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Diliberto–Straus algorithm for the uniform approximation by a sum of two algebras

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Abstract. In 1951, Diliberto and Straus [5] proposed a levelling algorithm for the uniform approximation of a bivariate function, defined on a rectangle with sides parallel to the coordinate axes, by sums of univariate functions. In the current paper, we consider the problem of approximation of a continuous function defined on a compact Hausdorff space by a sum of two closed algebras containing constants. Under reasonable assumptions, we show the convergence of the Diliberto–Straus algorithm. For the approximation by sums of univariate functions, it follows that Diliberto–Straus's original result holds for a large class of compact convex sets.

Keywords. Uniform approximation; levelling algorithm; proximity map; bolt.

Mathematics Subject Classifications: 41A30, 41A65, 46B28, 65D15.

1. Introduction

Let E be a Banach space and U and V be closed subspaces thereof. In addition, let A and B be proximity maps (best approximation operators) acting from E onto U and V, respectively. We are interested in algorithmic methods for computing the distance to a given element $z \in E$ from $U + V$. Historically, there is a procedure called the levelling algorithm. This procedure can be described as follows: Starting with $z_1 = z$ compute $z_2 = z_1 - Az_1$, $z_3 = z_2 - Bz_2$, $z_4 = z_3 - Az_3$, and so forth. Clearly, $z - z_n \in U + V$ and the sequence $\{\|z_n\|\}_{n=1}^{\infty}$ is nonincreasing. The question is if and when $\|z_n\|$ converges to the error of approximation from $U + V$?

In 1933, von Neumann [23] proved that in Hilbert spaces the levelling algorithm is always effective for any pair of closed subspaces. But in general, for Banach spaces, this method needs additional favorable conditions to be effective. A general result of Golomb [9] (see also, p. 57 of [18]) is formulated as follows

Theorem 1.1 [9]. *Let U and V be subspaces of a Banach space having central proximity maps A and B, respectively, and $U + V$ be closed. Then the sequence $\{z_n\}$ converges in norm to the error of approximation from $U + V$, that is, $\|z_n\| \downarrow \operatorname{dist}(z, U + V)$.*

A proximity map A from a Banach space E onto a subspace U is called a central proximity map if for all $z \in E$ and $x \in U$ (see Chap. 4 of [18]),

$$\|z - Az + x\| = \|z - Az - x\|.$$

It should be noted that in a uniformly convex and uniformly smooth Banach space it was proved by Deutsch [4] that the above algorithm converges for two closed subspaces U and V if the sum $U + V$ is closed. He also noted that in any normed linear space that is not smooth one can always construct two linear subspaces for which the algorithm does not converge.

In the Banach space setting, the levelling algorithm goes under various names. It is fairly often called the Diliberto–Straus algorithm. Diliberto and Straus [5] were the first to consider this algorithm in the space of continuous functions. They proved that for the problem of uniform approximation of a bivariate function, defined on a rectangle with sides parallel to the coordinate axes, by sums of univariate functions, the sequence produced by the levelling algorithm converges to the desired quantity. Later this algorithm was generalized to continuous functions defined on a Cartesian product of two compact spaces (see [17, 18]). Golomb [9] observed that the centrality property of proximity maps plays a key role in the analysis of Diliberto and Straus. It should be remarked that the concept 'central proximity map' is due to Golomb. We refer the readers to the monographs by Light and Cheney [18] and by Khavinson [14] for interesting discussions around the Diliberto-Straus algorithm.

Let X be a compact Hausdorff space. In the current paper, we consider the Diliberto–Straus algorithm in the problem of approximating from a sum of two closed subalgebras of $C(X)$ that contain the constants. Under mild and reasonable assumptions, we prove that the sequence produced by the levelling algorithm converges to the error of approximation.

2. Main result

Let X be a compact Hausdorff space, $C(X)$ be the space of real-valued continuous functions on X and $A_1 \subset C(X)$, $A_2 \subset C(X)$ be two closed algebras that contain the constants. Define the equivalence relation R_i, $i = 1, 2$, for elements in X by setting

$$a \overset{R_i}{\sim} b \text{ if } f(a) = f(b) \quad \text{for all } f \in A_i. \tag{2.1}$$

Then, for each $i = 1, 2$, the quotient space $X_i = X/R_i$ with respect to the relation R_i, equipped with the quotient space topology, is compact. In addition, the natural projections $s : X \to X_1$ and $p : X \to X_2$ are continuous. Note that the quotient spaces X_1 and X_2 are not only compact but also Hausdorff (see p. 54 of [14]). Obviously, in view of the Stone–Weierstrass theorem,

$$\begin{aligned} A_1 &= \{f(s(x)) : f \in C(X_1)\}, \\ A_2 &= \{g(p(x)) : g \in C(X_2)\}. \end{aligned}$$

In this paper, we consider the problem of approximation of a function $h \in C(X)$ from the set $A_1 + A_2$. More precisely, we are interested in algorithmic methods for computing the error of approximation

$$E(h) = \inf_{w \in A_1 + A_2} \|h - w\|.$$

In the sequel, we assume that the algebras A_1 and A_2 obey the following property, which we call the C-property: For any function $h \in C(X)$, the real functions

$$f_1(a) = \max_{\substack{x \in X \\ s(x)=a}} h(x), \quad f_2(a) = \min_{\substack{x \in X \\ s(x)=a}} h(x), \quad a \in X_1,$$
$$g_1(b) = \max_{\substack{x \in X \\ p(x)=b}} h(x), \quad g_2(b) = \max_{\substack{x \in X \\ p(x)=b}} h(x), \quad b \in X_2$$

are continuous on the appropriate sets X_1 and X_2. Note that for a given compact space X many subalgebras of $C(X)$ do not possess the C-property. For example, in the above special case of $\mathbb{R}^2$, the above max functions are not continuous if $X = [0, 1]\times[0, 2]\cup[1, 2]\times[0, 1]$ and $h(x, y) = xy$. In order for the C-property to be fulfilled, the natural quotient mappings $s(x)$ and $p(x)$ should satisfy certain conditions. The following proposition provides a sufficient condition of such type for compact sequential spaces. A sequential space is a topological space with the property that a set is open iff every sequence x_n converging to a point in the set is, itself, eventually in the set (i.e. there exists N such that x_n is in the set for all $n \geq N$). Note that some essential properties and full characterization of sequential spaces via quotient mappings were given in the fundamental papers of Franklin (see [7, 8]).

PROPOSITION 2.1

Let X be a compact sequential Hausdorff space and A be a closed subalgebra of $C(X)$ that contains the constants. Let X_1 be a quotient space generated by the equivalence relation (2.1) and $s : X \to X_1$ be the natural quotient mapping. Then the functions f_1 and f_2 are continuous on X_1 for any $h \in C(X)$ if for any two points x and y with $s(x) = s(y)$ and any sequence $\{x_n\}_{n=1}^{\infty}$ tending to x, there exists a sequence $\{y_n\}_{n=1}^{\infty}$ tending to y such that $s(y_n) = s(x_n)$, for all $n = 1, 2, \ldots$.

Proof. On the contrary, suppose that the above hypothesis on the quotient mapping s holds, but one of the functions f_1 and f_2 is not continuous. Without loss of generality assume that f_1 is not continuous on X_1. Let f_1 be discontinuous at a point $a_0 \in X_1$. Note that by the result of Franklin (Proposition 1.2 of [7]), a quotient image of a sequential space is sequential. Thus the quotient space X_1 is sequential. Then there exists a number $\varepsilon > 0$ and a sequence $\{a_n\}_{n=1}^{\infty} \subset X_1$ tending to a_0, such that

$$|f_1(a_n) - f_1(a_0)| > \varepsilon, \tag{2.2}$$

for all $n = 1, 2, \ldots$. Since the function h is continuous on X, there exist points $x_n \in X$, $n = 0, 1, 2, \ldots$, such that $h(x_n) = f_1(a_n)$, $s(x_n) = a_n$, for $n = 0, 1, 2, \ldots$. Thus the inequality (2.2) can be written as

$$|h(x_n) - h(x_0)| > \varepsilon, \tag{2.3}$$

for all $n = 1, 2, \ldots$. Since X is compact and sequential, it is sequentially compact (see Theorem 3.10.31 of [6]); hence the sequence $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence. Without loss of generality assume that $\{x_n\}_{n=1}^{\infty}$ itself converges to a point $y_0 \in X$. Then $s(x_n) \to s(y_0)$, as $n \to \infty$. But by the assumption, we also have $s(x_n) \to s(x_0)$, as $n \to \infty$. Therefore, since X_1 is Hausdorff, $s(y_0) = s(x_0) = a_0$. Note that x_0 and y_0

cannot be the same point, since the equality $x_0 = y_0$ violates the condition (2.3). By the hypothesis of the proposition, we must have a sequence $\{z_n\}_{n=1}^{\infty}$ such that $z_n \to x_0$ and

$$s(z_n) = s(x_n),$$

for all $n = 1, 2, \ldots$. Since $s(x_n) = a_n$, $n = 1, 2, \ldots$, and on each level set $\{x \in X : s(x) = a_n\}$, the function h takes its maximum value at x_n we obtain that

$$h(z_n) \le h(x_n), n = 1, 2, \ldots.$$

Taking the limit in the last inequality as $n \to \infty$, gives us the new inequality

$$h(x_0) \le h(y_0). \tag{2.4}$$

Recall that on the level set $\{x \in X : s(x) = a_0\}$, the function h takes its maximum at x_0. Thus from (2.4) we conclude that $h(x_0) = h(y_0)$. This last equality contradicts the choice of the positive ε in (2.3), since $h(x_n) \to h(y_0)$, as $n \to \infty$. The obtained contradiction shows that the function f_1 is continuous on X_1. In the same way one can prove that f_2 is continuous on X_1. □

Example 1. It is not difficult to see that the inner product function $s(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$, where $\mathbf{a}$ is a nonzero vector in $\mathbb{R}^d$, satisfies the hypothesis of Proposition 2.1, if $\mathbf{x}$ varies in a compact convex set $Q \subset \mathbb{R}^d$. Indeed, let $\mathbf{x}_0$ and $\mathbf{y}_0$ be any two points in Q such that $s(\mathbf{x}_0) = s(\mathbf{y}_0)$. Take any sequence $\{\mathbf{x}_n\}_{n=1}^{\infty} \subset Q$, $\mathbf{x}_n \to \mathbf{x}_0$, as $n \to \infty$. We must show that there exists a sequence $\{\mathbf{y}_n\}_{n=1}^{\infty} \subset Q$ with the properties that $\mathbf{y}_n \to \mathbf{y}_0$, as $n \to \infty$, and $s(\mathbf{x}_n) = s(\mathbf{y}_n)$, for $n = 1, 2, \ldots$. To show this, first note that we have points $\mathbf{z}_1, \mathbf{z}_2 \in Q$ such that $s(\mathbf{x}_n) \in [s(\mathbf{z}_1), s(\mathbf{z}_2)]$, for all $n = 1, 2, \ldots$. One of the numbers, or both, $s(\mathbf{z}_1)$ and $s(\mathbf{z}_2)$ may equal to $s(\mathbf{x}_0)$, and in this case we assume that $\mathbf{z}_1$ and/or $\mathbf{z}_2$ coincides with $\mathbf{y}_0$. Consider now the line segments $[\mathbf{z}_1, \mathbf{y}_0]$ and $[\mathbf{y}_0, \mathbf{z}_2]$ (which may be degenerated into the point $\mathbf{y}_0$). Set $L = [\mathbf{z}_1, \mathbf{y}_0] \cup [\mathbf{y}_0, \mathbf{z}_2]$. Since Q is convex, $L \subset Q$. The function s is continuous on L. Hence by the Intermediate Value Theorem, there exists a sequence $\{\mathbf{y}_n\}_{n=1}^{\infty} \subset L$ such that $s(\mathbf{y}_n) = s(\mathbf{x}_n)$, for $n = 1, 2, \ldots$. Since $s(\mathbf{x}_n) \to s(\mathbf{x}_0)$ and $s(\mathbf{x}_0) = s(\mathbf{y}_0)$, it follows that $s(\mathbf{y}_n) \to s(\mathbf{y}_0)$, as $n \to \infty$. Now from this fact and the inclusion $\{\mathbf{y}_n\}_{n=1}^{\infty} \subset L$ we easily derive that $\mathbf{y}_n \to \mathbf{y}_0$, as $n \to \infty$.

Example 2. Let Q be a compact convex set in $\mathbb{R}^2$. From Example 1 it follows that the C-property holds for the algebras of univariate functions $A_1 = \{f(x) : f \in Q_x\}$ and $A_2 = \{g(y) : g \in Q_y\}$, where Q_x and Q_y are projections of Q into the coordinate axes x and y, respectively.

Example 3. The hypothesis of Proposition 2.1 strictly depends on the considered space X. That is, the natural quotient mapping s may satisfy this hypothesis, but for many closed subsets $E \subset X$, it may happen that the restriction of s to E no longer satisfies it. For example, let K be the unit square in the xy plane and $K_1 = [0, 1] \times [0, \frac{1}{2}] \cup [0, \frac{1}{2}] \times [0, 1]$. Consider the algebra U of univariate functions depending only on the variable x. Clearly, the coordinate projection $s(x, y) = x$ satisfies the hypothesis if s is considered over K. This is not true if s is considered over the set K_1. Indeed, for the sequence $\{(\frac{1}{2} + \frac{1}{n+1}, \frac{1}{2})\}_{n=1}^{\infty} \subset K_1$, which tends to $(\frac{1}{2}, \frac{1}{2})$, we cannot find a sequence $\{(x_n, y_n)\}_{n=1}^{\infty} \subset K_1$ tending to $(\frac{1}{2}, 1)$ such that $x_n = \frac{1}{2} + \frac{1}{n+1}$, $n = 1, 2, \ldots$.

Define the following operators

$$F : C(X) \to A_1, \quad Fh(a) = \frac{1}{2}\left(\max_{\substack{x \in X \\ s(x)=a}} h(x) + \min_{\substack{x \in X \\ s(x)=a}} h(x) \right), \quad \text{for all } a \in X_1$$

and

$$G : C(X) \to A_2, \quad Gh(b) = \frac{1}{2}\left(\max_{\substack{x \in X \\ p(x)=b}} h(x) + \min_{\substack{x \in X \\ p(x)=b}} h(x) \right), \quad \text{for all } b \in X_2.$$

Since the algebras $A_i, i = 1, 2$, both enjoy the C-property, for each function $h \in C(X)$, the functions Fh and Gh are continuous on X_1 and X_2, respectively. Via the quotient mappings s and p, these functions Fh and Gh can be considered also as functions defined on X. Since s and p are continuous on X, the functions Fh and Gh are continuous on X and hence belong to the algebras A_1 and A_2 respectively.

The following theorem plays a key role in the proof of our main result (Theorem 2.3).

Theorem 2.2. *Assume that X is a compact Hausdorff space and A_i, $i = 1, 2$, are closed subalgebras of $C(X)$ that contain the constants. In addition, assume that the C-property holds for these subalgebras. Then the operators F and G are central proximity maps onto A_1 and A_2, respectively. In addition, these operators are non-expansive. That is,*

$$\|Fv_1 - Fv_2\| \le \|v_1 - v_2\| \quad \text{and} \quad \|Gv_1 - Gv_2\| \le \|v_1 - v_2\|,$$

for all $v_1, v_2 \in C(X)$.

Proof. We prove this theorem for the operator F. A proof for G can be carried out in the same way.

Clearly, on the level set $\{x \in X : s(x) = a\}$, the constant $(Fh)(a)$ is a best approximation to h, among all constants. Varying over $a \in X_1$, we obtain a best approximating function $Fh : X \to \mathbb{R}$, which is due to the C-property in the algebra A_1.

Now let us prove that the proximity map F is a central proximity map. In other words, we must prove that for any functions $h \in C(X)$ and $f = f(s(x)) \in A_1$,

$$\|h - Fh - f\| = \|h - Fh + f\|. \tag{2.5}$$

Put $u = h - Fh$. There exists a point $x_0 \in X$ such that

$$\|u + f\| = |u(x_0) + f(s(x_0))|.$$

First assume that $|u(x_0) + f(s(x_0))| = u(x_0) + f(s(x_0))$. Since $Fu = 0$,

$$\max_{\substack{x \in X \\ s(x)=a}} u(x) = - \min_{\substack{x \in X \\ s(x)=a}} u(x), \quad \text{for all } a \in X_1. \tag{2.6}$$

Let

$$\min_{\substack{x \in X \\ s(x)=s(x_0)}} u(x) = u(x_1). \tag{2.7}$$

Then from (2.6) and (2.7) it follows that

$$-u(x_1) \geq u(x_0).$$

Taking the last inequality and the equality $s(x_1) = s(x_0)$ into account, we may write

$$\|u - f\| \geq f(s(x_1)) - u(x_1) \geq f(s(x_0)) + u(x_0) = \|u + f\| . \tag{2.8}$$

Changing in (2.8) the function f to $-f$ gives the reverse inequality $\|u + f\| \geq \|u - f\|$. Thus (2.5) holds.

Note that if $|u(x_0) + f(s(x_0))| = -(u(x_0) + f(s(x_0)))$, then by replacing eq. (2.7) by

$$\max_{\substack{x \in X \\ s(x) = s(x_0)}} u(x) = u(x_1), \tag{2.9}$$

we will derive from (2.6) and (2.9) that $u(x_1) \geq -u(x_0)$. This inequality is then used to obtain the estimation

$$\|u - f\| \geq -(f(s(x_1) - u(x_1)) \geq -(f(s(x_0)) + u(x_0)) = \|u + f\| ,$$

which in turn yields (2.5). The centrality has been proven.

Now we prove that the operator F is non-expansive. First note that it is nondecreasing. That is, if for all $x \in X$, $h_1(x) \leq h_2(x)$, then $Fh_1(s(x)) \leq Fh_2(s(x))$. Besides, $F(h + c) = Fh + c$, for any real number c. Let now v_1 and v_2 be arbitrary functions in $C(X)$. Put $c = \|v_1 - v_2\|$. Then for any $x \in X$, we can write

$$v_2(x) - c \leq v_1(x) \leq v_2(x) + c$$

and further,

$$Fv_2(s(x)) - c \leq Fv_1(s(x)) \leq Fv_2(s(x)) + c.$$

From the last inequality, we obtain that

$$\|Fv_1 - Fv_2\| \leq c = \|v_1 - v_2\| .$$

Thus the operator F is non-expansive. □

Consider the iterations

$$h_1(x) = h(x),\ h_{2n} = h_{2n-1} - Fh_{2n-1},\ h_{2n+1} = h_{2n} - Gh_{2n},\ n = 1, 2, \ldots .$$

From Theorem 2.2 and the above-mentioned general result of Golomb (see Theorem 1.1) one can obtain the following theorem.

Theorem 2.3. *Let all the assumptions of Theorem* 2.2 *hold and* $A_1 + A_2$ *be closed in* $C(X)$. *Then* $\|h_n\|$ *converges to the error of approximation* $E(h)$.

Does the Diliberto and Straus algorithm converge without the closedness assumption on the sum $A_1 + A_2$? We do not yet know a complete answer to this question. Recall that we obtain Theorem 2.3 by using the general result of Golomb (see Theorem 1.1), in which the ‘closedness’ is a major hypothesis. Below we give a different (and independent of Golomb’s result) proof of this theorem, where it is shown how we use the mentioned closedness. It should be remarked that in the simplest case of approximation by sums of univariate functions, the known classical proofs of the Diliberto–Straus algorithm require that one could form a closed bolt (for this terminology, see below) from an arbitrary bolt $(x_1, \ldots, x_n)$ of a given compact set $Q \subset \mathbb{R}^2$ by adding a point $y \in Q$, whose first and second coordinates are equal to that of x_1 and x_n respectively (see [5, 14, 17]). Note that this point may not lie in Q, unless Q is a Cartesian product of two compact sets. The main idea behind our proof is to make use of weak* cluster points of some sequence of 'unclosed bolt functionals' instead of considering only 'closed bolt functionals'. We hope that this idea can be useful in future attempts to prove the convergence of the Diliberto–Straus algorithm without the closedness assumption.

For further analysis we need the following objects called 'bolts of lightning' or simply 'bolts'.

DEFINITION 2.2 [19, 20]

Let X be a compact space and A_i, $i = 1, 2$, be subalgebras of $C(X)$ that contain the constants. Let s and p be the natural quotient mappings generated by the equivalence relation (2.1). A finite ordered set $l = \{x_1, x_2, \ldots, x_n\} \subset X$, where $x_i \neq x_{i+1}$, with either $s(x_1) = s(x_2), p(x_2) = p(x_3), s(x_3) = s(x_4), \ldots$ or $p(x_1) = p(x_2), s(x_2) = s(x_3), p(x_3) = p(x_4), \ldots$ is called a bolt with respect to (A_1, A_2).

If in a bolt $\{x_1, \ldots, x_n, x_{n+1}\}$, $x_{n+1} = x_1$ and n is an even number, then the bolt $\{x_1, \ldots, x_n\}$ is said to be closed. Bolts, in the case when $X \subset \mathbb{R}^2$ and the algebras A_1 and A_2 coincide with the spaces of univariate functions $\varphi(x)$ and $\psi(y)$, respectively, are geometrically explicit objects. In this case, a bolt is a finite ordered set $\{x_1, x_2, \ldots, x_n\}$ in $\mathbb{R}^2$ with the line segments $[x_i, x_{i+1}]$, $i = 1, \ldots, n$, alternatively perpendicular to the x and y axes (see [1, 11–14, 19]). These objects were first introduced by Diliberto and Straus in [5] and they are called 'permissible lines'. They appeared further in a number of papers with several different names such as 'paths' [17, 18], 'trips' [20, 21], 'links' [3, 15, 16]. The term 'bolt of lightning' was due to Arnold [1].

With each bolt $l = \{x_1, \ldots, x_n\}$ with respect to (A_1, A_2), we associate the following bolt functional

$$r_l(h) = \frac{1}{n} \sum_{i=1}^{n} (-1)^{n+1} h(x_i).$$

It is an exercise to check that r_l is a linear bounded functional on $C(X)$ with the norm $\|r_l\| \leq 1$ and $\|r_l\| = 1$ if and only if the set of points x_i with odd indices i does not intersect with the set of points with even indices. Besides, if l is closed, then $r_l \in (A_1 + A_2)^{\perp}$, where $(A_1 + A_2)^{\perp}$ is the annihilator of the subspace $A_1 + A_2 \subset C(X)$. If l is not closed, then r_l is generally not an annihilating functional. However, it obeys the following important inequality

$$|r_l(f)| \leq \frac{2}{n} \|f\|, \tag{2.10}$$

for all $f \in A_i$, $i = 1, 2$. This inequality means that for bolts l with sufficiently large number of points, r_l behaves like an annihilating functional.

Now we are able to prove Theorem 2.3.

Proof. First, let us write the above iteration in the following form

$$h_1 = h, h_{n+1} = h_n - q_n,$$

where $q_n = Fh_n$, if n is odd; $q_n = Gh_n$, if n is even.

Introduce the functions

$$\begin{aligned} u_n &= q_1 + \cdots + q_{2n-1}, \\ v_0 &= 0, v_n = q_2 + \cdots + q_{2n}, n = 1, 2, \ldots. \end{aligned}$$

Clearly, $u_n \in A_1$ and $v_n \in A_2$. Besides, $h_{2n} = h - u_n - v_{n-1}$ and $h_{2n+1} = h - u_n - v_n$, for $n = 1, 2, \ldots$.

It is easy to see that the following inequalities hold

$$\|h_1\| \geq \|h_2\| \geq \|h_3\| \geq \cdots \geq E(h).$$

Therefore, there exists the limit

$$M = \lim_{n\to\infty} \|h_n\| \geq E(h).$$

It is a consequence of the Hahn–Banach extension theorem that

$$E(h) = \sup_{\substack{r \in (A_1 + A_2)^{\perp} \\ \|r\| \leq 1}} |r(h)|,$$

where the sup is attained by some functional. To complete the proof it is enough to show that for any $\varepsilon > 0$, there exists a functional r_0 such that $r_0 \in (A_1 + A_2)^{\perp}$, $\|r_0\| \leq 1$ and $|r_0(h)| \geq M - \varepsilon$.

Let ε be an arbitrarily small positive real number. For each positive integer $k = 1, 2, \ldots$, set $\delta_k = \frac{\varepsilon}{2^{2k}}$. There exists a number n_k such that for all $n \geq n_k$,

$$\|h_n\| \leq M + \delta_k.$$

Without loss of generality, we may assume that n_k is even. In the following, for each k, we are going to construct a bolt $l_k = \{x_1^k, \ldots, x_{2k+1}^k\}$ with the property that $|r_{l_k}(h_{n_k})| \geq M - \varepsilon$. This will lead us to the above mentioned functional r_0.

Since $Fh_{2m} = 0$, for $m = 1, 2, \ldots$, and $n_k + 2k$ is an even number,

$$\max_{\substack{x \in X \\ s(x) = a}} h_{n_k+2k}(x) = - \min_{\substack{x \in X \\ s(x) = a}} h_{n_k+2k}(x), \quad \text{for all } a \in X_1.$$

Then there exist points x_1 and x_2 such that $h_{n_k+2k}(x_1) = \|h_{n_k+2k}\|$, $h_{n_k+2k}(x_2) = -\left\|h_{n_k+2k}\right\|$ and $s(x_1) = s(x_2)$. This can be written in the form

$$h_{n_k+2k-1}(x_1) - Fh_{n_k+2k-1}(s(x_1)) = \|h_{n_k+2k}\|, \tag{2.11}$$

$$h_{n_k+2k-1}(x_2) - Fh_{n_k+2k-1}(s(x_2)) = -\|h_{n_k+2k}\|. \tag{2.12}$$

Therefore,

$$\begin{aligned} Fh_{n_k+2k-1}(s(x_1)) &= Fh_{n_k+2k-1}(s(x_2)) \\ &= h_{n_k+2k-1}(x_1) - \|h_{n_k+2k}\| \leq \|h_{n_k+2k-1}\| - M \leq \delta_k \end{aligned}$$

and

$$Fh_{n_k+2k-1}(s(x_1)) = Fh_{n_k+2k-1}(s(x_2))$$
$$= h_{n_k+2k-1}(x_2) + \|h_{n_k+2k}\| \geq -(\|h_{n_k+2k-1}\| - M) \geq -\delta_k,$$

i.e.,

$$-\delta_k \leq Fh_{n_k+2k-1}(s(x_1)) = Fh_{n_k+2k-1}(s(x_2)) \leq \delta_k. \tag{2.13}$$

From (2.11)–(2.13) we obtain that

$$h_{n_k+2k-1}(x_1) = Fh_{n_k+2k-1}(s(x_1)) + \|h_{n_k+2k}\| \geq M - \delta_k, \tag{2.14}$$
$$h_{n_k+2k-1}(x_2) = Fh_{n_k+2k-1}(s(x_1)) - \|h_{n_k+2k}\| \leq -M + \delta_k. \tag{2.15}$$

Since $Gh_{2m+1} = 0$, for $m = 1, 2, \ldots$,

$$\max_{\substack{x \in X \\ p(x) = b}} h_{n_k+2k-1}(x) = - \min_{\substack{x \in X \\ p(x) = b}} h_{n_k+2k-1}(x), \quad \text{for all } b \in X_2.$$

Then from (2.14) and (2.15) it follows that there exist points x_3 and x_3' satisfying the relations

$$h_{n_k+2k-1}(x_3) \geq M - \delta_k, \quad p(x_3) = p(x_2) \tag{2.16}$$

and

$$h_{n_k+2k-1}(x_3') \leq -M + \delta_k, \quad p(x_3') = p(x_1). \tag{2.17}$$

In the inequalities (2.14)–(2.17), replace h_{n_k+2k-1} by $h_{n_k+2k-2} - Gh_{n_k+2k-2}$. Then we have the following estimates.

$$h_{n_k+2k-2}(x_1) - Gh_{n_k+2k-2}(p(x_1)) \geq M - \delta_k, \tag{2.18}$$
$$h_{n_k+2k-2}(x_2) - Gh_{n_k+2k-2}(p(x_2)) \leq -M + \delta_k, \tag{2.19}$$
$$h_{n_k+2k-2}(x_3) - Gh_{n_k+2k-2}(p(x_3)) \geq M - \delta_k, \tag{2.20}$$
$$h_{n_k+2k-2}(x_3') - Gh_{n_k+2k-2}(p(x_3')) \leq -M + \delta_k. \tag{2.21}$$

The following inequalities are obvious.

$$h_{n_k+2k-2}(x_1) \leq \|h_{n_k+2k-2}\| \leq M + \delta_k,$$
$$h_{n_k+2k-2}(x_3) \leq \|h_{n_k+2k-2}\| \leq M + \delta_k,$$
$$h_{n_k+2k-2}(x_2) \geq -\|h_{n_k+2k-2}\| \geq -M - \delta_k,$$
$$h_{n_k+2k-2}(x_3') \geq -\|h_{n_k+2k-2}\| \geq -M - \delta_k.$$

Considering these obvious inequalities in (2.18)–(2.21), we can write that

$$-2\delta_k \leq Gh_{n_k+2k-2}(p(x_1)) = Gh_{n_k+2k-2}(p(x_3')) \leq 2\delta_k, \tag{2.22}$$
$$-2\delta_k \leq Gh_{n_k+2k-2}(p(x_2)) = Gh_{n_k+2k-2}(p(x_3)) \leq 2\delta_k. \tag{2.23}$$

Taking into account (2.22) and (2.23) in the estimates (2.18)–(2.21), we obtain that

$$h_{n_k+2k-2}(x_1) \geq M - 3\delta_k,$$
$$h_{n_k+2k-2}(x_2) \leq -M + 3\delta_k,$$

$$h_{n_k+2k-2}(x_3) \geq M - 3\delta_k, \tag{2.24}$$

$$h_{n_k+2k-2}(x_3') \leq -M + 3\delta_k. \tag{2.25}$$

Now since

$$\max_{\substack{x \in X \\ s(x)=a}} h_{n_k+2k-2}(x) = -\min_{\substack{x \in X \\ s(x)=a}} h_{n_k+2k-2}(x), \quad \text{for all } a \in X_1,$$

from (2.24) and (2.25) it follows that there exist points x_4 and x_4' satisfying

$$h_{n_k+2k-2}(x_4) \leq -M + 3\delta_k,\ s(x_4) = s(x_3)$$

and

$$h_{n_k+2k-2}(x_4') \geq M - 3\delta_k,\ s(x_4') = s(x_2).$$

Repeating the above process for the function $h_{n_k+2k-2}(x) = h_{n_k+2k-3}(x) - Fh_{n_k+2k-3}(s(x))$, we obtain that

$$-4\delta_k \leq Fh_{n_k+2k-3}(s(x_i)) \leq 4\delta_k, \quad i = 1, 2, 3, 4 \tag{2.26}$$

and

$$\begin{aligned}
h_{n_k+2k-3}(x_1) &\geq M - 7\delta_k, \\
h_{n_k+2k-3}(x_2) &\leq -M + 7\delta_k, \\
h_{n_k+2k-3}(x_3) &\geq M - 7\delta_k, \\
h_{n_k+2k-3}(x_4) &\leq -M + 7\delta_k, \\
h_{n_k+2k-3}(x_4') &\geq M - 7\delta_k.
\end{aligned}$$

As above, we can find a point x_5 such that $p(x_5) = p(x_4)$ and

$$h_{n_k+2k-3}(x_5) \geq M - 7\delta_k.$$

Continuing this process until we reach the function h_{n_k}, we obtain the points $x_1, x_2, \ldots, x_{2k+2}$ satisfying $s(x_1) = s(x_2)$, $p(x_2) = p(x_3), \ldots, s(x_{2k+1}) = s(x_{2k+2})$. Clearly, these points, in the given order, form a bolt, which we denote by l_k. Note that in the considered process, we also deal with the points $x_3', x_4', \ldots$, but these points play only an auxiliary role: they are needed in obtaining the inequalities (2.22), (2.23), (2.26),.... At the points of the bolt l_k, the values of h_{n_k} obey the estimates

$$h_{n_k}(x_i) \geq M - (2^{2k} - 1)\delta_k \geq M - \varepsilon, \text{for } i = 1, 3, \ldots, 2k+1$$

and

$$h_{n_k}(x_j) \leq -M + (2^{2k} - 1)\delta_k \leq -M + \varepsilon, \text{ for } j = 2, 4, \ldots, 2k+2.$$

Using these inequalities, we can estimate the absolute value of $r_{l_k}(h_{n_k})$ as follows

$$|r_{l_k}(h_{n_k})| \geq \frac{(k+1)(M-\varepsilon) - (k+1)(-M+\varepsilon)}{2k+2} = M - \varepsilon. \tag{2.27}$$

We can also write the following obvious inequality

$$\begin{aligned}
|r_{l_k}(h)| &= |r_{l_k}(h_{n_k} + u_{n_k/2} + v_{n_k/2-1})| \\
&\geq |r_{l_k}(h_{n_k})| - |r_{l_k}(u_{n_k/2})| - |r_{l_k}(v_{n_k/2-1})|.
\end{aligned} \tag{2.28}$$

Considering (2.27) and (2.10) in (2.28) we obtain that

$$|r_{l_k}(h)| \geq M - \varepsilon - \frac{2}{2k+2}(\|u_{n_k/2}\| + \|v_{n_k/2-1}\|). \tag{2.29}$$

At this stage we use our assumption of closedness of the subspace $A_1 + A_2$. Note that a subspace $B = B_1 + B_2$ of a Banach space is closed if and only if there exists $K < \infty$ such that each element b in B has a representation $b = b_1 + b_2$, where $b_1 \in B_1$, $b_2 \in B_2$ and $\max(\|b_1\|, \|b_2\|) \leq K\|b\|$ (see [20]). Now, since $A_1 + A_2$ is closed, there exists a constant K such that

$$\|u_n\| + \|v_{n-1}\| \leq K\|u_n + v_{n-1}\|, \; n = 1, 2, \ldots. \tag{2.30}$$

Note that the sequence $\{\|u_n + v_{n-1}\|\}_{n=1}^{\infty}$ is uniformly bounded. Indeed,

$$\|u_n + v_{n-1}\| = \|h - h_{2n}\| \leq 2\|h\|, \text{ for all } n = 1, 2, \ldots. \tag{2.31}$$

From (2.29), (2.30) and (2.31) it follows that there exists a constant $C > 0$ such that

$$|r_{l_k}(h)| \geq M - \varepsilon - \frac{C}{2k+2}. \tag{2.32}$$

Thus for each positive integer k, we constructed bolts l_k and bolt functionals r_{l_k}, for which the inequality (2.32) holds. Note that $\left\|r_{l_k}\right\| \leq 1$, for all k. By the well-known result of functional analysis (any bounded set in E^*, dual for a separable Banach space E, is precompact in the weak* topology), the sequence $\left\{r_{l_k}\right\}_{k=1}^{\infty}$ has a weak* cluster point. Denote this point by r_0. Then from (2.32) it follows that

$$|r_0(h)| \geq M - \varepsilon.$$

The above inequality completes the proof. □

Remark 1. In Theorem 2.3, we use the closedness assumption. For a given compact space X, the closedness of $A_1 + A_2$ in $C(X)$ strictly depends on the internal structure of X. There are several results on closedness of a sum of two algebras (see [14, 20, 21]). The most explicit and practical result is due to Medvedev [21]. He showed that the sum $A_1 + A_2$ is closed in $C(X)$ if and only if the lengths of all irreducible bolts of X are uniformly bounded. A bolt $\{x_1, \ldots, x_m\}$ is called irreducible if there does not exist another bolt $\{y_1, \ldots, y_l\}$ with $y_1 = x_1$, $y_l = x_m$ and $l < m$. For example, the set of functions $\varphi(x) + \psi(y)$ defined on a compact set $Q \subset \mathbb{R}^2$ is closed in the space $C(Q)$ if Q has a vertical or horizontal bar (a bar is a closed segment in Q, projection of which into the x or y axis coincides with the projection of Q into the same axis). This is because any two different points of Q can be connected by means of a bolt (with respect to the coordinate projections), using at most 2 points of a bar. That is, in this case, the lengths of irreducible bolts are not more than 4. A similar argument can be applied to a compact set Q, which contains a broken line L, with segments parallel to the coordinate axes, such that the projections of Q and L into one of the coordinate axis coincide. At the same time, there are compact sets of simple structure, for which the closedness result does not hold. Take, for example, the triangle ABC with $A = (0, 0)$, $B = (2, 2)$ and $C = (1, 0)$. In this case, there is no number bounding the lengths of irreducible bolts $(A, \ldots, X_n) \subset ABC$, $n = 1, 2, \ldots$, provided that X_n tends to B, as $n \to \infty$.

Remark 2. There is a difference between the convergence $\|h_n\| \to E(h)$, as $n \to \infty$, and the convergence of the sequence $\{h - h_n\}_{n=1}^{\infty}$ to a best approximation to h from $A_1 + A_2$. The latter is much stronger. Theorem 2.3 does not guarantee that the sequence $\{h - h_n\}$ converges in the latter strong sense. We do not know if it converges to a best approximation to h from $A_1 + A_2$. The question, for which compact spaces X and subalgebras A_1 and A_2, the sequence of functions produced by the Diliberto–Straus algorithm converges to a best approximation is fair, but too difficult to solve in this generality. It should be noted that Aumann [2] proved that for a rectangle $S = [a, b] \times [c, d]$ in $\mathbb{R}^2$ and for any function $h(x_1, x_2) \in C(S)$, the sequence $\{h - h_n\}$ converges uniformly to a best approximation to h from the subspace

$$D = \{\varphi(x) + \psi(y) : \varphi \in C[a, b],\ \psi \in C[c, d]\} .$$

Aumann's proof is based on the equicontinuity of some families of functions, namely the families $\{u_n\}$ and $\{v_n\}$ (see the proof of Theorem 2.3). To show the equicontinuity, Aumann substantially uses the non-expansivness of the averaging operators

$$(M_x f)(y) = \frac{\max_x f + \min_x f}{2} \quad \text{and} \quad (M_y f)(x) = \frac{\max_y f + \min_y f}{2}.$$

In case of the rectangle S, for any function $f \in C(S)$ and two fixed points $x_1, x_2 \in [a, b]$, we can always write

$$|M_y f_1 - M_y f_2| \leq \|f_1 - f_2\| , \tag{2.33}$$

where $f_1 = f(x_1, y)$, $f_2 = f(x_2, y)$. But inequality (2.33) is not generally valid for other compact sets $Q \subset \mathbb{R}^2$. This is because for some fixed points x_1 and x_2, the projections of the sets $\{(x_1, y) \in Q\}$ and $\{(x_2, y) \in Q\}$ into the y axis do not coincide; hence $M_y f_1$ and $M_y f_2$ are the averaging operators over two different sets.

Remark 3. Let us explain why we consider the sum of only two algebras. The problem is that for a sum of more than two algebras, the appropriate generalization of the Diliberto–Straus algorithm does not hold. In fact, it does not hold even in the simplest case of approximation by sums of univariate functions. To be more precise, let E be the unit cube in $\mathbb{R}^n$ and A_i be a best approximation operator from the space of continuous functions $C(E)$ to the subspace of univariate functions $G_i = \{g_i(x_i) : g_i \in C[0, 1]\}$, $i = 1, \ldots, n$.

That is, for each function $f \in C(X)$, the function $A_i f$ is a best approximation to f from G_i. Set

$$Tf = (I - A_n)(I - A_{n-1}) \cdots (I - A_1) f,$$

where I is the identity operator. It is clear that

$$Tf = f - g_1 - g_2 - \cdots - g_n,$$

where g_k is a best approximation from G_k to the function $f - g_1 - g_2 - \cdots - g_{k-1}$, $k = 1, \ldots, r$. Consider powers of the operator T: T^2, T^3 and so on. What is the limit of $\|T^n f\|$as $n \to \infty$? One may expect that the sequence $\{\|T^n f\|\}_{n=1}^{\infty}$ converges to $E(f)$ (the error of approximation from $G_1 + \cdots + G_n$), as in the case $n = 2$. This conjecture was first proposed in 1951 by Diliberto and Straus [5]. But later it was shown by Aumann [2], and independently by Medvedev [22], that the algorithm may not converge to $E(f)$ for the case $n > 2$.

It should be noted that in [10] it was proved that the algorithm converges as desired for any finite number of closed subspaces in the Hilbert space setting (without demanding closure of the sum of the subspaces). In fact, in a uniformly convex and uniformly smooth Banach space it was proved in [24] that if the sum is closed then the algorithm converges (in the strong sense of the previous remark) for any finite number of closed subspaces.

From Theorem 2.3 and Proposition 2.1 (see also Example 2) one can obtain the following corollary which is a generalization of the classical Diliberto and Straus theorem from rectangular sets to special compact convex sets of $\mathbb{R}^2$.

COROLLARY 2.4

Let $Q \subset \mathbb{R}^2$ *be a convex compact set with the property that any bolt* (*with respect to the coordinate projections*) *in* Q *can be made closed by adding only a fixed number of points of* Q. *Let*, *in addition*, $A_1 = \{f(x)\}$ *and* $A_2 = \{g(y)\}$ *be the algebras of univariate functions*, *which are continuous on the projections of* Q *into the coordinate axes* x *and* y, *respectively. Then for a given function* $h \in C(Q)$, *the sequence* $h_1 = h, h_{2n} = h_{2n-1} - Fh_{2n-1}, h_{2n+1} = h_{2n} - Gh_{2n}, n = 1, 2, \ldots$, *converges in norm to the error of approximation from* $A_1 + A_2$.

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