Proc. Indian Acad. Sci. (Math. Sci.) Vol. 127, No. 2, April 2017, pp. 375–381.
DOI 10.1007/s12044-016-0308-1

CrossMark

Eigenvalue estimates for submanifolds with bounded f-mean curvature

GUANGYUE HUANG[1,2] and BINGQING MA[1,2,*]

[1]College of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, People's Republic of China
[2]Henan Engineering Laboratory for Big Data Statistical Analysis and Optimal Control, Xinxiang 453007, People's Republic of China
[*]Corresponding author.
E-mail: hgy@henannu.edu.cn; bqma@henannu.edu.cn

MS received 17 March 2015; revised 3 June 2016

Abstract. In this paper, we obtain an extrinsic low bound to the first non-zero eigenvalue of the f-Laplacian on complete noncompact submanifolds of the weighted Riemannian manifold $(H^m(-1), e^{-f}dv)$ with respect to the f-mean curvature. In particular, our results generalize those of Cheung and Leung in *Math. Z.* **236** (2001) 525–530.

Keywords. Eigenvalue estimates; submanifolds; weighted manifolds; f-mean curvature.

2010 Mathematics Subject Classification. 53C21, 58C40.

1. Introduction

Let $(N^m, \bar{g}, d\mu)$ be an m-dimensional weighted Riemannian manifold, where $d\mu = e^{-f}dv$ and dv denotes the volume element induced by the metric $\bar{g}$ and f is a real-valued smooth function on N^m. The f-Laplacian (also called the weighted Laplacian) Δ_f is defined by

$$\Delta_f u = \mathrm{div}_f(\nabla u) = \Delta u - \nabla f \nabla u, \tag{1.1}$$

which is self-adjoint with respect to the weighted measure $d\mu$, where the f-divergence is given by

$$\mathrm{div}_f(\nabla u) = e^f \mathrm{div}(e^{-f}\nabla u).$$

For the concept of weighted measure space, see [1, 5] and references therein. Let $B_p(r)$ be a geodesic ball of radius r centred at p and $\lambda_1(r) > 0$ denotes the first eigenvalue of the Dirichlet boundary value problem

$$\Delta_f u = -\lambda u \ \text{ in } B_p(r); \qquad u = 0 \ \text{ on } \partial B_p(r). \tag{1.2}$$

By the domain monotonicity principle, $\lambda_1(r)$ is decreasing and has a limit independent of the choice of the centre p. That is, the first eigenvalue $\lambda_1^N(\Delta_f)$ of N^m is defined by

$$\lambda_1(\Delta_f) := \lim_{r\to\infty} \lambda_1(r). \tag{1.3}$$

Let us recall a famous theorem of McKean in [7]:

Theorem A. *Let M be a complete, simply connected manifold with sectional curvature bounded above by $-k^2$ for some non-zero constant k. Then*

$$\lambda_1(\Delta^M) \geq \frac{(n-1)^2k^2}{4}. \tag{1.4}$$

Let M be a submanifold in N^m. Let $\{e_1, \ldots, e_n\}$ be a local orthonormal basis of M with respect to the induced metric, and $\{e_{n+1}, \ldots, e_m\}$ is a local orthonormal frame which are perpendicular to M. Let $\bar{\nabla}$ be the connection on N induced by $\bar{g}$. We define the linear map A^α by

$$A^\alpha_{ij} = \langle A^\alpha(e_i), e_j\rangle := \langle \bar{\nabla}_{e_i} e_j, e_\alpha\rangle,$$

where α denotes the indices for the normal directions. Then the mean curvature vector field is given by

$$\mathbf{H} = \sum_{\alpha=n+1}^{m} H^\alpha e_\alpha,$$

where $H^\alpha = \sum_i A^\alpha_{ii}$. Observing that minimal submanifolds should retain some of the properties of the ambient space, Cheung and Leung [3] studied complete non-compact submanifolds in a hyperbolic space with the norm of their mean curvature vector bounded above by a constant. They proved as follows.

Theorem B. *Let M be an n-dimensional complete noncompact submanifold of the hyperbolic space $H^m(-1)$. If $\sup_M |\mathbf{H}| < n-1$, then*

$$\lambda_1(\Delta^M) \geq \frac{((n-1) - \sup_M |\mathbf{H}|)^2}{4}. \tag{1.5}$$

Let M be a submanifold in N^m. The f-mean curvature vector field (see p. 398 of [8] or [6]) is given by

$$\mathbf{H}_f = \mathbf{H} + (\bar{\nabla} f)^\perp, \tag{1.6}$$

where $\perp$ is the orthogonal projection onto the bundle of M. M is called an f-minimal submanifold in N^m if $\mathbf{H}_f$ vanishes everywhere.

In this paper, we will generalize Theorem B of Cheung and Leung to submanifolds of weighted manifolds with respect to the f-Laplacian. We will prove the following results:

Theorem 1.1. *Let M be an n-dimensional complete noncompact submanifold of the weighted Riemannian manifold $(H^m(-1), \mathrm{e}^{-f}\mathrm{d}v)$. If $\sup_M |\mathbf{H}_f - \bar{\nabla} f| < n-1$, then*

$$\lambda_1(\Delta_f^M) \geq \frac{((n-1) - \sup_M |\mathbf{H}_f - \bar{\nabla} f|)^2}{4}. \tag{1.7}$$

Equivalently,

$$\lambda_1(\Delta_f^M) \geq \frac{\left((n-1) - \sup_M \sqrt{|\mathbf{H}|^2 + |(\bar{\nabla} f)^{\top}|^2}\right)^2}{4}, \tag{1.8}$$

where $\top$ is the projection onto the tangent bundle of M.

In particular, for f-minimal submanifolds of hyperbolic space $H^m(-1)$, we have

COROLLARY 1.2

Let M be an n-dimensional f-minimal submanifold of the weighted Riemannian manifold $(H^m(-1), \mathrm{e}^{-f}\mathrm{d}v)$. If $\sup_M |\bar{\nabla} f| < n-1$, then

$$\lambda_1(\Delta_f^M) \geq \frac{((n-1) - \sup_M |\bar{\nabla} f|)^2}{4}. \tag{1.9}$$

Remark 1.3. In particular, when f is a constant, our Theorem 1.1 becomes Theorem 2 and Corollary 1.2 becomes Corollary 3 in [3], respectively.

Remark 1.4. Batista *et al.* in [2] obtained an extrinsic upper bound to the first non-zero eigenvalue of the f-Laplacian on closed submanifolds of weighted manifolds. In this paper, we obtain an extrinsic low bound to the first non-zero eigenvalue of the f-Laplacian on complete noncompact submanifolds of the weighted Riemannian manifold $(H^m(-1), \mathrm{e}^{-f}\mathrm{d}v)$. Furthermore, as pointed out by Batista *et al.* in [2], the term $|\mathbf{H}_f - \bar{\nabla} f|$ in Theorem 1.1 can not be replaced by $|\mathbf{H}_f|$.

2. Proofs of results

In order to achieve our main result, we need the following lemmas:

Lemma 2.1 *Let u be a smooth function on an m-dimensional weighted Riemannian manifold $(N, \mathrm{e}^{-f}\mathrm{d}v)$ and M an n-dimensional submanifold of $(N, \mathrm{e}^{-f}\mathrm{d}v)$. That is, we denote the dimension of N by $m = d(N)$ and the dimension of M by $n = d(M)$. Let $\bar{\Delta}$ be Laplacian on $(N, \mathrm{e}^{-f}\mathrm{d}v)$ and Δ the Laplacian on M, respectively. Then the following formula holds:*

$$\Delta_f(u|_M) = (\bar{\Delta}_f u)|_M + \langle \mathbf{H}_f, \bar{\nabla} u\rangle|_M - \sum_{\alpha=n+1}^{m} \bar{\nabla}^2 u(e_\alpha, e_\alpha)\Big|_M, \tag{2.1}$$

where $\{e_{n+1}, \ldots, e_m\}$ is a local orthonormal frame field which is perpendicular to M.

Proof of Lemma 2.1. Let $\{e_1, \ldots, e_n\}$ be a local orthonormal basis of M with respect to the induced metric, and $\{e_{n+1}, \ldots, e_m\}$ is a local orthonormal frame which are perpendicular to M. By the definition of Laplacian, we have

$$\begin{aligned}
\bar{\Delta}_f(u|_N) &= \bar{\Delta}(u|_N) - \langle \bar{\nabla} f, \bar{\nabla} u\rangle \\
&= \sum_{A=1}^{m}(e_A e_A - \bar{\nabla}_{e_A} e_A)u - \sum_{A=1}^{m} f_A u_A \\
&= \sum_{i=1}^{n}(e_i e_i - \bar{\nabla}_{e_i} e_i)u + \sum_{\alpha=n+1}^{m}(e_\alpha e_\alpha - \bar{\nabla}_{e_\alpha} e_\alpha)u \\
&\quad - \sum_{i=1}^{n} f_i u_i - \sum_{\alpha=n+1}^{m} f_\alpha u_\alpha \\
&= (\Delta u)|_M - \langle \mathbf{H}, \bar{\nabla} u\rangle + \sum_{\alpha=n+1}^{m} \bar{\nabla}^2 u(e_\alpha, e_\alpha) \\
&\quad - \sum_{i=1}^{n} f_i u_i - \langle (\bar{\nabla} f)^{\perp}, \bar{\nabla} u\rangle \\
&= (\Delta_f u)|_M - \langle \mathbf{H}_f, \bar{\nabla} u\rangle + \sum_{\alpha=n+1}^{m} \bar{\nabla}^2 u(e_\alpha, e_\alpha),
\end{aligned} \tag{2.2}$$

where we used the relation (see Lemma 2 of [4])

$$\sum_{i=1}^{n} \bar{\nabla}_{e_i} e_i = \mathbf{H} + \sum_{i=1}^{n} \nabla_{e_i} e_i$$

and

$$\bar{\nabla}^2 u(e_i, e_i) = \langle \bar{\nabla}_{e_i} \bar{\nabla} u, e_i\rangle.$$

This concludes the proof of Lemma 2.1. □

Lemma 2.2 (*Lemma* 3 *of* [4]). *Let* $\bar{\nabla}$ *be the connection of* $(H^m(-1), \mathrm{e}^{-f}\mathrm{d}v)$ *and* $\bar{g}$ *be its metric tensor. Then we have*

$$\bar{\nabla}^2 \cosh r = (\cosh r)\bar{g}, \tag{2.3}$$

where r *is the distance function on* $H^m(-1)$ *measured from a fixed point in* $H^m(-1)$.

Lemma 2.3. *Let* M *be an* n*-dimensional submanifold of the weighted Riemannian manifold* $(H^m(-1), \mathrm{e}^{-f}\mathrm{d}v)$. *Then*

$$\Delta_f(\cosh r) = n\cosh r + (\sinh r)\langle \mathbf{H}_f - \bar{\nabla} f, \bar{\nabla} r\rangle, \tag{2.4}$$

where r *is the distance function measured from a fixed point in* $H^m(-1)\backslash M$.

Proof of Lemma 2.3. Using Lemma 2.2, we obtain

$$\bar{\Delta}\cosh r = m(\cosh r), \qquad \sum_{\alpha=n+1}^{m} (\bar{\nabla}^2 \cosh r)(e_\alpha, e_\alpha) = (m-n)\cosh r. \quad (2.5)$$

Hence, we have from (2.1) by replacing $u = \cosh r$ that

$$\begin{aligned}
\Delta_f(\cosh r) &= \bar{\Delta}_f(\cosh r) + \langle \mathbf{H}_f, \bar{\nabla}(\cosh r)\rangle - \sum_{\alpha=n+1}^{m} (\bar{\nabla}^2 \cosh r)(e_\alpha, e_\alpha) \\
&= \bar{\Delta}(\cosh r) - \langle \bar{\nabla} f, \bar{\nabla}(\cosh r)\rangle + \langle \mathbf{H}_f, \bar{\nabla}(\cosh r)\rangle \\
&\quad - \sum_{\alpha=n+1}^{m} (\bar{\nabla}^2 \cosh r)(e_\alpha, e_\alpha) \\
&= n\cosh r - (\sinh r)\langle \bar{\nabla} f, \bar{\nabla} r\rangle + (\sinh r)\langle \mathbf{H}_f, \bar{\nabla} r\rangle, \quad (2.6)
\end{aligned}$$

which concludes the proof of Lemma 2.3. □

Lemma 2.4. *Let M be an n-dimensional submanifold of the weighted Riemannian manifold $(H^m(-1), e^{-f} dv)$. Then*

$$\Delta_f r = (n - |\nabla r|^2)\coth r + \langle \mathbf{H}_f - \bar{\nabla} f, \bar{\nabla} r\rangle, \quad (2.7)$$

where r is the distance function measured from a fixed point in $H^m(-1)\backslash M$.

Proof of Lemma 2.4. Note that

$$\Delta(\cosh r) = \mathrm{div}(\nabla \cosh r) = \mathrm{div}(\sinh r \nabla r) = (\sinh r)\Delta r + (\cosh r)|\nabla r|^2.$$

It follows that

$$\begin{aligned}
\Delta_f(\cosh r) &= \Delta(\cosh r) - \nabla f \nabla(\cosh r) \\
&= \Delta(\cosh r) - (\sinh r)\langle \nabla f, \nabla r\rangle \\
&= (\sinh r)\Delta_f r + (\cosh r)|\nabla r|^2, \quad (2.8)
\end{aligned}$$

which combining with (2.4) gives

$$(\sinh r)\Delta_f r + (\cosh r)|\nabla r|^2 = n\cosh r + (\sinh r)\langle \mathbf{H}_f - \bar{\nabla} f, \bar{\nabla} r\rangle. \quad (2.9)$$

Then the desired result follows by dividing both sides of (2.9) by $\sinh r$. □

Now we are in a position to prove Theorem 1.1. Recall that $|\bar{\nabla} r|^2 = 1$. Thus, we obtain

$$|\nabla r|^2 \le |\bar{\nabla} r|^2 = 1.$$

By virtue of (2.7), we have

$$\begin{aligned}
\Delta_f r &= (n - |\nabla r|^2)\coth r + \langle \mathbf{H}_f - \bar{\nabla} f, \bar{\nabla} r\rangle \\
&\ge (n-1)\coth r - |\mathbf{H}_f - \bar{\nabla} f||\bar{\nabla} r| \\
&\ge (n-1) - \sup_M |\mathbf{H}_f - \bar{\nabla} f|. \quad (2.10)
\end{aligned}$$

For any $u \in C_0^\infty(M)$, a direction yields

$$\begin{aligned} \mathrm{div}_f(u^2\nabla r) &= \langle \nabla u^2, \nabla r\rangle + u^2\Delta_f r \\ &\geq -|\nabla u^2||\nabla r| + ((n-1) - \sup_M |\mathbf{H}_f - \bar{\nabla} f|)u^2 \\ &\geq -|\nabla u^2| + ((n-1) - \sup_M |\mathbf{H}_f - \bar{\nabla} f|)u^2 \\ &= -2|u||\nabla u| + ((n-1) - \sup_M |\mathbf{H}_f - \bar{\nabla} f|)u^2. \end{aligned} \tag{2.11}$$

For any positive constant $\varepsilon > 0$, applying the Young's inequality

$$-2|u||\nabla u| \geq -\varepsilon u^2 - \frac{1}{\varepsilon}|\nabla u|^2$$

into (2.11) gives

$$\mathrm{div}_f(u^2\nabla r) \geq -\frac{1}{\varepsilon}|\nabla u|^2 + ((n-1) - \sup_M |\mathbf{H}_f - \bar{\nabla} f| - \varepsilon)u^2. \tag{2.12}$$

Integrating (2.12) on M, one gets

$$\frac{1}{\varepsilon}\int_M |\nabla u|^2 \,\mathrm{d}\mu \geq ((n-1) - \sup_M |\mathbf{H}_f - \bar{\nabla} f| - \varepsilon)\int_M u^2 \,\mathrm{d}\mu. \tag{2.13}$$

That is,

$$\int_M |\nabla u|^2 \,\mathrm{d}\mu \geq \varepsilon((n-1) - \sup_M |H_f - \bar{\nabla} f| - \varepsilon)\int_M u^2 \,\mathrm{d}\mu. \tag{2.14}$$

Maximizing the quadratic function $\varepsilon[(n-1) - \sup_M |H_f - \bar{\nabla} f| - \varepsilon]$ by taking

$$\varepsilon = \frac{(n-1) - \sup_M |\mathbf{H}_f - \bar{\nabla} f|}{2}$$

in (2.14) gives

$$\int_M |\nabla u|^2 \,\mathrm{d}\mu \geq \frac{((n-1) - \sup_M |\mathbf{H}_f - \bar{\nabla} f|)^2}{4}\int_M u^2 \,\mathrm{d}\mu. \tag{2.15}$$

Therefore, we have from (2.15),

$$\lambda_1(\Delta_f) \geq \frac{((n-1) - \sup_M |\mathbf{H}_f - \bar{\nabla} f|)^2}{4}, \tag{2.16}$$

which concludes the proof of the estimate (1.7) in Theorem 1.1. Since $\mathbf{H}_f = \mathbf{H} + (\bar{\nabla} f)^\perp$, we have

$$|\mathbf{H}_f - \bar{\nabla} f|^2 = |\mathbf{H} - (\bar{\nabla} f)^\top|^2 = |\mathbf{H}|^2 + |(\bar{\nabla} f)^\top|^2.$$

Hence, (2.16) can be written as

$$\lambda_1(\Delta_f) \geq \frac{\left((n-1) - \sup_M \sqrt{|\mathbf{H}|^2 + |(\bar{\nabla} f)^\top|^2}\right)^2}{4}. \tag{2.17}$$

We complete the proof of Theorem 1.1.

Acknowledgements

The authors would like to thank the referee for some helpful comments which made this paper more readable. The research of the first author is supported by NSFC (Nos 11371018, 11171091, 11671121), Henan Provincial Key Teacher (No. 2013GGJS-057) and IRTSTHN (14IRTSTHN023). The research of the second author is supported by NSFC (No. 11401179) and Henan Provincial Education Department (No. 14B110017).

References

[1] Bakry D and Emery M, Diffusion hypercontractives, *Sém. Prob. XIX Lect. Notes in Math.* **1123** (1985) 177–206

[2] Batista M, Cavalcante M P and Pyo J, Some isoperimetric inequalities and eigenvalue estimates in weighted manifolds, *J. Math. Anal. Appl.* **419** (2014) 617–626

[3] Cheung L F and Leung P F, Eigenvalue estimates for submanifolds with bounded mean curvature in the hyperbolic space, *Math. Z.* **236** (2001) 525–530

[4] Choe J and Gulliver R, Isoperimetric inequalities on minimal submanifolds of space forms, *Manuscr. Math.* **77** (1992) 169–189

[5] Li X-D, Liouville theorems for symmetric diffusion operators on complete Riemannian manifolds, *J. Math. Pures Appl.* **84** (2005) 1361–1995

[6] Li H Z and Wei Y, f-minimal surface and manifold with positive m-Bakry–Émery Ricci curvature, *J. Geom. Anal.* **25** (2015) 421–435

[7] McKean H P, An upper bound to the spectrum of Δ on a manifold of negative curvature, *J. Diff. Geom.* **4** (1970) 359–366

[8] Wei G and Wylie W, Comparison geometry for the Bakry–Émery Ricci tensor, *J. Diff. Geom.* **83** (2009) 377–405

COMMUNICATING EDITOR: Mj Mahan