose $g \in G_0$ is an element of order* 3. *Let u, v and w be the number of groups of type (I), (II) and (III) fixed points of g respectively, and let $t = b_2(X/\langle g \rangle)$. Then $2u + 3v = 12$, $w \leq 6$ and $t \geq 10$. Moreover, $t = 10$ iff $(u, v, w) = (6, 0, 0)$.*

On the existence of symplectic finite group actions on K3 surface, Mukai [8] showed the following restriction.

Theorem 2.6 [8]. *Let G be a finite group which acts (effectively and) symplectically on $K3$ surface X. Then the whole cohomology group $H^*(X, \mathbb{Q})$ is a Mathieu representation of G over $\mathbb{Q}$ and* $\dim\ H^*(X, \mathbb{Q})^G \geq 5$.

We also need the following G-signature theorems and Lefschetz fixed point theorem.

Theorem 2.7 (G-signature theorem [5]).

$$|G|\text{sign}(X/G) = \text{sign}(X) + \sum_{g \in G-1} \text{sign}(g, X).$$

Theorem 2.8 [5]. *If the fixed point set of g only consists of isolated fixed points, then*

$$\text{sign}(g, X) = -\sum_{P \in \text{Fix}(g)} \cot\frac{\theta_1}{2}\cot\frac{\theta_2}{2},$$

where θ_1, θ_2 are the rotation numbers at fixed point P.

Theorem 2.9 (Lefschetz fixed point theorem [1]). *$L(g, X) = \chi(F)$, where $\chi(F)$ is the Euler characteristic of the fixed-point set F and $L(g, X)$ is the Lefschetz number of the map $g : X \to X$, which is defined by*

$$L(g, X) = \sum_{k=0}^{4} (-1)^k \text{tr}(g)|_{H^k(X;\mathbb{R})}.$$

Suppose X is a symplectic homotopy K3 surface. The Seiberg–Witten invariant of a simply-connected, closed, oriented, smooth 4-manifold X with $b_2^+ \geq 2$ is a map

$$SW_X : \beta \in H^2(X; \mathbb{Z})|\beta \equiv \omega_2(TX) \text{ mod } 2 \to \mathbb{Z}.$$

If $SW_X(\beta) \neq 0$, then β is called a basic class. For a homotopy K3 surface, $\beta = 0$ is always a basic class by Morgan and Szabò [7]. For symplectic homotopy K3 surface X, we set

$$L_X \equiv \text{Span}(\beta \in H^2(X; \mathbb{Z})|SW_X(\beta) \neq 0) \subset H^2(X; \mathbb{Z}).$$

Denote the rank of L_X by r_X. Taubes [10] showed that $c_1(K_X) \in L_X$ and $c_1(K_X) = 0$ iff $r_X = 0$. As we know, $c_1(K_X) = 0$ is a characterization of minimally exotic symplectic homotopy K3 surface [3]. Thus we can detect the exoticness of the smooth structure of X by determining whether r_X is zero.

Suppose G is a finite group which acts on X via symplectic symmetries. Then there is an induced action of G on the set of basic classes, which extends to a linear action on the lattice L_X. In particular, $c_1(K_X)$ is fixed under this action of G and

$$SW_X(g \cdot \beta) = SW_X(\beta),$$

for any basic class β and any $g \in G$.

Table 1. Conjugate class of S_5.

Conjugate class	(1)	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)
Order	1	2	3	2	4	6	5
Number of element	1	10	20	15	30	20	24

Besides $r_X \leq \min(b_2^+, b_2^-) = 3$ (cf. Proposition 4.1 of [3]), we can obtain the inequality

$$\dim(L_X \otimes_{\mathbb{Z}} \mathbb{R})^G \leq \min(b_2^+(X/G), b_2^+(X/G)),$$

which allows one to obtain the information about $\dim(L_X \otimes_{\mathbb{Z}} \mathbb{R})^G$ and r_X and decide the upper bound on the exoticness r_X.

Lastly, note that the conjugate classes of elements in S_5 are as in Table 1.

3. Main results

In the following, we always denote the symplectic homotopy K3 surface by X and the symmetric group S_5 by G.

Proof of Theorem 1.2. Let G_0 be the maximal subgroup of G such that $b_2^+(X/G_0) = 3$. Then in our case, $G_0 = G$. Since $g \in G_0$ for any $g \in G$, every order 2 element has 8 isolated fixed points and every order 4 element has 4 isolated fixed points by Lemma 2.2. By the similar proof as in Lemma 2.4, every order 5 element of G has 4 isolated fixed points and the order 3 element has 6 type (I) fixed points or 12 type (II) fixed points.

For order 6 element $g \in S_5$, note that $\mathrm{Fix}(g) \subset \mathrm{Fix}(g^2) \bigcap \mathrm{Fix}(g^3)$. Since $\mathrm{Fix}(g^2)$ and $\mathrm{Fix}(g^3)$ are all isolated, $\mathrm{Fix}(g)$ is also isolated. Let μ^i be the eigenvalue of the action generated by g^i on $H^2(X;\mathbb{R})$, V_{g^i} be the respective eigenspace and t_i be the dimension of V_{g^i}, $i = 1, 2, \dots, 5$. Then it is easy to know $V_{g^2} = V_{g^4}$, $V_g = V_{g^5}$. Thus

$$t_2 = t_4, \ t_1 = t_5. \tag{1}$$

We discuss the number of the fixed points of g in the following two cases.

Case 1. Assume $\mathrm{Fix}(g^2)$ consists of 6 type (I) isolated fixed points.

In this case, the local representations of $\mathrm{Fix}(g)$ must be $(1, 5)$. Denote the number of these fixed points by a. By Lemma 2.5, the dimension of the 1-eigenspace of g^2 in $H^2(X;\mathbb{R})$ is 10, thus

$$t_0 + t_3 = 10. \tag{2}$$

Then the Lefschetz fixed point theorem and (1) and (2) give

$$2 + t_0 + t_1\mu + t_2\mu^2 + (10 - t_0)\mu^3 + t_2\mu^4 + t_1\mu^5 = a. \tag{3}$$

Simplifying (3), we get

$$2t_0 - 8 + t_1 - t_2 = a. \tag{4}$$

Besides, the contribution of the fixed points of type $(1, 5)$ of g to the G-signature is as follows:

$$\operatorname{sign}(g, X) = \operatorname{sign}(g^5, X) = -\sum_{P\in \operatorname{Fix}(g)} \cot\frac{\pi}{6}\cot\frac{5\pi}{6} = 3a,$$

$$\operatorname{sign}(g^2, X) = \operatorname{sign}(g^4, X) = -\sum_{P\in \operatorname{Fix}(g^2)} \cot\left(\frac{1}{2}\cdot\frac{2\pi}{6}\cdot 2\right)\cdot\cot\left(\frac{1}{2}\cdot\frac{2\pi}{6}\cdot 10\right) = 2,$$

$$\operatorname{sign}(g^3, X) = 0.$$

By G-signature theorem, we have

$$6(3-(t_0-3)) = -16 + 2\cdot(3a) + 4. \tag{5}$$

Combining (4), (5) and the fact $t_1 + t_2 = 6$, we obtain $a = 2$ or 4. Since

$$\dim(H^*(X;\mathbb{R}))^G = \frac{1}{|G|}\sum_{g\in G}\operatorname{tr}(g) \geq 5$$

(cf. Lemma 2.6), we have $a = 2$.

Case 2. Suppose $\operatorname{Fix}(g^2)$ consists of 12 type (II) isolated fixed points.

In this case, the local representations of $\operatorname{Fix}(g)$ must be of $(1, 1)$ or $(5, 5)$ and the dimension of the 1-eigenspace of g^2 in $H^2(X;\mathbb{R})$ is 14. Denote the number of the two types of fixed points by b and c respectively. By the Lefschetz fixed point theorem,

$$2 + t_0 + t_1\mu + t_2\mu^2 + (14 - t_0)\mu^3 + t_2\mu^4 + t_1\mu^5 = b + c. \tag{6}$$

Simplifying (6), we get

$$2t_0 - 12 + t_1 - t_2 = b + c. \tag{7}$$

Next we consider the contribution of fixed points with local representations $(1, 1)$ and $(5, 5)$ to the G-signature respectively.

For fixed points with local representation $(1, 1)$, we have

$$\operatorname{sign}(g, X) = \operatorname{sign}(g^5, X) = -3b,\ \operatorname{sign}(g^2, X) = \operatorname{sign}(g^4, X) = -\frac{d}{3},$$
$$\operatorname{sign}(g^3, X) = 0,$$

where d denotes the number of the fixed points in $\operatorname{Fix}(g^2)$ with local representations $(1, 1)$.

For the fixed points with local representation $(5, 5)$, we have

$$\operatorname{sign}(g, X) = \operatorname{sign}(g^5, X) = -3c,$$

$$\operatorname{sign}(g^2, X) = \operatorname{sign}(g^4, X) = -\frac{e}{3},$$

$$\operatorname{sign}(g^3, X) = 0,$$

where e denotes the number of the fixed points in $\operatorname{Fix}(g^2)$ with local representations $(2, 2)$.

By the G-signature theorem, we have

$$6(3-(t_0-3))=-16+2(-3b-3c)+2\left(-\frac{1}{3}d-\frac{1}{3}e\right). \tag{8}$$

Note that $d+e=12$, thus (8) can be simplified to be

$$b+c=t_0-10. \tag{9}$$

Combining equations (7) and (9), we get $b+c=t_2-t_1-8$. Since $t_2+t_1=4$, $-4\leq t_2-t_1\leq 4$. Hence we get a contradiction, which means there is no such g action on X.

In summary, the G action has the same fixed point set as does a symplectic holomorphic action of G. As a consequence, this implies that

$$\dim(H^2(X;\mathbb{R}))^G=\dim(H^*(X;\mathbb{R}))^G-2=5-2=3.$$

Since $b_2^+(X/G)=3$, $H^2(X;\mathbb{R})^G$ must be positive-definite. Note that $c_1(K_X)\in H^2(X;\mathbb{R})^G$ and $c_1(K_X)\cdot c_1(K_X)=0$, thus we have $c_1(K_X)=0$ which is equivalent to X being minimally exotic. Theorem 3.1 is proved.

□

Next we suppose $b_2^+(X/G)=1$. For $G=S_5$, the commutator $[G,G]=A_5$. By Lemma 2.1, the commutator $[G,G]$ is contained in G_0. Thus $G_0=A_5$. In the following text, we always denote $\tau\in G$ as any element of order 4 and $g\in G$ as any element of order 6. Then τ^2 is an involution in G_0, g^3 is an involution which is not in G_0 and g^2 is an order 3 element in G_0. Since the order 2 element g^3 is not contained in $G_0=A_5$, $b_2^+(X/<g^3>)=1$ by [2, 3]. Thus $b_2^+(X/G)=0$ or 1.

In the case $b_2^+(X/G)=1$, we study the fixed-point set structure of the symplectic G actions on X as follows.

Lemma 3.2. *Suppose* $b_2^+(X/G)=1$. *Let* $\tau\in G$ *be an element of order* 4. *Then one of the following is true*:

(1) *When* $b_2^+(X/\tau)=1$, *then the set* $\mathrm{Fix}(\tau)$ *consists of* 2, 4, 6 *or* 8 *isolated fixed points with local representation* (1, 1) *or* (3, 3).
(2) *When* $b_2^+(X/\tau)=3$, *then the set* $\mathrm{Fix}(\tau)$ *consists of* 4 *isolated fixed points with local representation contained in* $SL_2(\mathbb{C})$.

Proof. For order 4 element τ, τ^2 generates an involution in G_0. Thus $\mathrm{Fix}(\tau^2)$ consists of 8 isolated fixed points (cf. [2, 3]). Hence the fixed point of τ must also be isolated fixed points. Denote by s_+, s_- the number of fixed points of types (1, 3) and (1, 1) or (3, 3) of τ. Note that τ induces an involution τ^2 on $H^2(X;\mathbb{R})$ with the dimension of 1-eigenspace being 14 and the dimension of –1-eigenspace being 8. Denote by $t_\pm$ the dimension of the $\pm$1-eigenspace of τ in $H^2(X;\mathbb{R})$. Then $t_++t_-=14$. Suppose $t_0=b_2^+(X/\langle\tau\rangle)$, then the Lefschetz fixed point theorem and the G-signature theorem give rise to the following system of equations:

$$2+t_+-(14-t_+)=s_++s_-,$$

$$4(t_0-(t_+-t_0))=-16+2s_+-2s_-,$$

where we use the fact that the signature contribution of a fixed point of types (1, 3) and type (3, 3) or (1, 1) is 2, -2 respectively. Solving these equations, we get

$$t_0 = 1 + s_+/2. \tag{10}$$

Since the action of τ can be lifted to the spin structure and τ^2 is of even type, τ is also of even type. Then by [6], $b_2^+(X/\langle\tau\rangle) = 1$ or $b_2^+(X/\langle\tau\rangle) = 3$. If $t_0 = b_2^+(X/\langle\tau\rangle) = 1$, then $s+ = 0$, $s- = 0, 2, 4, 6$ or 8. If $t_0 = b_2^+(X/\langle\tau\rangle) = 3$, then $s+ = 4$, $s- = 0$. □

By discussing the spin number of τ action as in [3], we can rule out the case $s_+ = 0$, $s_- = 0$ and the lemma follows.

Lemma 3.3. *Suppose* $b_2^+(X/G) = 1$. *Let* $g \in G$ *be an element of order* 6. *Then one of the following is true*:

(1) *When* $\mathrm{Fix}(g^2)$ *consists of* 6 *isolated fixed points*, *then the set* $\mathrm{Fix}(g)$ *is empty.*
(2) *When* $\mathrm{Fix}(g^2)$ *consists of* 12 *isolated fixed points*, *then the set* $\mathrm{Fix}(g)$ *consists of* 2 *or* 4 *isolated fixed points.*

Proof. For order 6 element, note that $\mathrm{Fix}(g) \subset \mathrm{Fix}(g^2) \bigcap \mathrm{Fix}(g^3)$. Thus $\mathrm{Fix}(g)$ consists of isolated fixed points. Since g^2 generates an order 3 cyclic group action on X, the local representations of $\mathrm{Fix}(g)$ are of the following two types:

Type 1: (1, 1), (1, 4), (2, 2), (2, 5), (4, 4), (5, 5).
Type 2: (1, 2), (1, 5), (2, 4), (4, 5).

Note that Type 1 fixed points correspond to the fixed points of g^2 with local representation (1, 1) or (2, 2), while the Type 2 fixed points correspond to the fixed points of g^2 with local representation (1, 2).

Case 1. Suppose $\mathrm{Fix}(g^2)$ consists of 6 isolated fixed points, then the local representations of $\mathrm{Fix}(g)$ is of Type 2. Denote the number of the fixed points with local representation (1, 2), (1, 5), (2, 4), (4, 5). by p, q, r, l.

For the number p fixed points of type (1, 2), their contributions to the G-signature are as follows:

$$\mathrm{sign}(g, X) = \mathrm{sign}(g^5, X) = -p,\ \mathrm{sign}(g^2, X) = \mathrm{sign}(g^4, X) = \frac{1}{3} \cdot 6,$$
$$\mathrm{sign}(g^3, X) = 0.$$

For the number q fixed points of type (1, 5), their contributions to the G-signature are as follows:

$$\mathrm{sign}(g, X) = \mathrm{sign}(g^5, X) = 3q,\ \mathrm{sign}(g^2, X) = \mathrm{sign}(g^4, X) = \frac{1}{3} \cdot 6,$$
$$\mathrm{sign}(g^3, X) = 0.$$

For the number r fixed points of type (2, 4), their contributions to the G-signature are as follows:

$$\mathrm{sign}(g, X) = \mathrm{sign}(g^5, X) = \frac{1}{3}r,\ \mathrm{sign}(g^2, X) = \mathrm{sign}(g^4, X) = \frac{1}{3} \cdot 6,$$
$$\mathrm{sign}(g^3, X) = 0.$$

For the number l fixed points of type (4, 5), their contributions to the G-signature are as follows:

$$\operatorname{sign}(g, X) = \operatorname{sign}(g^5, X) = -l,\ \operatorname{sign}(g^2, X) = \operatorname{sign}(g^4, X) = \frac{1}{3} \cdot 6,$$
$$\operatorname{sign}(g^3, X) = 0.$$

Then the G-signature theorem and Lefschetz fixed point theorem give rise to the following system of equations.

$$6(1 - (t_0 - 1)) = -16 + 2(3q - p + \frac{1}{3}r - l) + 8,$$

$$2 + t_0 + t_1\mu + t_2\mu^2 + (10 - t_0)\mu^3 + t_2\mu^4 + t_1\mu^5 = p + q + r + l.$$

Solving these equations, we get $-12 + 3(t_1 - t_2) = p + 9q + \frac{11}{3}r + l$. On one hand, note that $t_1 + t_2 = 6$, then $t_1 - t_2$ must be an even integer and satisfy $-6 \leq t_1 - t_2 \leq 6$. Then we obtain $r = q = 0$ and $p + l = 0$ or 6. On the other hand, since the 6 isolated fixed points of g^2 must be on 2 tori (cf. [4]), $\operatorname{Fix}(g)$ is also on these 2 tori. While as we know, the action generated by g on torus must be free or has only 1 isolated fixed point. Thus $\operatorname{Fix}(g)$ is empty.

Case 2. Suppose $\operatorname{Fix}(g^2)$ consists of 12 isolated fixed points. Denote the number of fixed points of types (1, 1) and (2, 2) of order 3 element by m and n respectively, then $m + n = 12$. In this case, the fixed points of g is of Type 1. Denote the number of the fixed points with local representation (1, 1), (1, 4), (2, 2), (2, 5), (4, 4), (5, 5), by a, b, c, d, e, f. Then the signature contributions of every fixed points of g are as follows.

For the number a fixed points of type (1, 1), their contributions to the G-signature are as follows:

$$\operatorname{sign}(g, X) = \operatorname{sign}(g^5, X) = -3a,\ \operatorname{sign}(g^2, X) = \operatorname{sign}(g^4, X) = -\frac{1}{3} \cdot 12$$
$$= -\frac{1}{3}m,\ \operatorname{sign}(g^3, X) = 0.$$

For the number b fixed points of type (1, 4), their contributions to the G-signature are as follows:

$$\operatorname{sign}(g, X) = \operatorname{sign}(g^5, X) = b,\ \operatorname{sign}(g^2, X) = \operatorname{sign}(g^4, X) = -\frac{1}{3}m,$$
$$\operatorname{sign}(g^3, X) = 0.$$

For the number c fixed points of type (2, 2), their contributions to the G-signature are as follows:

$$\operatorname{sign}(g, X) = \operatorname{sign}(g^5, X) = -\frac{1}{3}c,\ \operatorname{sign}(g^2, X) = \operatorname{sign}(g^4, X) = -\frac{1}{3}n,$$
$$\operatorname{sign}(g^3, X) = 0.$$

For the number d fixed points of type (2, 5), their contributions to the G-signature are as follows:

$$\operatorname{sign}(g, X) = \operatorname{sign}(g^5, X) = d,\ \operatorname{sign}(g^2, X) = \operatorname{sign}(g^4, X) = -\frac{1}{3}n,$$
$$\operatorname{sign}(g^3, X) = 0.$$

For the number e fixed points of type (4, 4), their contributions to the G-signature are as follows:

$$\operatorname{sign}(g, X) = \operatorname{sign}(g^5, X) = -\frac{1}{3}e,\ \operatorname{sign}(g^2, X) = \operatorname{sign}(g^4, X) = -\frac{1}{3}m,$$
$$\operatorname{sign}(g^3, X) = 0.$$

For the number f fixed points of type (5, 5), their contributions to the G-signature are as follows:

$$\operatorname{sign}(g, X) = \operatorname{sign}(g^5, X) = -3e,\ \operatorname{sign}(g^2, X) = \operatorname{sign}(g^4, X) = -\frac{1}{3}n,$$
$$\operatorname{sign}(g^3, X) = 0.$$

By G-signature theorem, we have

$$6(1 - (t_0 - 1)) = -16 + 2(-3(a + f) + (b + d) - \frac{1}{3}(c + e)) - 24. \quad (11)$$

Besides, the Lefschetz fixed point theorem gives rise to

$$2 + t_0 + t_1\mu + t_2\mu^2 + (14 - t_0)\mu^3 + t_2\mu^4 + t_1\mu^5 = a + b + c + d + e + f. \quad (12)$$

Solving equations (11) and (12), we get

$$16 + 3(t_1 - t_2) = -3(a + f) + 5(b + d) + \frac{7}{3}(c + e).$$

On one hand, note that $t_1 + t_2 = 4$, then $t_1 - t_2$ must be even integer and satisfy $-4 \leq t_1 - t - 2 \leq 4$. Then from (12), $\operatorname{Fix}(g)$ consists of even number isolated fixed points. On the other hand, since $\operatorname{Fix}(g^2)$ consists of 12 isolated fixed points, in our case, these fixed points must be on 4 tori. Hence, $\operatorname{Fix}(g)$ are also on these 4 tori. Since the action generated by g on torus must be free or has only one fixed point, we get $\operatorname{Fix}(g)$ consists of 2 or 4 isolated fixed points. Then the lemma follows.

□

Note that $\dim H^*(X; \mathbb{R})^G = \frac{1}{|G|}\sum_{g\in G} \operatorname{tr}(g) \geq 5$ and it is an integer. Then by Lemmas 3.2, 3.3 and 3.4, we get the following result.

Theorem 3.4 *Let X be a symplectic homotopy $K3$ surface and $G = S_5$ act on X symplectically. If $b_2^+(X/G) = 1$ and $\operatorname{Fix}(g^3)$ is empty or a disjoint union of embedded J-holomorphic tori, then $\operatorname{Fix}(g^2)$ consists of 6 isolated fixed points, $\operatorname{Fix}(g)$ is empty and $\operatorname{Fix}(\tau)$ consists of 8 isolated fixed points.*

Remark 3.5. If $b_2^+(X/G) = 0$, then we can obtain the same fixed-point set structure as Theorem 3.4 by similar discussion as above. In this case,

$$\dim(L_X \otimes_{\mathbb{Z}} \mathbb{R})^G \leq \min(b_2^+(X/G), b_2^+(X/G)) = 0.$$

Thus $\operatorname{rank}(L_X^G) = 0$. Since $c_1(K_X) \in L_X^G$, we obtain $c_1(K_X) = 0$ which means X is minimally exotic.

Remark 3.6. Under the conditions of Theorem 3.4, $\dim\ H^*(X;\mathbb{R})^G = \frac{1}{|G|}\sum_{g\in G}\mathrm{tr}(g) = 5$. Then

$$\dim(L_X \otimes_{\mathbb{Z}} \mathbb{R})^G \le \min(b_2^+(X/G), b_2^+(X/G)) = \min(1,2) = 1$$

and so $\mathrm{rank}(L_X^G) \le 1$. Since $c_1(K_X) \in L_X^G$, we can not determine whether $c_1(K_X)$ is 0. Thus the exoticness of the smooth structure of X can not be determined in this case.

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