Proc. Indian Acad. Sci. (Math. Sci.) Vol. 127, No. 2, April 2017, pp. 323–335.
DOI 10.1007/s12044-017-0336-5

Analysing the Wu metric on a class of eggs in $\mathbb{C}^n$ – I

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MS received 26 November 2015; revised 6 April 2016

Abstract. We study the Wu metric on convex egg domains of the form

$$E_{2m} = \{z \in \mathbb{C}^n : |z_1|^{2m} + |z_2|^2 + \cdots + |z_{n-1}|^2 + |z_n|^2 < 1\},$$

where $m \geq 1/2$, $m \neq 1$. The Wu metric is shown to be real analytic everywhere except on a lower dimensional subvariety where it fails to be C^2-smooth. Overall however, the Wu metric is shown to be continuous when $m = 1/2$ and even C^1-smooth for each $m > 1/2$, and in all cases, a non-Kähler Hermitian metric with its holomorphic curvature strongly negative in the sense of currents. This gives a natural answer to a conjecture of S. Kobayashi and H. Wu for such E_{2m}.

Keywords. Wu metric; Kobayashi metric; negative holomorphic curvature.

2010 Mathematics Subject Classification. Primary: 32F45; Secondary: 32Q45.

1. Introduction

Connections between various forms of the Schwarz lemma and hyperbolicity in complex analysis i.e., negativity of curvature of a biholomorphically invariant metric, are too well known to need any elaboration (a brief overview can be found in [12]). Nevertheless, certain fundamental conjectures towards building a theory of hyperbolicity pertinent to complex analysis in standard differential geometric terms, still remain unsettled. The present article is one of a two-part note, concerning the following conjecture of Kobayashi [10] and its modified form by Wu [13].

Conjecture (*K-W*): On every Kobayashi complete hyperbolic complex manifold, there exists a C^k-smooth (for some $k \geq 0$) complete Hermitian metric with its holomorphic curvature (the term 'holomorphic curvature' stands precisely for the holomorphic *sectional* curvature and is said to be strongly negative, if it is bounded above by a negative constant) bounded above by a negative constant in the sense of currents.

Needless to say, for the above conjecture to contribute to complex analysis, the metric must transform in a 'nice' manner under holomorphic mappings. The Kobayashi metric is the largest pseudo-distance with the nice feature of being distance decreasing under all holomorphic mappings. However, the Kobayashi metric fails to be Hermitian in general, obstructing applicability of standard (Hermitian) differential geometric techniques.

A method proposed by Wu in [13] to redress this problem is as follows. For an infinitesimal metric η, form a new metric whose unit ball about the origin in each tangent space, is the ellipsoid of minimal volume containing the unit ball with respect to η. By an ellipsoid, we mean the image of a ball in the standard l^2-norm, under a linear automorphism of $\mathbb{C}^n$. As ellipsoids furnish the simplest compact convex sets with (quadratic) polynomial boundary, Wu's methodology is perhaps the first best attempt at constructing Hermitian (invariant) metrics. In the present article, we work with η as the infinitesimal Kobayashi metric (we shall reserve the term 'metric' for what was termed as a sub-metric in [13] and use the word 'distance' for what is termed as a 'metric' in general topology and metric geometry) and refer to the resultant metric as the Wu metric.

An explicit description of the Wu metric on the Thullen domains (these are also referred to as 'complex ellipsoids' in the complex analysis literature; however, since the ellipsoid in Wu's construction is defined by a polynomial of degree two, we adopt a different terminology) namely $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^{2m} + |z_2|^2 < 1\}$, where $m > 0$, was obtained by Cheung and Kim in [5] and [6]. This is a natural example arising in the pursuit of well-adapted metrics in complex analysis, the background for which can be found in [13] and [9]. Let us make a few remarks. Although the statement of the Conjecture K-W as it stands, had been affirmed for smoothly bounded Thullen domains way back in the 1980's owing to the fact that the curvatures of the Bergman and the Kähler-Einstein metric do lie trapped between a pair of negative constants [1, 4], these metrics are not *functorial*: the Hermitian metric sought for, in the Conjecture K-W is desired to transform holomorphic mappings into Lipschitz mappings with respect to the distance induced by the metric, *wherever such a metric is defined*. Being functorial and the simplest modified (Hermitian) form of the Kobayashi metric, the Wu metric is the first natural candidate to be investigated for K-W. However, the Wu metric may fail to be upper semicontinuous [8], notwithstanding the fact that the Kobayashi metric is always upper semicontinuous. But then again, the Wu metric seems to reflect the boundary geometry *better* in special instances. For instance, the holomorphic curvature for the Kobayashi metric is a well defined notion (see [7]) and for any bounded convex domain in $\mathbb{C}^n$, it is identically a negative constant. On the other hand, for C^2-smooth convex Thullen domains in $\mathbb{C}^2$, the Wu metric is a C^1-smooth Hermitian whose curvature is a negative constant only on an appropriate neighbourhood of the strongly pseudoconvex portion of the boundary (see [5]). The complement of the closure of this neighbourhood forms an approach region for a thin subset of the boundary consisting of weakly pseudoconvex points and the curvature of Wu metric is nowhere constant on this region. Nevertheless, in all cases, the Wu metric retains the strong negativity of curvature.

The purpose of this note and its counterpart [2], is to carry forward the case study of the Wu metric in conjunction with K-W as in [5] and [6], to egg domains of the form

$$E_{2m} = \left\{ z \in \mathbb{C}^n : |z_1|^{2m} + |z_2|^2 + \cdots + |z_{n-1}|^2 + |z_n|^2 < 1 \right\}. \tag{1.1}$$

More specifically, we reinstate the viewpoint of [5, 6]: being functorial and *respecting Kobayashi hyperbolicity*, the Wu metric is conceivably the most natural answer to K-W when restricted to a suitable class of domains – the class for this article being the eggs E_{2m} as above, with $m \geq 1/2$. This condition on m is precisely, to ensure the convexity of such E_{2m}. By Lempert's work [11], all invariant metrics on convex bounded domains, which turn holomorphic maps into Lipschitz 1-maps, coincide with the Kobayashi metric. Analysis of the non-convex eggs is dealt in a separate article [2], for clarity.

We now provide a description of our results. First let $m > 1$. Then ∂E_{2m} is atleast C^2-smooth. Split up E_{2m} as a disjoint union of the following subsets.

$$Z=\{f\big((0,\hat{0})\big)|f \in \text{Aut } E_{2m}\},$$
$$M^-=\{f\big((p_1,\hat{0})\big)|f \in \text{Aut } E_{2m}, 0 < p_1 < 2^{-1/2m}\},$$
$$M^0=\{f\big((2^{-1/2m},\hat{0})\big)|f \in \text{Aut } E_{2m}\}$$

and

$$M^+=\{f\big((p_1,\hat{0})\big)|f \in \text{Aut } E_{2m}, 2^{-1/2m} < p_1 < 1\},$$

where Aut E_{2m} denotes the group of holomorphic automorphisms of E_{2m}. Notice that each of the above sets is a union of orbits under the action of Aut E_{2m}. As orbits of distinct points either coincide or never intersect, it follows that the above sets are disjoint. The subdomain M^+ is a 'pinched' one-sided neighbourhood of the strongly pseudoconvex piece of the boundary, pinched along the weakly pseudoconvex points of the boundary which form the real algebraic subvariety of ∂E_{2m} given by

$$\omega = \{(z_1, z_2, \ldots, z_n) : z_1 = 0, |z_2|^2 + \cdots + |z_n|^2 = 1\}.$$

The boundary of M^+ is the union of ∂E_{2m} and M^0. The significance of these sets is that the Wu metric is real analytic, Kählerian of constant negative curvature on M^+. All these characteristics break down on M^0 while just ensuring an overall C^1-smoothness of the Wu metric therein. The real analyticity is restored again on M^-, which lies 'inside' M^0. Smoothness may be lost along the remaining seam inside M^0 namely, Z which however encodes nuances about the smoothness of the boundary near ω, as reported below.

Theorem 1.1. *Let m be any real number bigger than one. The Wu metric on E_{2m} is a C^1-smooth Hermitian metric which is real analytic on $E_{2m} \setminus (M^0 \cup Z)$. It is of class C^1 but not C^2 on M^0. At points of Z, the Wu metric is Hölder of class $C^{[2m],2m-[2m]}$ if m is not an integer whereas it is real analytic in case m is an integer.*

On M^+, the Wu metric is Kähler of constant negative holomorphic curvature, whereas on M^- it is a non-Kähler Hermitian metric with its holomorphic curvature non-constant and bounded between a pair of negative constants. Overall on E_{2m}, the Wu metric has a strongly negative holomorphic curvature in the sense of currents.

Note that the set Z is a complex hypersurface in E_{2m} while M^0 is a real analytic hypersurface in E_{2m}. Indeed, M^0 is described explicitly by

$$\{z \in E_{2m} \ : \ 2|z_1|^{2m} + |z_2|^2 + \cdots + |z_n|^2 = 1\}$$

while Z is the intersection of E_{2m} with the hyperplane $\{z_1 = 0\}$. As the parameter m drops below one, the containment $E_{2m} \supset \mathbb{B}^n$ reverses and the 'tangential approach' region to ω through M^+, where the Wu metric is Kähler of constant negative curvature, then disappears altogether.

Theorem 1.2. *Let $1/2 < m < 1$. Then ∂E_{2m} is C^1 but not C^2-smooth. The Wu metric on E_{2m} is a C^1-smooth Hermitian metric which is real analytic on $E_{2m} \setminus Z$. At points of Z, the Wu metric is $C^{1,2m-1}$ but not of class C^2.*

For $m = 1/2$, the Wu metric on E_{2m} is a continuous Hermitian metric, real analytic on $E_{2m} \setminus Z$.

For $1/2 \le m < 1$, the Wu metric on E_{2m} is nowhere Kähler. Its holomorphic curvature is bounded above by a negative constant independent of m, in the sense of currents.

2. Preliminaries

Write $z \in \mathbb{C}^n$ as $z = (z_1, \hat{z})$, where $\hat{z} = (z_2, \ldots, z_n)$. Any invariant metric, in particular, the Wu metric and the Kobayashi metric, can be written down explicitly on E_{2m} once it is known at points of $S \cup \{0\}$, where $S = \{p \in \mathbb{C}^n \ : \ p = (p_1, \hat{0}), \ 0 < p_1 < 1\}$. Indeed, the automorphism

$$\Phi(z) = \Phi_p(z) = \left(\frac{|p_1|}{p_1} \frac{s^{1/m}}{(1 - \langle \hat{z}, \hat{p} \rangle)^{1/m}} z_1, \Psi(\hat{z}) \right) \tag{2.1}$$

of E_{2m} takes $p = (p_1, \hat{p})$ to $(\tilde{p}_1, \hat{0})$, where $\tilde{p}_1 = |p_1|/s^{1/m}$ with $s = \sqrt{1 - |\hat{p}|^2}$ and Ψ is an automorphism of $\mathbb{B}^{n-1}$ taking $\hat{p}$ to the origin. The Kobayashi metric for E_{2m} for $m \ge 1/2$, was worked out in [3], but its optimal smoothness was not determined therein. This is contained in the final section of the present article.

Theorem 2.1 (Theorem 1.6 of [3]). *The Kobayashi metric for E_{2m} at the point $(p_1, \hat{0})$ for $0 < p_1 < 1$, denoted $K((p_1, \hat{0}), (v_1, \hat{v}))$ is given by*

$$\begin{cases} \left(\frac{m^2 p_1^{2m-2} |v_1|^2}{(1 - p_1^{2m})^2} + \frac{|v_2|^2}{1 - p_1^{2m}} + \cdots + \frac{|v_n|^2}{1 - p_1^{2m}} \right)^{1/2} & \text{for } u \le p_1, \\ \frac{m\alpha(1-t)|v_1|}{p_1(1-\alpha^2)(m(1-t)+t)} & \text{for } u > p_1, \end{cases}$$

where $t = 2m^2 p_1^2 (u^2 + 2m(m-1)p_1^2 + u(u^2 + 4m(m-1)p_1^2))^{-1/2}$, $u = m|v_1|/(|v_2|^2 + \cdots + |v_n|^2)^{1/2}$ and α is the unique positive solution of

$$\alpha^{2m} - t\alpha^{2m-2} - (1-t)p_1^{2m} = 0.$$

Moreover, K is C^1-smooth on $E_{2m} \times (\mathbb{C}^n \setminus \{0\})$ for $m > 1/2$.

The Kobayashi metric at the origin is given by $K(0, v) = q_{E_{2m}}(v)$, where $q_{E_{2m}}$ denotes the *Minkowski functional* of E_{2m}. Overall, for $m > 1$, the Kobayashi metric is C^2 but not C^3-smooth, as will be shown in section 6. The first step in investigating the smoothness of the Wu metric is to determine its unit sphere in tangent spaces. In terms of the Euclidean coordinates on the tangent bundle $E_{2m} \times \mathbb{C}^n$, the unit sphere of the Wu metric in $T_p E_{2m}$ for $p \in S$, is given by

$$r_1 |v_1|^2 + r_2(|v_2|^2 + \cdots + |v_n|^2) = 1$$

where r_1 and r_2 are positive real-valued continuous functions of p_1. The main objective now is

(*) to find $r_1, r_2 > 0$ such that the ellipsoid

$$\{v \in \mathbb{C}^n : r_1 |v_1|^2 + r_2(|v_2|^2 + \cdots + |v_n|^2) < 1\}$$

has the smallest volume while containing the set

$$I(E_{2m},(p_1,\hat{0})) = \{v \in \mathbb{C}^n : K((p_1,\hat{0}),v) < 1\}.$$

To this end, first note that the expressions for the best fitting ellipsoid (the unit ball in any given tangent space with respect to the Wu metric will be referred to as the 'best fitting ellipsoid' or the Wu ellipsoid) at $(p_1,\hat{0})$ and the Kobayashi indicatrix at $(p_1,\hat{0})$ involve terms of the form $|v_1|^2$, $|v_2|^2,\ldots,|v_n|^2$ only. Consequently, the problem reduces to finding $r_1, r_2 > 0$ such that the set

$$\{(v_1,\ldots,v_n) : v_1 \geq 0,\ldots,v_n \geq 0, r_1 v_1^2 + r_2(v_2^2 + \cdots + v_n^2) = 1\}$$

encloses the smallest volume with the coordinate axes and contains the set

$$\{(v_1,\ldots,v_n) : v_1 \geq 0,\ldots,v_n \geq 0, K((p_1,\hat{0}),v) \leq 1\}.$$

We reformulate problem (*) using the concept of square-convexity from [5]. Write $x = v_2^2 + \ldots + v_n^2$ and $y = v_1^2$, so that the problem of finding the best fitting ellipsoid at $(p_1,\hat{0}) \in E_{2m}$ can be restated as follows:

(**) Find r_1, r_2 such that the line segment $r_1 y + r_2 x = 1$ in the first quadrant bounds the smallest area with the coordinate axes and contains the set

$$\{(x,y) : v_1 \geq 0,\ldots,v_n \geq 0, K((p_1,\hat{0}),v) \leq 1\}.$$

The boundary of this set is determined by the equation $K^2((p_1,\hat{0}),v) = 1$, which splits into two parts, as is evident from Theorem 2.1. Henceforth, the portion of the curve $K^2((p_1,\hat{0}),v) = 1$ for $u \leq p_1$ and $u > p_1$ will be referred to as the lower K-curve C_{low} and the upper K-curve C_{up} respectively. In x-y coordinates, the lower K-curve C_{low} is given by

$$m^2 p_1^{2m-2}(1 - p_1^{2m})^{-2} y + (1 - p_1^{2m})x = 1, \tag{2.2}$$

which represents a straight line. Denote by $y = \left(f_{\text{low}}(\sqrt{x})\right)^2$, the function representing the lower K-curve C_{low} in the first quadrant of $\mathbb{R}^2$. C_{low} may also be parametrized as

$$x(\alpha) = (1 - p_1^{2m})^2 \alpha, \tag{2.3}$$

$$y(\alpha) = m^{-2} p_1^{-2m+2}(1 - p_1^{2m})^2(1 - \alpha(1 - p_1^{2m})),$$

for $1 \leq \alpha < (1 - p_1^{2m})^{-1}$. We shall only present the upper K-curve C_{up} parametrized by α, for $0 < \alpha < 1$ with $\alpha^{2m-1} > p_1^{2m}$, as

$$3x(\alpha) = \alpha^{-4m+2}(\alpha^{4m-2} + p_1^{4m} - p_1^{2m}\alpha^{2m-2} - p_1^{2m}\alpha^{2m}) \text{ and} \tag{2.4}$$

$$y(\alpha) = p_1^2 m^{-2} \alpha^{-4m+2}(m\alpha^{2m-2} - (m-1)\alpha^{2m} - p_1^{2m})^2.$$

Let f_{up} denote the function representing the upper K-curve, so that $y(\alpha) = \left(f_{\text{up}}\left(\sqrt{x(\alpha)}\right)\right)^2$ is the equation of the upper curve.

PROPOSITION 2.2

The function f_{low} *is both square convex and square concave. The function* f_{up} *is strictly square convex for* $1/2 \leq m < 1$ *and strictly square concave for* $m > 1$.

Here and in what follows, we shall not give full-fledged proofs. Rather we only provide a summary of our work and report our results.

We finish off the preliminaries with a recap of what negativity of holomorphic sectional curvature for a *continuous* Hermitian metric means. Let g be a continuous Hermitian metric on a complex manifold X i.e., it can be represented in local coordinates as $g = \sum_{i,j=1}^{n} g_{i\bar{j}} dz_i \otimes d\bar{z}_j$ with the coefficients $g_{i\bar{j}}$ being continuous functions. We say that the holomorphic curvature of g is bounded from above by a negative constant c in the sense of currents if, every embedded Riemann surface S in X with $g|_S = G\, d\xi \otimes d\bar{\xi}$, satisfies

$$\Delta_\xi \log G = \frac{\partial^2 \log G}{\partial\xi \partial\bar{\xi}} > -cG\, \partial\xi \wedge \partial\bar{\xi},$$

in the sense of currents. In case g is atleast C^2-smooth, this coincides with the standard notion of curvature: the holomorphic curvature of G along the direction of $\xi = (\xi_1, \ldots, \xi_n) \in T_p X$ at $p \in X$ is given by

$$\frac{\sum R_{i\bar{j}k\bar{l}}(p)\xi_i\bar{\xi}_j\xi_k\bar{\xi}_l}{\sum g_{i\bar{j}}(p) g_{k\bar{l}}(p)\xi_i\bar{\xi}_j\xi_k\bar{\xi}_l},$$

where $R_{i\bar{j}k\bar{l}}$ are the components of the curvature tensor given by

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial\bar{z}_l} + \sum_{\alpha,\beta} g^{\alpha\bar{\beta}} \frac{\partial g_{i\bar{\beta}}}{\partial z_k} \frac{\partial g_{\alpha\bar{j}}}{\partial\bar{z}_l},$$

and $(g^{\alpha\bar{\beta}})$ denotes the inverse of the matrix $(g_{\alpha\bar{\beta}})$.

We shall alternate between these definitions when the metric is atleast C^2-smooth, especially towards the end of the article.

3. The Wu metric on E_{2m} for $1/2 \leq m < 1$

To find the best fitting ellipsoid at $p \in S$, first note that in the x-y coordinates, the lower K-curve and the upper K-curve are both convex, by Proposition 2.2. Moreover, it follows from C^1-smoothness of the Kobayashi metric on E_{2m} (refer Theorem 2.1) that the upper and lower K-curve together yield a C^1-smooth convex curve in the x-y coordinates. It can then be deduced that the line segment obtained by joining the x-intercept of lower K-curve C_{low} and y-intercept of the upper K-curve C_{up}, bounds the smallest area with the coordinate axes while containing the set $\{K((p_1, \hat{0}), v) \leq 1\}$. It is evident from (2.2) that the x-intercept of the lower K-curve is $1 - p_1^{2m}$ while the y-intercept of the upper K-curve is $(1 - p_1^2)^2$ since $K((p_1, \hat{0}), (v_1, \hat{0})) = (1 - p_1^2)^{-1}|v_1|$. The line segment obtained by joining these two intercepts is given by

$$(1 - p_1^{2m})^{-1}x + (1 - p_1^2)^{-2}y = 1.$$

Observe that the slope of this line segment is greater than the slope of the lower K-curve C_{low}. To summarize, $r_1 = (1 - p_1^2)^{-2}$ and $r_2 = (1 - p_1^{2m})^{-2}$ provide a solution to the extremal problem (**) and subsequently, the expression for the Wu metric $h_{E_{2m}}$ at reference points is

$$h_{E_{2m}}(p_1, \hat{0}) = \frac{dz_1 \otimes d\bar{z}_1}{(1 - p_1^2)^2} + \frac{dz_2 \otimes d\bar{z}_2}{1 - p_1^{2m}} + \cdots + \frac{dz_n \otimes d\bar{z}_n}{1 - p_1^{2m}}. \tag{3.1}$$

Explicit expression for the Wu metric at $p \in E_{2m}$ with $p_1 \neq 0$, is obtained using the automorphisms Φ_p. Continuity of the Wu metric allows its determination at the remaining points $(0, p_2, \ldots, p_n) \in E_{2m}$.

Theorem 3.1. *For $1/2 \leq m < 1$, the Wu metric on E_{2m} at the point $(z_1, \ldots, z_n)$ is given by $\sum_{i,j=1}^{n} h_{i\bar{j}}(z_1, \ldots, z_n)\, dz_i \otimes d\bar{z}_j$, where*

$$h_{1\bar{1}} = \frac{(1 - |\hat{z}|^2)^{1/m}}{((1 - |\hat{z}|^2)^{1/m} - |z_1|^2)^2},$$

$$h_{1\bar{j}} = \frac{(1 - |\hat{z}|^2)^{-1+1/m}\bar{z}_1 z_j}{m((1 - |\hat{z}|^2)^{1/m} - |z_1|^2)^2}$$

for $2 \leq j \leq n$ and $h_{j\bar{1}} = \overline{h_{1\bar{j}}}$,

$$h_{j\bar{j}} = \left(\frac{(1 - |\hat{z}|^2)^{-2+1/m}|z_1|^2|z_j|^2}{m^2((1 - |\hat{z}|^2)^{1/m} - |z_1|^2)^2} + \frac{1 - |\hat{z}|^2 + |z_j|^2}{(1 - |\hat{z}|^2)(1 - |\hat{z}|^2 - |z_1|^{2m})} \right),$$

for $2 \leq j \leq n$. For $2 \leq i, j \leq n$ with $i \neq j$, we have

$$h_{i\bar{j}} = \left(\frac{(1 - |\hat{z}|^2)^{-2+1/m}|z_1|^2\bar{z}_i z_j}{m^2((1 - |\hat{z}|^2)^{1/m} - |z_1|^2)^2} + \frac{\bar{z}_i z_j}{(1 - |\hat{z}|^2)(1 - |\hat{z}|^2 - |z_1|^{2m})} \right).$$

It follows that the Wu metric is real analytic on the set $\{(z_1, \ldots, z_n) \in E_{2m} : z_1 \neq 0\}$. Verifying $\partial h_{1\bar{2}}/\partial z_2 \neq \partial h_{2\bar{2}}/\partial z_1$ at points of S, by a direct calculation, we infer that the Wu metric is non-Kähler. Using Theorem 3.1, one can also verify that the holomorphic curvature of the Wu metric is bounded between a pair of negative constants on the domain $U_Z = E_{2m} \setminus Z$. The strong negativity of the holomorphic curvature current across the remaining thin set Z (surrounded by U_Z) follows using arguments similar to those outlined in § 5 (for $1/2 < m < 1$) and § 3 of [2] (when $m = 1/2$).

Remark 3.2. Explicit expression of the Wu metric as above, yields information about the overall order of its smoothness, which relies on the smoothness of the power function of one variable $t \to |t|^\alpha$, where α is a positive real number. Such a function is infinitely differentiable if α is an integer > 1, while it is only $[\alpha]$-smooth if α is not an integer in which case, the $[\alpha]$-th derivative lies in the $C^{0,\alpha-[\alpha]}$-Hölder class. We are concerned about the regularity of such power functions in a neighbourhood of the origin. Note that $(1-|\hat{z}|^2)^{1/m}$ is real analytic on E_{2m} as it never vanishes there. Barring $m = 1/2$, note that the integral part of $2m$ is 1, while its fractional part is $2m - 1$. It follows from Theorem

3.1 that the Wu metric is *not* C^2-smooth at points of Z. However, it is $C^{1,2m-1}$-smooth at Z. So the Wu metric is C^1-smooth overall, unless $m = 1/2$.

4. The Wu metric on M^+ for $m > 1$

Theorem 4.1. *The Wu metric tensor at* $(p_1, \dots, p_n) \in M^+$ *is given by*

$$\frac{1}{(s^2 - |p_1|^{2m})^2}\left(m^2 s^2 |p_1|^{2m-2} dp_1 \otimes d\bar{p}_1 + \sum_{j=2}^{n}\left(s^2 + |p_j|^2 - |p_1|^{2m}\right) dp_j \otimes d\bar{p}_j + 2\Re\left(\sum_{j=2}^{n} m|p_1|^{2m-2} p_1 \bar{p}_j dp_j \otimes d\bar{p}_1\right) + 2\Re\left(\sum_{\substack{j,k=2\\ k<j}}^{n} p_k \bar{p}_j dp_j \otimes d\overline{p}_k\right)\right).$$

The smooth function $\rho(z) = -\log(1 - (|z_1|^{2m} + |z_2|^2 + \cdots + |z_n|^2))$ is a Kähler potential for the Wu metric of E_{2m} on M^+. It is also straightforward to verify that the Wu metric is Kähler with constant holomorphic curvature -2 on M^+.

5. The Wu metric on M^- for $m > 1$

PROPOSITION 5.1

For $0 < p_1 < 2^{-1/2m}$, *the unit sphere of the Wu metric in* $T_{(p_1,\hat{0})}E_{2m}$ *does not intersect the lower K-curve* C_{low} *and intersects the upper K-curve* C_{up} *in a unique point* (x^*, y^*) *with* $x^* \neq 0$.

Real analytic dependence of (x^*, y^*) on the base point, can be deduced by the arguments involving Fritz John's generalization of the Lagrange multiplier method to study the variation of the best fitting ellipsoid (as we move from one tangent space to another), as in [5]. The 'Lagrangian' functional adapted to study all the constraints and parameters jointly, is of the form

$$H\big(p, R_1(p), R_2(p), V_1(p), V_2(p), \Theta(p)\big) : (0, 2^{-1/2m}) \times (0, \infty)^5 \to \mathbb{R}^5 \tag{5.1}$$

for our case as well. This will be useful for analysing the regularity of the Wu metric across the thin sets M^0 and Z at the boundary of M^-. For the analysis of curvature, it is enough to determine the curvature tensor at points of the segment S where the matrix representing the Wu metric is diagonal. This requires knowledge about the metric tensor in a neighbourhood of S which we record below. First, let α^* be the value of the parameter α, where the upper K-curve meets the square transform of the best fitting ellipsoid. Writing $X = (\alpha^*)^2$, we remark that X may also be defined as the unique solution in the interval $(0, 1)$ of the equation

$$s^4 X^{2m-1} - (m+1)|p_1|^{2m} s^2 X^{m-1} + (m-2)|p_1|^{2m} s^2 X^m + 2|p_1|^{4m} = 0.$$

PROPOSITION 5.2

The Wu metric at $p \in M^-$ *is given by*

$$\frac{s^2 X^{2m-1}}{2F_s^2}\left(\frac{m^2 s^2}{|p_1|^2} dp_1 \otimes d\bar{p}_1 + \sum_{j=2}^{n}\left(\frac{F_{s_j}}{|p_1|^{2m}} - 1\right) dp_j \otimes d\bar{p}_j \right.$$
$$+ 2\Re \sum_{j=2}^{n} \frac{m\bar{p}_j}{\bar{p}_1} dp_j \otimes d\bar{p}_1 + \frac{mX^{m-1} - (m-1)X^m}{|p_1|^{2m}}$$
$$\left. \times \sum_{j,k=2, j\neq k}^{n} \bar{p}_j p_k dp_j \otimes d\bar{p}_k \right),$$

where $F_s = ms^2 X^{m-1} - (m-1)s^2 X^m - |p_1|^{2m}$, $s_j^2 = s^2 + |p_j|^2 = 1 - (|\hat{p}|^2 - |p_j|^2)$ *and* $F_{s_j} = ms_j^2 X^{m-1} - (m-1)s_j^2 X^m$. *Moreover, the Wu metric is real analytic on the set* M^-.

It turns out that on M^-, each component of the curvature tensor either vanishes identically or is a strictly negative real-valued function. We now list the components $R_{i\bar{j}k\bar{l}}$ of the curvature tensor, with (standard) notations as in [5], which are negative. First, consider the case where atmost two of the indices (i, j, k, l) are distinct and one of them is 1. Let $\gamma \in \{2, 3, \ldots, n\}$. It follows from [5] that if (i, j, k, l) is one of

$$(\gamma, \gamma, \gamma, \gamma), (\gamma, \gamma, 1, 1), (\gamma, 1, 1, \gamma), (1, \gamma, \gamma, 1), (1, 1, \gamma, \gamma), (1, 1, 1, 1)$$

then $R_{i\bar{j}k\bar{l}}$ is strictly negative. For quadruples of $1, \gamma$ not listed above, the corresponding curvature component vanishes. It can be checked that $R_{1\bar{1}\gamma\bar{\gamma}} \neq R_{\gamma\bar{1}1\bar{\gamma}}$ at $(p_1, \hat{0})$ for all $p_1 \in (0, 2^{-1/2m})$, and hence the Wu metric is nowhere Kähler on the orbit of the segment $(0, 2^{-1/2m})$, i.e. on M^-. Next, if $i = j > 1$ and $k = l > 1$, then $R_{i\bar{i}k\bar{k}}$ is negative. In case, both the indices i, j are distinct and > 1, it turns out that $R_{i\bar{j}k\bar{l}}$ is negative, iff $(k, l) = (j, i)$. In *all* other cases, $R_{i\bar{j}k\bar{l}} = 0$. It is then possible to establish assertions about the curvature on M^- as in Theorem 1.2.

6. The Wu metric along and across the thin sets Z and M^0

Analysis of the Wu metric in the remaining thin subsets Z and M^0 of E_{2m}, relies firstly on the smoothness of elementary power functions, as mentioned in Remark 3.2. We shall deal only with the case $m \notin \mathbb{N}$, as the complementary case is easier. Let us begin by analysing the smoothness of the Wu metric *at* the origin. This entails studying the behaviour of how the Kobayashi metric is changing at the origin. We recall that

$$K((0, 0, \ldots, 0), (v_1, \ldots, v_n)) = \frac{1}{\tilde{\alpha}},$$

where $\tilde{\alpha}$ is the unique positive solution of the equation

$$|v_1|^{2m} \tilde{\alpha}^{2m} + (|v_2|^2 + \cdots + |v_n|^2)\tilde{\alpha}^2 = 1.$$

It follows that $\tilde{\alpha}$ is $C^{[2m]}$-smooth as a function of $|v_1|$ and real analytic in $|v_2|, \ldots, |v_n|$. Hence, $\tilde{\alpha}$ is $C^{[2m]}$-smooth function of $|v_1|, |v_2|, \ldots, |v_n|$. Indeed, these conclusions can be derived by consulting the assertions concerning the regularity of the solution of C^k-smooth equations, in the implicit function theorem. We now examine the regularity of $K(p, v)$, when p varies across Z. It suffices to focus attention on the case when p varies through $S^\epsilon = \{(p_1, \hat{0}) : p_1 \in [0, \epsilon)\}$ for some $\epsilon > 0$. We shall simultaneously deal with joint regularity in the variables p, v as well though we shall be terse; in this regard, observe that the Kobayashi metric at a point p in a small neighbourhood of the origin, is of the form $K(\Phi_p(p), D\Phi_p(v))$ with $\Phi_p(z)$ as in (2.1). Note that $\Phi_p(z)$ is jointly real analytic in the variables z and (the parameter) p. Consequently, the $C^{[2m]}$-smoothness of $K(\cdot, \cdot)$ in a neighbourhood of the origin follows, as soon as we verify that $K(p, v)$ is $C^{[2m]}$-smooth for p varying in S^ϵ. As for v, it suffices to restrict attention in a neighbourhood of the set of points of contact of the Wu ellipsoid with the Kobayashi indicatrix in $T_p E_{2m}$. Recall from Proposition 5.1 that on M^-, the contact point of the square transforms of the Kobayashi indicatrix and the Wu ellipsoid, lies on the upper K-curve where the Kobayashi metric is described as follows (cf. eq. (7.29) of [3]):

$$K((p_1, \hat{0}), v) = \tilde{\alpha}|v_1|^2 \big/ \tilde{t}(|v_1|^2\tilde{\alpha}^2 - p_1^2\tilde{\alpha}^{2m}|v_1|^{2m}). \tag{6.1}$$

Here, $x = \tilde{\alpha}$ satisfies the equation $F(v, p, x) = 0$ with $p = (p_1, \hat{0})$, where

$$\tilde{t}^2 = \frac{2|v_1|^2}{|v_1|^2 + 2(1-1/m)|\hat{v}|^2 p_1^2 + |v_1|\sqrt{|v_1|^2 + 4(1-1/m)p_1^2|\hat{v}|^2}} \tag{6.2}$$

and

$$F(v, p, x) = \left(1 - \tilde{t}^2 \frac{|\hat{v}|^2 p_1^2}{|v_1|^2}\right)|v_1|^{2m}x^{2m} + \tilde{t}^2|\hat{v}|^2x^2 - 1.$$

An application of the implicit function theorem to $F(v, p, x)$ will enable us to write $\tilde{\alpha}$ as $C^{[2m]}$-smooth function of p and v, provided we verify $\frac{\partial F}{\partial x}(w, p, \tilde{\alpha}) \neq 0$, where $w = (w_1, w_2, \ldots, w_n)$ is any of the points corresponding to the point (x^*, y^*), obtained in Proposition 5.1; so, $w_1^2 = y^*$ and $w_2^2 + \cdots + w_n^2 = x^*$. To obtain a contradiction, assume that $\partial F/\partial x(w, p, \tilde{\alpha}) = 0$, i.e.,

$$\partial F/\partial x(w, p, \tilde{\alpha}) = 2m\big(1 - \tilde{t}^2|w_1|^{-2}|\hat{w}|^2p_1^2\big)|w_1|^{2m}x^{2m-1} + 2|\hat{w}|^2\tilde{t}^2x = 0.$$

Since $\tilde{\alpha} \neq 0$, it follows from the above equation that

$$(1 - \tilde{t}^2|w_1|^{-2}|\hat{w}|^2p_1^2)|w_1|^{2m}\tilde{\alpha}^{2m} = -|\hat{w}|^2\tilde{t}^2\tilde{\alpha}^2/m, \tag{6.3}$$

so that the defining equation for $\tilde{\alpha}$ can be rewritten as $\tilde{\alpha}^2 = m(m-1)^{-1}|\tilde{t}\hat{w}|^{-2}$. Substituting this into (6.3) and dividing throughout by $\tilde{t}^2|\hat{w}|^2p_1^2$ yields

$$\left(\frac{1}{p_1^2\tilde{t}^2}\frac{|w_1|^2}{|\hat{w}|^2} - 1\right)|w_1|^{2m-2} = -\frac{(m-1)^{m-1}}{m^m}\frac{(p_1^2\tilde{t}^2|\hat{w}|^2)^{m-1}}{p_1^2}\left(\frac{1}{p_1^2}\right)^{m-1}$$

Define $f(p_1, w) = |w_1|^2/p_1^2\tilde{t}^2|\hat{w}|^2$, so as to write the last equation as

$$(f(p_1, w) - 1)(f(p_1, w))^{m-1} = -(m-1)^{m-1}/m^m p_1^{2m}. \tag{6.4}$$

Denote by R the ratio $|w_1|/|\hat{w}|$, so as to write

$$f(p_1, w) = \frac{R^2}{2p_1^2} + \frac{R}{2p_1}\left(\frac{R^2}{p_1^2} + 4\left(1 - \frac{1}{m}\right)\right)^{1/2} + 1 - \frac{1}{m}.$$

At this point, recall that we are in the case $u \geq p_1$ (where $u = m|w_1|/|\hat{w}|$) or equivalently, $R/p_1 > 1/m$. Using this in the above equation gives $f(p_1, w) > 1$ which is a contradiction, since the right hand side of (6.4) is negative. Hence, we conclude that our assumption $\partial F/\partial x(w, p, \tilde{\alpha}) = 0$ must be wrong. This enables an application of the implicit function theorem to deduce that $\tilde{\alpha}$ is a $[2m]$-smooth function of (p, v). Next, (6.2) shows that $\tilde{t}$ is a smooth function. Further, notice that the expression (6.1) of the Kobayashi metric is a rational function (whose denominator is well-defined on E_{2m}) involving elementary power functions of $v, p, \alpha, \tilde{t}$. Hence, K^2 is a $C^{[2m]}$-smooth function of (p, v) near the origin. Next, another application of the implicit function theorem, this time to the functional H, mentioned at (5.1), leads to the desired smoothness of the Wu metric near Z. It may seem that in doing so, there may be a loss of a degree of smoothness for the Wu metric as compared to the $C^{[2m]}$-smoothness of the Kobayashi metric. However, it is possible to discern the optimal order of smoothness of the Wu metric to be $C^{[2m]}$ (as in Theorem 4.1 of [9]), owing to the fact that the meeting point of the Wu ellipse with the K-curve in the tangent spaces near the origin, lies away from the coordinate axes; this ensures the real analytic dependence of K^2 on the variables v. This together with an analysis of the explicit expression of K^2 yields the $C^{[2m]}$-smoothness of the component functions of H; thereby the $C^{[2m]}$-smoothness of the Wu metric is near the origin, subsequently, near Z. To remark now about the case $m \in \mathbb{N}$, similar analysis yields the conclusion that the Wu metric is real analytic near Z in this case.

In order to analyse the Wu metric on M^0, we do *not* need to assume that $m \notin \mathbb{N}$ for convenience. We only provide a couple of details to support the arguments in [5], for passing through to our present setting. Firstly, we claim that the Kobayashi indicatrix is C^2-smooth but not C^3-smooth on E_{2m}, for $m > 1$. To establish this, it suffices to analyse the smoothness at the joining point $u = p_1$ of the upper and lower K-curves. Note that the lower K-curve (cf. eq. (2.2)) is a straight line and hence $\partial^l y/\partial x^l \equiv 0$ for $l \geq 2$. We now show that, for the upper K-curve $\big(x(\alpha), y(\alpha)\big)$, $\partial^l y/\partial x^l$ is zero for $l = 1, 2$ but not for $l = 3$ at the joining point. To this end, note that the joining point $u = p_1$ corresponds to the limiting value 1 for the parameter α.

$$\partial^2 y/\partial x^2 = \dot{x}(\alpha)\ddot{y}(\alpha) - \dot{y}(\alpha)\ddot{x}(\alpha)/\big(\dot{x}(\alpha)\big)^3. \tag{6.5}$$

A direct computation shows that the numerator of the right-hand side above is given by

$$-8m^{-1}(m-1)p_1^{2m+2}\alpha^{-6m-1}(1-\alpha^2)(m\alpha^{2m-2}+(m-1)\alpha^{2m}-(2m-1)p_1^{2m})^2,$$

which as a function of p_1 vanishes identically when $\alpha = 1$. This confirms that the values of $\partial^2 y/\partial x^2$ of the upper and lower K-curves match at the 'joining point' and leads to the C^2-smoothness of the Kobayashi indicatrix in $T_p E_{2m}$ for $p \in S$. Next, we evaluate $\partial^3 y/\partial x^3$ and show that the values fail to match up. Indeed, as the numerator in (6.5) is zero at $\alpha = 1$, it can be calculated that at the joining point, $\partial^3 y/\partial x^3$ is given by

$$\big(\dot{x}(1)\dddot{y}(1) - \dot{y}(1)\dddot{x}(1)\big)/(\dot{x}(1))^4 \tag{6.6}$$

provided $\dot{x}(1) \neq 0$ – it turns out that $\dot{x}(1) \neq 0$ precisely when $m = 1/2$. The numerator of (6.6) can be evaluated to be $16m^{-1}(m-1)(2m-1)^2 p_1^{2m+2}(1 - p_1^{2m})^2$. As $1 - p_1^{2m}$ never vanishes on E_{2m} and we are in the case $m \neq 1$ and more importantly $m \neq 1/2$, this proves that the Kobayashi metric is not C^3 smooth on E_{2m} when $m > 1$. For the case $m < 1$, we only need to look at the indicatrix at the origin, to conclude that the

Kobayashi metric is not of class C^2. Getting back again to the case $m > 1$, it remains to study how these indicatrices vary with respect to p, to establish C^2-smoothness of the Kobayashi metric. Indeed, first note the real analyticity in the expression for the 'joining point' given by $((1-p_1^{2m})^2, p_1^2(1-p_1^{2m})^2/m^2)$. While this provides a clue about the C^2-smoothness, for a proof it remains to check that $\partial^2/\partial p_1^2$ and the $\partial^2/\partial p_1 \partial x$ derivatives of $(f_{\rm up}(\sqrt{x}))^2$ and $(f_{\rm low}(\sqrt{x}))^2$ (which are actually functions of both x and p_1) agree when evaluated at the joining point. This indeed happens and involves several implicit differentiations. For this we note that (2.3), (2.4) do *not* together form a parametrization of the K-curve, which is of desired smoothness at the joining point. Nevertheless, a calculus with these parametrizations confirms that the K-curves do coalesce to form a regular surface in the (x, y, p)-space, which is C^2-smooth. More precisely, *all* second order partial derivatives of $(f_{\rm up}(\sqrt{x}, p_1))^2$ and $(f_{\rm low}(\sqrt{x}, p_1))^2$ match as required. This leads to the C^2-smoothness of the aggregate of the indicatrices, thereby yielding C^2-smoothness of the Kobayashi metric, first at reference points of the form $(p_1, \hat{0})$ for $0 < p_1 < 1$; thereafter, also on $E_{2m} \setminus Z$ which is the orbit of the segment S under the real analytic action of ${\rm Aut}(E_{2m})$. Since we have already discussed the smoothness of the Kobayashi metric in a neighbourhood of the origin, together with the foregoing observations, this completes the verification that the optimal overall smoothness of the Kobayashi metric is indeed C^2, when $m > 1$. One may now proceed with arguments employed in [5] (past Proposition 7 therein) to conclude the regularity-analysis on M^0.

We now conclude the article with remarks, for completeness, on the curvature of the Wu metric on the critical sets M^0 and Z. It is enough to focus attention near the point $p = (2^{-1/2m}, 0)$ in M^0. As M^0 is a finite type hypersurface, it cannot contain any non-trivial germ of a complex analytic variety. In particular, any Riemann surface through p can intersect M^0 only at p or along a smooth curve. This together with the fact that the holomorphic curvature is bounded above by a negative constant on either sides of M^0 namely, M^- and M^+, paves the way for applying the lemma in Appendix B of [5], thereby rendering the negativity of the holomorphic curvature current on M^0. Recall that Z is a complex hypersurface and consequently, Z intersects an embedded Riemann surface in E_{2m} only in a discrete set of points, unless it is contained in (as an open subset of) Z. In the former case, the required negativity of the holomorphic curvature current follows from the arguments using the lemma in Appendix B of [5], as the holomorphic curvature is strongly negative in any small (deleted) neighbourhood of Z consisting of points of M^-. In the latter case, we may assume that the Riemann surface is an open subset U of Z containing the origin. Pick any one-dimensional complex subspace l in Z – recall that Z is in fact a complex hyperplane. So $l_U = l \cap U$ is contained in l. We restrict attention to a small disc D about the origin with $D \subset l_U$. Now, by what we had observed earlier about the regularity of the Wu metric near Z, we know that the Wu metric is atleast C^2-smooth (as $m > 1$) in some small neighbourhood of each point of Z (which consists only of points of Z or M^-). Also recall that Z is the orbit of the origin under ${\rm Aut}(E_{2m})$. More explicitly, Z is the set of points z in E_{2m} where $z_1 = 0$ and therefore Z is precisely the set of fixed points for the automorphism $(z_1, \hat{z}) \to (-z_1, \hat{z})$ of E_{2m}. As automorphisms of E_{2m} are isometries of E_{2m} equipped with its Wu metric $h_{E_{2m}}$ and fixed point sets for isometries are totally geodesic, we conclude that Z is a totally geodesic complex submanifold of $(E_{2m}, h_{E_{2m}})$. (We recall that though holomorphic maps are not exactly contractions in the Wu metric, biholomorphisms are isometries with respect to the Wu metric. This follows by noting that derivatives of biholomorphisms render linear isomorphisms between tangent spaces and therefore transform ellipsoids into ellipsoids. Indeed, this sets up a correspondence between ellipsoids on the tangent spaces which correspond under the biholomorphism. The fact that Kobayashi indicatrices are preserved by the derivatives of biholomorphisms combined with the uniqueness of the minimal volume ellipsoid (containing any particular Kobayashi indicatrix in a tangent space), shows that the Wu ellipsoids are preserved as well. This yields the invariance of the Wu metric under biholomorphisms.) This assures us that the holomorphic curvature in the direction of l is realized by the Gaussian curvature of the Wu metric restricted to D and the desired negativity of the holomorphic curvature follows.

Acknowledgements

The authors would like to thank their advisor, Kaushal Verma for suggesting them this problem namely, the study of the Wu metric on the class of egg domains as stated in the abstract of the present article. Both the authors were supported by the DST-INSPIRE Fellowship of the Government of India.

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COMMUNICATING EDITOR: Gadadhar Misra