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Infinitely many sign-changing solutions of an elliptic problem involving critical Sobolev and Hardy–Sobolev exponent

MOUSOMI BHAKTA

Department of Mathematics, Indian Institute of Science Education and Research, Pashan, Pune 411 008, India
E-mail: mousomi@iiserpune.ac.in

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Abstract. We study the existence and multiplicity of sign-changing solutions of the following equation

$$\begin{cases} -\Delta u = \mu|u|^{2^\star-2}u + \frac{|u|^{2^*(t)-2}u}{|x|^t} + a(x)u \quad \text{in}\, \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $0 \in \partial\Omega$, all the principal curvatures of $\partial\Omega$ at 0 are negative and $\mu \geq 0$, $a > 0$, $N \geq 7$, $0 < t < 2$, $2^\star = \frac{2N}{N-2}$ and $2^\star(t) = \frac{2(N-t)}{N-2}$.

Keywords. Sign-changing solution; multiple critical exponent; Hardy–Sobolev; infinitely many solutions.

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1. Introduction

In this article, we study the following elliptic problem:

$$\begin{cases} -\Delta u = \mu|u|^{2^\star-2}u + \frac{|u|^{2^\star(t)-2}u}{|x|^t} + a(x)u \quad \text{in}\, \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\partial\Omega$ is C^3, $0 \in \bar{\Omega}$ and

$$\mu \geq 0,\ a \in C^1(\bar{\Omega}),\ a > 0,\ 0 < t < 2 \qquad \text{and} \qquad 2^\star(t) = \frac{2(N-t)}{N-2}. \tag{1.2}$$

Here $2^\star$ is the usual critical Sobolev exponent $\frac{2N}{N-2}$. By a solution of the above equation, we mean $u \in H_0^1(\Omega)$ satisfying

$$\int_\Omega \nabla u \cdot \nabla v \mathrm{d}x = \mu \int_\Omega |u|^{2^\star-2}uv\mathrm{d}x + \frac{|u|^{2^\star(t)-2}uv}{|x|^t}\mathrm{d}x + \int_\Omega a(x)uv\mathrm{d}x, \quad \forall\, v \in H_0^1(\Omega).$$

Equivalently, u is a critical point of the functional I,

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \frac{1}{2}\int_\Omega a(x)|u|^2 \, dx - \frac{\mu}{2^*}\int_\Omega |u|^{2^*} \, dx - \frac{1}{2^*(t)}\int_\Omega \frac{|u|^{2^*(t)}}{|x|^t} \, dx. \tag{1.3}$$

I is a well defined C^1 functional on $H_0^1(\Omega)$ for any open subset of $\mathbb{R}^N$, thanks to the following Hardy–Sobolev inequality,

Hardy–Sobolev inequality. Let $N \geq 3$, $0 \leq t < 2$. Then there exists a positive constant $C = C(N, t)$ such that

$$\left(\int_{\mathbb{R}^N} \frac{|u|^{2^\star(t)}}{|x|^t} dx\right)^{\frac{2}{2^*(t)}} \leq C \int_{\mathbb{R}^N} |\nabla u|^2 dx \quad \forall\, u \in C_0^\infty(\mathbb{R}^N). \tag{1.4}$$

Equation (1.1) involves multiple critical exponents, namely, critical Sobolev exponent and Hardy–Sobolev exponent. In recent years, a lot of attention has been given to the existence of nontrivial solutions of problem (1.1). As it is mentioned in [16], one can apply the pioneering idea of Brezis and Nirenberg [3] to obtain a positive solution of (1.1).

When $\Omega = \mathbb{R}^N$ and the function a is singular at the origin, existence of positive solution to more general type equations involving multiple critical exponents was studied by Fillippucci *et al.* [8] using Mountain Pass lemma of Ambrosetti and Rabinowitz [1]. There the crucial step is to show that the Mountain Pass value is strictly less than the first energy level at which the Palais–Smale condition fails. For the existence of the Mountain Pass solution of (1.1), we also refer [13] and the references therein. As pointed out in [16], when $0 \in \partial\Omega$, the mean curvature of $\partial\Omega$ at 0 plays an important role in the existence of Mountain Pass solutions (see [5, 10, 11, 13]).

In [16], Yan and Yang considered the problem (1.1) with $a \in C^1(\bar{\Omega})$ and $a(0) > 0$. They proved the existence of infinitely many solutions using the compactness of the solutions of Brezis–Nirenberg type problem established by Devillanova and Solimini [7] for $N \geq 7$. But [16] does not have any information about the existence and multiplicity of sign changing solutions. Also, it is worth mentioning that one cannot obtain the existence of infinitely many sign changing solutions of (1.1) by adopting the method of [16]. Therefore a natural question which is still open is whether (1.1) has infinitely many sign changing solutions for any $a \in C^1(\bar{\Omega})$ such that $a > 0$.

A very important result by Schechter and Zou [15] asserts that there exists infinitely many sign changing solutions to the Brezis–Nirenberg problem in higher dimension. Very recently this kind of technique was also used in [6] and [9] to prove the existence of infinitely many sign changing solutions of Hardy–Sobolev–Maz'ya type equations and Brezis–Nirenberg problem in hyperbolic space respectively.

So far in the literature the only two papers that deal with the sign-changing solution of (1.1) are [4] and [17]. In [4], a pair of sign-changing solutions and in [17] infinitely many sign changing solutions were obtained for (1.1) when $t = 2$, the function a being a constant and $0 \in \Omega$. More precisely in [17], the following problem was studied:

$$\begin{cases} -\Delta u = |u|^{2^\star - 2}u + \mu\frac{u}{|x|^2} + \lambda u & \text{in}\, \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.5}$$

where $\lambda > 0$ is a constant, $0 \in \Omega$. We would like to point out as in [16] that, there are some differences between the case $t = 2$ and $t \in (0, 2)$. If $t = 2$, solutions of (1.5) are singular at 0 and this was the main reason to impose the condition $\mu \in \left(0, \frac{(N-2)^2}{4} - 4\right)$ in [17]. If $t \in (0, 2)$, no such conditions are needed. Also, there are some differences between the cases when a is a constant function and when it is not a constant function.

Our main theorem is the following:

Theorem 1.1. *Let $N \geq 7$, $0 \in \partial\Omega$, all the principal curvatures of $\partial\Omega$ at 0 be negative and (1.2) hold. Then eq. (1.1) has infinitely many sign changing solutions.*

We will prove this theorem by applying an abstract theorem by Schechter and Zou theorem 2 of [15]. However we can not apply the theorem directly as I, that is, the variational problem corresponding to (1.1) does not satisfy Palais–Smale condition. To overcome this difficulty we consider the perturbed subcritical problem

$$\begin{cases} -\Delta u = \mu |u|^{2^\star - 2 - \epsilon_n} u + \dfrac{|u|^{2^\star(t) - 2 - \epsilon_n} u}{|x|^t} + a(x) u & \text{in } \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.6}$$

where $0 < \epsilon_n \downarrow 0$. We will prove that for each ϵ_n, (1.6) has a sequence of sign changing solution $\{u_{n,l}\}_{l=1}^{\infty}$ and we will show that Morse index of $\{u_{n,l}\}_{l=1}^{\infty}$ has a lower bound. Then we will prove that for fixed l, $\sup_{n \in \mathbb{N}} ||u_{n,l}||_{H_0^1(\Omega)} < \infty$.

We organize the paper as follows. In §2, we prove the existence and the estimate of the Morse index of sign-changing solution of (1.6). Using this, in §3, we prove Theorem 1.1. In §4, we prove a nonexistence result in star-shaped domain under some condition on the function a.

Notation. Throughout this paper, we denote the norm in $H_0^1(\Omega)$ by $||u|| = \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{1/2}$ and $|u|_{q,t,\Omega} := \left(\int_\Omega \frac{|u|^q}{|x|^t} \mathrm{d}x\right)^{1/q}$. We say $u \in L_t^q(\Omega)$ if $|u|_{q,t,\Omega} < \infty$.

2. Existence of sign-changing critical points

Consider the weighted eigenvalue problem:

$$-\Delta u = \lambda a(x) u \quad \text{in} \quad \Omega; \qquad u = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

Since $a \in C^1(\bar{\Omega})$ and is strictly positive, the above operator has infinitely many eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ such that $0 < \lambda_1(\Omega) < \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \cdots \leq \lambda_l(\Omega) \leq \cdots$. Therefore, we can write

$$\lambda_1 = \inf_{u \in H_0^1(\Omega), u \neq 0} \frac{\displaystyle\int_\Omega |\nabla u|^2 \mathrm{d}x}{\displaystyle\int_\Omega a(x) u^2 \mathrm{d}x}. \tag{2.2}$$

Let ϕ_i be the orthonormal eigenvectors corresponding to λ_i where we know that $\phi_1 > 0$. We denote $E_k = \operatorname{span}\{\phi_1, \ldots, \phi_k\}$. Then $E_k \subset E_{k+1}$ and $H_0^1(\Omega) = \overline{\cup_{k=1}^{\infty} E_k}$ (see [12]).

Lemma 2.1. *Suppose all the assumptions in Theorem* 1.1 *hold. In addition, if* $\lambda_1 \leq 1$, *then eq.* (1.1) *has infinitely many sign-changing solutions.*

Proof. By multiplying eq. (1.1) by ϕ_1 and integrating by parts, it is easy to check that if $\lambda_1 \leq 1$, then any nontrivial solution of (1.1) has to change sign. Also by Theorem 1.2 of [16], it follows that eq. (1.1) has infinitely many solutions. Therefore the lemma follows. □

From now onwards we assume that $\lambda_1 > 1$. Fix $\epsilon_0 > 0$ small enough and choose a sequence $\epsilon_n \in (0, \epsilon_0)$ such that $\epsilon_n \downarrow 0$ in (1.6). We define the energy functional corresponding to (1.6) as

$$I_n(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, \mathrm{d}x - \frac{1}{2}\int_\Omega a(x)|u|^2 \, \mathrm{d}x - \frac{\mu}{2^* - \epsilon_n}\int_\Omega |u|^{2^*-\epsilon_n} \, \mathrm{d}x - \frac{1}{2^*(t) - \epsilon_n}\int_\Omega \frac{|u|^{2^*(t)-\epsilon_n}}{|x|^t} \, \mathrm{d}x. \tag{2.3}$$

Then I_n is an even C^2 functional on $H_0^1(\Omega)$. Also, I_n satisfies the Palais–Smale condition for each n. Therefore by Ambrosetti and Rabinowitz [1], (1.6) has infinitely many critical points $\{u_{n,l}\}_{l=1}^{\infty}$. More precisely, it follows from [14] that there are positive numbers $c_{n,l}, l = 1, 2, \ldots,$ with $c_{n,l} \uparrow \infty$ as $l \uparrow \infty$ and $I_n(u_{n,l}) = c_{n,l}$. We define the augmented Morse index of $u_{n,l}$ by $m^*(u_{n,l})$ as follows:

$$m^*(u_{n,l}) := \max\{\dim\ H : H \subset H_0^1(\Omega) \text{ is a subspace such that } I_n''(u_{n,l})(v, v) \leq 0 \quad \forall\, v \in H\}.$$

For each ϵ_n, we define

$$||u||_{*,n} = \mu|u|_{L^{2^*-\epsilon_n}(\Omega)} + \left(\int_\Omega \frac{|u|^{2^*(t)-\epsilon_n}}{|x|^t}\right)^{\frac{1}{2^*(t)-\epsilon_n}}; \quad \forall\, u \in H_0^1(\Omega).$$

Here we state two lemmas as in [6]. Therefore we omit the proof.

Lemma 2.2. *If* Ω *is a bounded domain in* $\mathbb{R}^N$, $0 \leq t < 2$, $N \geq 3$ *and* $1 \leq q \leq p < \infty$, *then* $L_t^p(\Omega) \subset L_t^q(\Omega)$ *and the inclusion is continuous.*

Lemma 2.3. *Let* $1 \leq q < 2^*(t), 0 \leq t < 2$ *and* $N \geq 3$. *Then the embedding* $H_0^1(\Omega) \subset L_t^q(\Omega)$ *is compact.*

Therefore by (1.4), Lemma 2.2 and Lemma 2.3, we have $||u||_{*,n} \leq C||u||_{H_0^1(\Omega)}$, where C is independent of n and $||u_l - u||_{*,n} \to 0$ whenever $u_l \rightharpoonup u$ in $H_0^1(\Omega)$. Hence (A_0) of [15] is satisfied.

We define $\mathcal{P} := \{u \in H_0^1(\Omega) : u \geq 0\}$ and $\mathcal{K}_n := \{u \in H_0^1(\Omega) : I_n'(u) = 0\}$. For each $\delta > 0$, we define $\mathcal{D}(\delta) := \{u \in H_0^1(\Omega) : \mathrm{dist}(u, \mathcal{P}) < \delta\}$. The gradient I_n' is of the form $I_n'(u) = u - K_n(u)$, where $K_n : H_0^1(\Omega) \to H_0^1(\Omega)$ is a continuous operator. In the next proposition, we will see how the operator K_n behaves on $\mathcal{D}(\delta)$.

PROPOSITION 2.4

Let $\lambda_1 > 1$ and (1.2) hold. Then for any $\delta_0 > 0$ small enough, $K_n(\pm\mathcal{D}(\delta_0)) \subset \pm\mathcal{D}(\delta) \subset \pm\mathcal{D}(\delta_0)$ for some $\delta \in (0, \delta_0)$. Moreover, $\pm\mathcal{D}(\delta_0) \cap \mathcal{K}_n \subset \mathcal{P}$.

Proof. We first note that $K_n(u)$ can be decomposed as $K_n(u) = L(u) + G_n(u)$, where $L(u), G_n(u) \in H_0^1(\Omega)$ are the unique solutions of the following equation:

$$-\Delta(L(u)) = a(x)u; \quad -\Delta(G_n(u)) = \mu|u|^{2^*-2-\epsilon_n}u + \frac{|u|^{2^*(t)-2-\epsilon_n}u}{|x|^t}.$$

In other words, $L(u)$ and $G_n(u)$ are uniquely determined by

$$\langle L(u), v\rangle_{H_0^1(\Omega)} = \int_\Omega a(x)uv\mathrm{d}x, \tag{2.4}$$

$$\langle G_n(u), v\rangle_{H_0^1(\Omega)} = \mu\int_\Omega |u|^{2^*-2-\epsilon_n}uv\mathrm{d}x + \int_\Omega \frac{|u|^{2^*(t)-2-\epsilon_n}uv}{|x|^t}\mathrm{d}x. \tag{2.5}$$

We claim that, if $u \in \mathcal{P}$ then $L(u), G_n(u) \in \mathcal{P}$. Let $u \in \mathcal{P}$. Then

$$-\int_\Omega |\nabla L(u)^-|^2\mathrm{d}x = \left\langle L(u), L(u)^-\right\rangle_{H_0^1(\Omega)} = \int_\Omega a(x)uL(u)^- \geq 0,$$

which immediately implies that $L(u) \in \mathcal{P}$. Similary, we have $G_n(u) \in \mathcal{P}$. Using (2.4) we see that

$$\begin{aligned} ||L(u)||^2 = \langle L(u), L(u)\rangle_{H_0^1(\Omega)} = \int_\Omega a(x)uL(u) \leq \left(\int_\Omega a(x)u^2\mathrm{d}x\right)^{\frac{1}{2}} \\ \times \left(\int_\Omega a(x)L(u)^2\mathrm{d}x\right)^{\frac{1}{2}}. \end{aligned} \tag{2.6}$$

Therefore using (2.2) in the above expression, we obtain

$$||L(u)||^2_{H_0^1(\Omega)} \leq \frac{1}{\lambda_1}||u||_{H_0^1(\Omega)}||L(u)||_{H_0^1(\Omega)},$$

which yields $||L(u)||_{H_0^1(\Omega)} \leq \frac{1}{\lambda_1}||u||_{H_0^1(\Omega)}$. For any $u \in H_0^1(\Omega)$, we consider $v \in \mathcal{P}$ such that $\mathrm{dist}(u, \mathcal{P}) = ||u - v||$. Then

$$\mathrm{dist}(L(u), \mathcal{P}) \leq ||L(u) - L(v)|| \leq \frac{1}{\lambda_1}||u - v|| \leq \frac{1}{\lambda_1}\mathrm{dist}(u, \mathcal{P}). \tag{2.7}$$

Next,

$$\begin{aligned} \mathrm{dist}(G_n(u), \mathcal{P})||G_n(u)^-|| &\leq ||G_n(u) - G_n(u)^+||||G_n(u)^-|| = ||G_n(u)^-||^2 \\ &\leq -\langle G_n(u), G_n(u)^-\rangle_{H_0^1(\Omega)} \\ &= -\mu\int_\Omega |u|^{2^*-2-\epsilon_n}uG_n(u)^- - \int_\Omega \frac{|u|^{2^*(t)-2-\epsilon_n}uG_n(u)^-}{|x|^t} \\ &\leq \mu\int_\Omega |u|^{2^*-2-\epsilon_n}u^-G_n(u)^- + \int_\Omega \frac{|u|^{2^*(t)-2-\epsilon_n}u^-G_n(u)^-}{|x|^t} \end{aligned}$$

$$= \mu \int_\Omega |u^-|^{2^*-1-\epsilon_n} G_n(u)^- + \int_\Omega \frac{|u^-|^{2^*(t)-1-\epsilon_n} G_n(u)^-}{|x|^t}$$
$$\leq \mu \left(\int_\Omega |u^-|^{2^*-\epsilon_n} \mathrm{d}x\right)^{\frac{2^*-1-\epsilon_n}{2^*-\epsilon_n}} \left(\int_\Omega |G_n(u)^-|^{2^*-\epsilon_n} \mathrm{d}x\right)^{\frac{1}{2^*-\epsilon_n}}$$
$$+ \left(\int_\Omega \frac{|u^-|^{2^*(t)-\epsilon_n}}{|x|^t} \mathrm{d}x\right)^{\frac{2^*(t)-1-\epsilon_n}{2^*(t)-\epsilon_n}} \left(\int_\Omega \frac{|G_n(u)^-|^{2^*(t)-\epsilon_n}}{|x|^t} \mathrm{d}x\right)^{\frac{1}{2^*(t)-\epsilon_n}}. \tag{2.8}$$

By using Lemma 2.2 and the Sobolev inequality, the last term in the RHS of the above expression (2.8) can be shown less than

$$C\left(|u^-|^{2^*-1-\epsilon_n}_{L^{2^*-\epsilon_n}} + |u^-|^{2^*(t)-1-\epsilon_n}_{L_t^{2^*(t)-\epsilon_n}}\right) ||G_n(u)^-||_{H_0^1(\Omega)}.$$

Therefore we obtain from (2.8),

$$\mathrm{dist}(G_n(u), \mathcal{P}) \leq C\left(|u^-|^{2^*-1-\epsilon_n}_{L^{2^*-\epsilon_n}} + |u^-|^{2^*(t)-1-\epsilon_n}_{L_t^{2^*(t)-\epsilon_n}}\right).$$

Using (1.4), it is easy to check that, from the above equation we can obtain

$$\mathrm{dist}(G_n(u), \mathcal{P}) \leq C(\mathrm{dist}(u, \mathcal{P})^{2^*-1-\epsilon_n} + \mathrm{dist}(u, \mathcal{P})^{2^*(t)-1-\epsilon_n}) \quad \forall\, u \in H_0^1(\Omega),$$

(see [6, 17]). As $\lambda_1 > 1$, we choose $\nu \in (\frac{1}{\lambda_1}, 1)$. Then there exists $\delta_0 > 0$ such that if $\delta \leq \delta_0$, we have

$$\mathrm{dist}(G_n(u), \mathcal{P}) \leq \left(\nu - \frac{1}{\lambda_1}\right) \mathrm{dist}(u, \mathcal{P}) \quad \forall\, u \in \mathcal{D}(\delta). \tag{2.9}$$

Combining (2.7) and (2.9), we obtain

$$\mathrm{dist}(K_n(u), \mathcal{P}) \leq \mathrm{dist}(L(u), \mathcal{P}) + \mathrm{dist}(G_n(u), \mathcal{P}) \leq \nu\, \mathrm{dist}(u, \mathcal{P}) \quad \forall\, u \in \mathcal{D}(\delta).$$

Hence we get $K_n(\mathcal{D}(\delta_0) \subset \mathcal{D}(\delta)) \subset \mathcal{D}(\delta_0))$ for some $\delta \in (0, \delta_0)$. Also if, $\mathrm{dist}(u, \mathcal{P}) < \delta_0$ and $I_n'(u) = 0$, i.e., $u = K_n(u)$, then we have, $\mathrm{dist}(u, \mathcal{P}) = \mathrm{dist}(K_n(u), \mathcal{P}) \leq \nu \mathrm{dist}(u, \mathcal{P})$, which immediately implies $u \in \mathcal{P}$. Similarly we can prove $K_n(-\mathcal{D}(\delta_0)) \subset -\mathcal{D}(\delta) \subset -\mathcal{D}(\delta_0)$ for some $\delta \in (0, \delta_0)$ and $-\mathcal{D}(\delta_0) \cap \mathcal{K}_n \subset \mathcal{P}$. This completes the proof. □

Lemma 2.5. *Let* $\lambda_1 > 1$ *and* (1.2) *hold. Then for each* k, $\lim_{||u|| \to \infty, u \in E_k} I_n(u) = -\infty$.

Lemma 2.6. *Let* $\lambda_1 > 1$ *and* (1.2) *hold. Then for any* $\alpha_1, \alpha_2 > 0$, *there exists an* α_3 *depending on* α_1 *and* α_2 *such that* $||u|| \leq \alpha_3$ *for all* $u \in I_n^{\alpha_1} \cap \{u \in H_0^1(\Omega) : ||u||_{*,n} \leq \alpha_2\}$, *where* $I_n^{\alpha_1} = \{u \in H_0^1(\Omega) : I_n(u) \leq \alpha_1\}$.

The above two lemmas are quite standard and can be proved by using similar techniques as in [6].

Theorem 2.7. *Let $\lambda_1 > 1$ and (1.2) hold. Then for each n, eq. (1.6) has infinitely many sign-changing solutions $\{u_{n,l}\}_{l=1}^{\infty}$ such that for each l, the sequence $\{u_{n,l}\}$ is bounded in $H_0^1(\Omega)$ and the augmented Morse index of $u_{n,l}$ is greater than or equal to l.*

Proof. By applying Proposition 2.4, Lemma 2.5 and Lemma 2.6, we see that I_n satisfies all the assumptions $(A_1)-(A_3)$ in Theorem 2 of [15]. Hence by Theorem 2 of [15], I_n has a sign-changing critical point $u_{n,l}$ at the level $c_{n,l}$, where $c_{n,l} \leq \sup_{E_{l+1}} I_n$ and $m^*(u_{n,l}) \geq l$.

Claim. There exists a positive constant T_1, independent of n and l such that

$$c_{n,l} \leq T_1 \lambda_{l+1}^{\frac{2^*(t)-\epsilon_0}{2(2^*(t)-\epsilon_0-2)}}.$$

To see this, we note that since $2^*(t) - \epsilon_0 > 2$,

$$||u||^2 \leq \lambda_{l+1}|u|^2_{L^2(\Omega)} \leq C\lambda_{l+1}|u|^2_{L_t^{2^*(t)-\epsilon_0}(\Omega)} \quad \forall\, u \in E_{l+1}, \tag{2.10}$$

where $C > 0$ is a constant independent of n, l. Since $0 < t < 2$, $2^*(t)-\epsilon_0 < 2^*(t)-\epsilon_n$, by Hölder inequality, it is easy to check that there exists constants $D_1, D_1' > 0$, independent of n, l such that $|u|_{L_t^{2^*(t)-\epsilon_0}(\Omega)} < D_1|u|_{L_t^{2^*(t)-\epsilon_n}(\Omega)} + D_1'$. Therefore,

$$\begin{aligned} I_n(u) &\leq \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \frac{1}{2^*(t)-\epsilon_n}\int_\Omega \frac{|u|^{2^*(t)-\epsilon_n}}{|x|^t}\mathrm{d}x \\ &\leq \frac{1}{2}||u||^2 - D_2\int_\Omega \frac{|u|^{2^*(t)-\epsilon_0}}{|x|^t}\mathrm{d}x + D_3, \end{aligned}$$

where $D_2, D_3 > 0$ are constants independent of n, l. Using (2.10) in the above expression, we have for all $u \in E_{l+1}$,

$$\begin{aligned} I_n(u) &\leq \frac{1}{2}||u||^2 - D_4\lambda_{l+1}^{-\frac{2^*(t)-\epsilon_0}{2}}||u||^{L^{2^*(t)-\epsilon_0}} + D_3 \leq D_5\lambda_{l+1}^{\frac{2^*(t)-\epsilon_0}{2(2^*(t)-\epsilon_0-2)}} \\ &\quad + D_3 \leq T_1\lambda_{l+1}^{\frac{2^*(t)-\epsilon_0}{2(2^*(t)-\epsilon_0-2)}}, \end{aligned}$$

where D_i, $i = 1, \ldots, 5$ and T_1 are positive constants independent of n, l. Since energy of any critical point is non-negative, we conclude that $I_n(u_{n,l}) \in [0, T_1\lambda_{l+1}^{\frac{2^*(t)-\epsilon_0}{2(2^*(t)-\epsilon_0-2)}}]$. Also we see that

$$\begin{aligned} I_n(u_{n,l}) &= I_n(u_{n,l}) - \frac{1}{2^*(t)-\epsilon_n}I_n'(u_{n,l})(u_{n,l}) \\ &= \left(\frac{1}{2} - \frac{1}{2^*(t)-\epsilon_n}\right)\int_\Omega (|\nabla u_{n,l}|^2 - a(x)u_{n,l}^2)\mathrm{d}x \\ &\quad + \mu\left(\frac{1}{2^*(t)-\epsilon_n} - \frac{1}{2^*-\epsilon_n}\right)\int_\Omega |u_{n,l}|^{2^*-\epsilon_n}\mathrm{d}x \\ &\geq \left(\frac{1}{2} - \frac{1}{2^*(t)-\epsilon_0}\right)\int_\Omega (|\nabla u_{n,l}|^2 - a(x)u_{n,l}^2)\mathrm{d}x \\ &\geq \left(\frac{1}{2} - \frac{1}{2^*(t)-\epsilon_0}\right)\left(1 - \frac{1}{\lambda_1}\right)||u_{n,l}||^2. \end{aligned}$$

As $\lambda_1 > 1$, coefficient in the RHS is strictly positive. Hence $\{u_{n,l}\}_{n=1}^{\infty}$ is bounded in $H_0^1(\Omega)$ for each l, which completes the proof. □

3. Proof of Theorem 1.1

We start this section by quoting a theorem from Yan and Yang (Theorem 1.1 of [16]).

Theorem 3.1. *Let $a \in C^1(\bar{\Omega})$, $a(0) > 0$ and $0 \in \partial\Omega$, all the principal curvatures of $\partial\Omega$ at 0 are negative. If $N \geq 7$, $\mu \geq 0$, then for any u_n which is a solution of (1.6) with $\epsilon = \epsilon_n \to 0$, satisfying $||u_n|| \leq C$, for some constant independent of n, u_n converges strongly in $H_0^1(\Omega)$.*

Proof of Theorem 1.1. Combining Theorem 2.7 and Theorem 3.1, we obtain $u_{n,l} \to u_l$ in $H_0^1(\Omega)$ as $n \to \infty$. Then $\{u_l\}_{l=1}^{\infty}$ is a sequence of solution to eq. (1.1) with energy $c_l \in [0, T_1\lambda_{l+1}^{\frac{2^*(t)-\epsilon_0}{2(2^*(t)-\epsilon_0-2)}}]$. Next, we claim that u_l is sign changing for each l. To see this, we note that as $I_n'(u_{n,l}) = 0$, we get

$$\int_\Omega (|\nabla u_{n,l}^{\pm}|^2 - a(x)|u_{n,l}^{\pm}|^2)\mathrm{d}x = \mu \int_\Omega |u_{n,l}^{\pm}|^{2^*-\epsilon_n}\mathrm{d}x + \int_\Omega \frac{|u_{n,l}^{\pm}|^{2^*(t)-\epsilon_n}}{|x|^t}\mathrm{d}x$$

Therefore, using (2.2) we have

$$\left(1 - \frac{1}{\lambda_1}\right) ||u_{n,l}^{\pm}||^2 \leq \mu \int_\Omega |u_{n,l}^{\pm}|^{2^*-\epsilon_n}\mathrm{d}x + \int_\Omega \frac{|u_{n,l}^{\pm}|^{2^*(t)-\epsilon_n}}{|x|^t}\mathrm{d}x.$$

Since $\left(1 - \frac{1}{\lambda_1}\right) < 1$, by using Lemma 2.2 and Sobolev inequality (1.4) in the above expression, we obtain $||u_{n,l}^{\pm}|| \geq C > 0$, where C is independent of n. This in turn implies that $||u_l^{\pm}|| \geq C' > 0$. Hence the claim follows.

To complete the proof, it remains to show that infinitely many u_l's are different. This is equivalent to proving that $\lim_{l\to\infty} I(u_l) = \lim_{l\to\infty} c_l = \infty$. We prove this by the method of contradiction. Suppose $\lim_{l\to\infty} c_l \leq c < \infty$. For each l, we find $n_l > l$ such that $|c_{n_l,l} - c_l| < \frac{1}{l}$. Therefore, $\lim_{l\to\infty} c_{n_l,l} = \lim_{l\to\infty} c_l < c < \infty$. Since $I_n'(u_{n_l,l}) = 0$, once again it proves that $\{u_{n_l,l}\}$ is bounded in $H_0^1(\Omega)$. Therefore by Theorem 3.1, $\{u_{n_l,l}\}$ converges in $H_0^1(\Omega)$ and the augmented Morse index of $\{u_{n_l,l}\}_{l=1}^{\infty}$ remains bounded, which is a contradiction to the fact that $m^*(u_{n_l,l}) \geq l$. This completes the proof. □

4. Nonexistence result

Theorem 4.1. *Suppose $N \geq 3$, $a \in C^1(\bar{\Omega})$ and $(a(x) + \frac{1}{2}x \cdot \nabla a) \leq 0$ for every $x \in \Omega$. Then eq. (1.1) does not have any nontrivial solution in a domain which is star-shaped with respect to the origin.*

Proof. We will prove this using the Pohozaev identity in the spirit of [2]. For $\epsilon > 0$ and $R > 0$, define $\phi_{\epsilon,R} := \phi_\epsilon(x)\psi_R(x)$, where $\phi_\epsilon(x) = \phi(\frac{|x|}{\epsilon})$, $\psi_R(x) := \psi(\frac{|x|}{R})$, ϕ and ψ being smooth functions in $\mathbb{R}$ with properties $0 \leq \phi, \psi \leq 1$, with the support of ϕ, ψ in $(1, \infty)$ and $(-\infty, 2)$ respectively and $\phi(t) = 1$ for $t \geq 2$ and $\psi = 1$ for $t \leq 1$.

Let u be any solution of eq. (1.1). Then u is smooth away from the origin and hence $(x\cdot\nabla u)\phi_{\epsilon,R}\in C_c^2(\bar\Omega)$. Multiplying eq. (1.1) by $(x\cdot\nabla u)\phi_{\epsilon,R}$ and integrating by parts, we obtain

$$\begin{aligned}
&\int_\Omega \nabla u\nabla\big((x\cdot\nabla u)\phi_{\epsilon,R}\big)\mathrm{d}x-\int_{\partial\Omega}\frac{\partial u}{\partial\nu}(x\cdot\nabla u)\phi_{\epsilon,R}\mathrm{d}S\\
&\quad=\mu\int_\Omega |u|^{2^*-2}u(x\cdot\nabla u)\phi_{\epsilon,R}\mathrm{d}x+\int_\Omega\frac{|u|^{2^*(t)-2}u}{|x|^t}(x\cdot\nabla u)\phi_{\epsilon,R}\mathrm{d}x\\
&\qquad+\int_\Omega a(x)u(x\cdot\nabla u)\phi_{\epsilon,R}\mathrm{d}x. \qquad (4.1)
\end{aligned}$$

The RHS can be simplified as follows:

$$\begin{aligned}
\text{RHS} &= \frac{\mu}{2^*}\int_\Omega\nabla(|u|^{2^*})\cdot x\phi_{\epsilon,R}\mathrm{d}x+\frac{1}{2^*(t)}\int_\Omega\nabla(|u|^{2^*(t)})\cdot x\frac{\phi_{\epsilon,R}}{|x|^t}\mathrm{d}x\\
&\quad+\frac12\int_\Omega a(x)\nabla(|u|^2)\cdot x\phi_{\epsilon,R}\mathrm{d}x\\
&= -\mu\left(\frac{N-2}{2}\right)\int_\Omega |u|^{2^*}\phi_{\epsilon,R}\mathrm{d}x-\frac{\mu}{2^*}\int_\Omega|u|^{2^*}(x\cdot\nabla\phi_{\epsilon,R})\mathrm{d}x\\
&\quad-\frac{N-2}{2}\int_\Omega\frac{|u|^{2^*(t)}}{|x|^t}\phi_{\epsilon,R}\mathrm{d}x-\frac{1}{2^*(t)}\int_\Omega\frac{|u|^{2^*(t)}}{|x|^t}(x\cdot\nabla\phi_{\epsilon,R})\mathrm{d}x\\
&\quad-\frac{N}{2}\int_\Omega a(x)|u|^2\phi_{\epsilon,R}\mathrm{d}x-\frac12\int_\Omega a(x)|u|^2(x\cdot\nabla\phi_{\epsilon,R})\mathrm{d}x\\
&\quad-\frac12\int_\Omega|u|^2(x\cdot\nabla a)\phi_{\epsilon,R}\mathrm{d}x. \qquad (4.2)
\end{aligned}$$

As $|x\cdot\nabla\phi_{\epsilon,R}|=|x(\psi_R\nabla\phi_\epsilon+\phi_\epsilon\nabla\psi_R)|\le C$, by using dominated convergence theorem, it is easy to check that

$$\begin{aligned}
\lim_{R\to\infty}\lim_{\epsilon\to0}\text{RHS} &= -\left(\frac{N-2}{2}\right)\left(\mu\int_\Omega|u|^{2^*}\mathrm{d}x+\int_\Omega\frac{|u|^{2^*(t)}}{|x|^t}\mathrm{d}x\right)\\
&\quad-\frac{N}{2}\int_\Omega a(x)u^2\mathrm{d}x-\frac12\int_\Omega|u|^2(x\cdot\nabla a)\mathrm{d}x. \qquad (4.3)
\end{aligned}$$

Following the calculation in Theorem 4.1 of [2], LHS of (4.1) can be estimated as

$$\begin{aligned}
\text{LHS} &= -\left(\frac{N-2}{2}\right)\int_\Omega|\nabla u|^2\phi_{\epsilon,R}\mathrm{d}x-\frac12\int_{\partial\Omega}\left(\frac{\partial u}{\partial\nu}\right)^2(x\cdot\nu)\phi_{\epsilon,R}\mathrm{d}S\\
&\quad-\frac12\int_\Omega|\nabla u|^2(x\cdot\nabla\phi_{\epsilon,R})\mathrm{d}x+\int_\Omega(x\cdot\nabla u)(\nabla u\cdot\nabla\phi_{\epsilon,R})\mathrm{d}x. \qquad (4.4)
\end{aligned}$$

Here we use the fact that $x\cdot\nabla u=x\cdot\nu\frac{\partial u}{\partial\nu}$ on $\partial\Omega$, since $u=0$ on $\partial\Omega$. The first three terms on the right-hand side of (4.4) can be estimated as before. For the last term, we see that

$$\begin{aligned}
&\lim_{R\to\infty}\lim_{\epsilon\to0}\Big|\int_\Omega(x\cdot\nabla u)(\nabla u\cdot\nabla\phi_{\epsilon,R})\mathrm{d}x\Big|\\
&\quad=\lim_{R\to\infty}\lim_{\epsilon\to0}\Big|\int_\Omega(x\cdot\nabla u)\big(\psi_R(\nabla u\cdot\nabla\phi_\epsilon)+\phi_\epsilon(\nabla u\cdot\nabla\psi_R)\big)\mathrm{d}x\Big|
\end{aligned}$$

$$\leq \lim_{\epsilon\to 0} C_1 \int_{\epsilon\leq|x|\leq 2\epsilon} |\nabla u|^2 \mathrm{d}x + \lim_{R\to\infty} C_2 \int_{R\leq|x|\leq 2R} |\nabla u|^2 \mathrm{d}x$$
$$= 0.$$

Therefore from (4.4), we get

$$\lim_{R\to\infty}\lim_{\epsilon\to 0} \mathrm{LHS} = -\left(\frac{N-2}{2}\right)\int_{\Omega} |\nabla u|^2 \mathrm{d}x - \frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial u}{\partial \nu}\right)^2 (x\cdot\nu)\mathrm{d}S. \quad (4.5)$$

Combining (4.3) and (4.5), and using eq. (1.1), we obtain

$$-\int_{\Omega}\left(a(x) + \frac{1}{2}x\cdot\nabla a\right)|u|^2 \mathrm{d}x + \frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial u}{\partial \nu}\right)^2 (x\cdot\nu)\mathrm{d}S = 0.$$

Since Ω is star-shaped with respect to the origin, the second term in the LHS of the above expression is non-negative and by the assumptions of this theorem, the first term is also non-negative. Hence by the principle of unique continuation, $u = 0$ in Ω. This completes the proof. □

Remark 4.2. We give some examples satisfying the assumptions on a in the last theorem:

(i) $a(x) = -\alpha|x|^p,\ p \geq 2,\ \alpha > 0$,
(ii) $a(x) = -e^{\alpha|x|^p}$, $p \geq 2, \alpha > 0$,
(iii) $a(x) = -|x|^\alpha - e^{|x|^\beta}$, $\alpha \geq 2,\ \beta \geq 2$,
(iv) $a(x) = -|x|^\alpha e^{|x|^\beta}$, $\alpha \geq 2,\ \beta \geq 2$.

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