



Abelianization of the F -divided fundamental group scheme

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Abstract. Let (X, x_0) be a pointed smooth proper variety defined over an algebraically closed field. The Albanese morphism for (X, x_0) produces a homomorphism from the abelianization of the F -divided fundamental group scheme of X to the F -divided fundamental group of the Albanese variety of X . We prove that this homomorphism is surjective with finite kernel. The kernel is also described.

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1. Introduction

Let X be a proper and smooth variety defined over an algebraically closed field k . Once a k -point x_0 of X is chosen, we have the Albanese morphism

$$\alpha : X \longrightarrow A,$$

as explained in p. 10 of [13]. Let π_1^{et} stand for the étale fundamental group as defined in [12] and use the superscript ‘ab’ to denote the abelianization. Since $\pi_1^{\text{et}}(A, \alpha(x_0))$ is abelian (XI, 2.1, p. 222 of [12]) the homomorphism of fundamental groups induced by α factors through the quotient $\pi_1^{\text{et}}(X, x_0) \longrightarrow \pi_1^{\text{et}}(X, x_0)^{\text{ab}}$. It is known that the resulting homomorphism

$$\alpha_{\#} : \pi_1^{\text{et}}(X, x_0)^{\text{ab}} \longrightarrow \pi_1^{\text{et}}(A, 0)$$

is surjective with finite kernel; the kernel can also be described (see Lemma 5 of [5]).

Analogous considerations can be made for the essentially finite fundamental group scheme π_1^{ef} [10], since it is known that $\pi_1^{\text{ef}}(A, \alpha(x_0))$ is abelian [11]. This work was undertaken by Antei, who showed that the homomorphism $\alpha_{\#} : \pi_1^{\text{ef}}(X, x_0)^{\text{ab}} \longrightarrow \pi_1^{\text{ef}}(A, 0)$ is surjective with finite kernel (see [1]). Langer [7] treated the analogous property in the setting of the S -fundamental group scheme.

Our aim here is to address the question for the F -divided fundamental group scheme [2] and generalize the analysis made in loc. cit. More precisely, we prove the following (see Theorem 4.1):

- The homomorphism from the abelianized F -divided fundamental group scheme of X to the F -divided fundamental group scheme of A (which is abelian), induced by α , is surjective.
- The kernel of the above surjective homomorphism is finite.

We also describe this kernel.

Notation and standard terminology

- (1) Let k be an algebraically closed field of characteristic $p > 0$.
- (2) If Y is any k -scheme, let $F : Y \rightarrow Y$ stand for the *absolute* Frobenius morphism: on the underlying topological space it is just the identity and $F^\# : \mathcal{O}_Y \rightarrow \mathcal{O}_Y$ is the usual Frobenius morphism of rings in characteristic p .
- (3) The F -divided sheaves are the *flat* sheaves of Definition 1.1 on p. 3 of [3]. We adopt the terminology introduced in p. 695, Definition 4 of [2].
- (4) The *rank* of an F -divided sheaf $\{\mathcal{E}_i, \sigma_i\}_{i \in \mathbb{N}}$ is the rank of \mathcal{E}_0 (which are locally free due to Lemma 6 of [2]).
- (5) The fundamental group scheme for the F -divided sheaves on a pointed smooth k -scheme (Y, y_0) , call it $\Pi(Y, y_0)$, is the one introduced in p. 696, Definition 7 of [2]: the category of F -divided sheaves is equivalent, as a Tannakian category, to the category of finite dimensional representations of $\Pi(Y, y_0)$.
- (6) Let G be an affine group scheme over k . A *quotient* of G is a quotient in the sense of Waterhouse (p. 114, 15.1 of [16]). A quotient morphism of affine group schemes $G \rightarrow H$ is also said to be surjective.
- (7) Given an affine group scheme G over k , we let G^{ab} , G^{uni} and G^{diag} stand respectively for the largest abelian, largest unipotent, and largest diagonal quotient of G .
- (8) Given any abelian group Λ , we write $\text{Diag}(\Lambda)$ for the affine group scheme defined in Part 1, 2.5, p.26 of [4]. It is simply the ‘torus’ having Λ as its group of characters.
- (9) If Y is a smooth and proper variety over k , its *Picard scheme* will be denoted by $\mathbf{Pic}(Y)$. We reserve the symbol $\text{Pic}(Y)$ for the *Picard group*. The connected component of the identity of $\mathbf{Pic}(Y)$ will be denoted by the familiar $\mathbf{Pic}^0(Y)$; we write $\text{Pic}^0(Y)$ for its group of k -points. At present, the best general reference for the Picard scheme seems to be [6].
- (10) Finite groups will be identified with their associated finite affine group schemes (§2.3 of [16]). The same identifications are extended to profinite groups.
- (11) For any abelian group scheme G and any positive integer m , we write $G[m]$ for the kernel of multiplication by m .
- (12) For an abelian variety B over k , we let $T_p B$ stand for the pro-étale group scheme $\varprojlim B[p^n](k)$. The reader should bear in mind that $T_p B$ is an *unipotent* group scheme.

2. The Albanese variety

Let X be a proper and smooth variety defined over k . In what follows, the underlying reduced subscheme of $\mathbf{Pic}^0(X)$ is denoted by $\mathbf{Pic}_{\text{red}}^0(X)$. We fix a closed point $x_0 \in X$. Consider the unique Poincaré line bundle on $X \times \mathbf{Pic}^0(X)$ which is trivial on $x_0 \times \mathbf{Pic}^0(X)$.

Restrict the Poincaré bundle to $X \times \mathbf{Pic}_{\text{red}}^0(X)$. Viewing this restriction as a line bundle on $\mathbf{Pic}_{\text{red}}^0(X)$ parametrized by X , we get a morphism

$$\alpha : X \longrightarrow A := \mathbf{Pic}^0(\mathbf{Pic}_{\text{red}}^0(X)). \quad (2.1)$$

The abelian variety A and the morphism α defined above enjoy an universal property characterizing them as the Albanese variety and Albanese morphism in the sense of [14]. Since we have no use for this universal property, the reader is only required to bear in mind the definition in (2.1).

PROPOSITION 2.1

The homomorphism

$$\alpha^* : \mathbf{Pic}^0(A) \longrightarrow \mathbf{Pic}_{\text{red}}^0(X)$$

induced by the Albanese morphism α in (2.1) is an isomorphism.

Proof. For an abelian variety B , we have $\mathbf{Pic}^0(\mathbf{Pic}^0(B)) = B$, with the identification given by the Poincaré line bundle on $B \times \mathbf{Pic}^0(B)$. More precisely, the classifying morphism $B \longrightarrow \mathbf{Pic}^0(\mathbf{Pic}^0(B))$ associated to the above Poincaré line bundle is an isomorphism. In particular, we have $\mathbf{Pic}^0(A) = \mathbf{Pic}_{\text{red}}^0(X)$. Now it is straightforward to check that α^* is the identity morphism of $\mathbf{Pic}_{\text{red}}^0(X)$. \square

2.1 The group of isomorphism classes of F -divided sheaves of rank one

Define

$$\vartheta(X) := \left\{ \begin{array}{l} F\text{-divided sheaves} \\ \text{of rank one on } X \end{array} \right\} / \text{isomorphisms}. \quad (2.2)$$

Under the tensor product of line bundles, $\vartheta(X)$ is an abelian group. Since X is assumed to be proper, the structure of $\vartheta(X)$ is easily determined from the Picard group of X by means of the following construction (Section 3 of [2]).

DEFINITION 2.3

For an abelian group G , let $[p] : G \longrightarrow G$ be the homomorphism $z \longmapsto p \cdot z$. We define $G\langle p \rangle$ as the following projective limit:

$$\varprojlim (\dots \xrightarrow{[p]} G \xrightarrow{[p]} G \xrightarrow{[p]} \dots).$$

For each invertible sheaf \mathcal{L} of X , we write $[\mathcal{L}]$ for its class in the Picard group $\text{Pic}(X)$.

PROPOSITION 2.2 (cf. p. 7, Theorem 1.8 of [3] and p. 706, Lemma 20 of [2])

Let $\mathcal{L} = (\mathcal{L}_0, \mathcal{L}_1, \dots)$ be an F -divided sheaf of rank one. Write

$$\tau(\mathcal{L}) = ([\mathcal{L}_0], [\mathcal{L}_1], \dots) \in \text{Pic}(X)\langle p \rangle.$$

Then, τ induces an isomorphism between the abelian groups $\vartheta(X)$ (see (2.2)) and $\text{Pic}(X)\langle p \rangle$ (see Definition 2.3).

Proof. If \mathcal{L} and \mathcal{M} are isomorphic F -divided sheaves of rank one, then clearly $\tau(\mathcal{L}) = \tau(\mathcal{M})$, and hence τ is indeed a homomorphism from $\vartheta(X)$ to $\text{Pic}(X)\langle p \rangle$. It is obvious that τ is surjective, while its injectivity is an observation due to p. 6, Proposition 1.7 of [3]. \square

COROLLARY 2.3

Let $\text{NS}(X)$ stand for the Néron–Severi group of X and $\text{NS}(X)'$ for its subgroup of elements whose order is finite and prime to p . Then $\vartheta(X)$ sits in a short exact sequence

$$0 \longrightarrow \text{Pic}^0(X)\langle p \rangle \longrightarrow \vartheta(X) \longrightarrow \text{NS}(X)' \longrightarrow 0.$$

Proof. By definition, we have a short exact sequence of abelian groups

$$0 \longrightarrow \text{Pic}^0(X) \longrightarrow \text{Pic}(X) \longrightarrow \text{NS}(X) \longrightarrow 0$$

which gives rise to an exact sequence of the projective systems considered in Definition 2.3. Applying the projective limit functor and using that

$$[p] : \text{Pic}^0(X) \longrightarrow \text{Pic}^0(X)$$

is surjective (p. 59 of [8]), we conclude that the sequence

$$0 \longrightarrow \text{Pic}^0(X)\langle p \rangle \longrightarrow \text{Pic}(X)\langle p \rangle \longrightarrow \text{NS}(X)\langle p \rangle \longrightarrow 0$$

is exact (§3.5 of [17]). As $\text{NS}(X)$ is finitely generated (Exp. 13, Theorem 5.1 of [15]) and the functor $G \mapsto G\langle p \rangle$ annihilates finitely generated free abelian groups, the corollary follows from Proposition 2.2 and the easily verified by isomorphisms $\text{NS}(X)' \simeq \text{NS}(X)'\langle p \rangle \simeq \text{NS}(X)\langle p \rangle$. \square

PROPOSITION 2.4

The natural homomorphism

$$\alpha^* : \vartheta(A) \longrightarrow \vartheta(X)$$

is injective, and its cokernel is the group $\text{NS}(X)'$ introduced in Corollary 2.3. In particular, the kernel of

$$\alpha_{\#} : \Pi(X, x_0)^{\text{diag}} \longrightarrow \Pi(A, \alpha(x_0))^{\text{diag}}$$

is isomorphic to $\text{Diag}(\text{NS}(X)')$.

Proof. We have a commutative diagram

$$\begin{array}{ccc} \vartheta(A) & \xrightarrow{\alpha^*} & \vartheta(X) \\ \sigma \downarrow & & \downarrow \tau \\ \text{Pic}(A)\langle p \rangle & \longrightarrow & \text{Pic}(X)\langle p \rangle \\ j \uparrow & & \uparrow i \\ \text{Pic}^0(A)\langle p \rangle & \longrightarrow & \text{Pic}^0(X)\langle p \rangle . \end{array}$$

From Proposition 2.2, we know that σ and τ are isomorphisms. From Corollary 2.3 and Corollary 2, Ch. IV, Section 19, p. 165 of [8], we know that j is an isomorphism. The same Corollary 2.3 also guarantees that i is injective with co-kernel $\text{NS}(X)'$. Since, by Proposition 2.1, the homomorphism $\text{Pic}^0(A)\langle p \rangle \longrightarrow \text{Pic}^0(X)\langle p \rangle$ is an isomorphism, we are done. The statement concerning group schemes follows easily from the fact that the functor Diag takes exact sequences of abelian groups to exact sequences of affine abelian group schemes. \square

3. The group $\pi^{\text{et}}(X, x_0)^{\text{ab}}$

Since $\Pi(X, x_0)^{\text{uni}}$ is proetale (Corollary 16, p. 704 of [2]), the following considerations are in order.

PROPOSITION 3.1 (cf. p. 308, Lemma 5 of [5])

The homomorphism

$$\pi_1^{\text{et}}(X, x_0)^{\text{ab, uni}} \longrightarrow \pi_1^{\text{et}}(A, \alpha(x_0))^{\text{uni}} = T_p(A)$$

is surjective and its kernel K is the group of k -points of the Cartier dual of the local affine group scheme

$$\mathbf{Pic}^0(X)/\mathbf{Pic}_{\text{red}}^0(X).$$

Proof. This can be easily extracted from p. 308, Lemma 5 of [5], but we give details for the convenience of the reader.

We will consider the category of finite groups as a full subcategory of the category of finite group schemes over k . In the same spirit, the category of pro-finite groups is regarded as a subcategory of the category of pro-etale group schemes over k . We know that

$$\text{Hom}(\pi_1^{\text{et}}(X, x_0)^{\text{uni}}, \mathbb{Z}/p^n\mathbb{Z}) \simeq H_{\text{et}}^1(X, \mathbb{Z}/p^n\mathbb{Z}),$$

and the latter is due to p. 131, Corollary 4.18 of [9], isomorphic to

$$\text{Hom}(\mu_{p^n}, \mathbf{Pic}(X)) = \text{Hom}(\mu_{p^n}, \mathbf{Pic}^0(X)).$$

Let Q stand for the quotient $\mathbf{Pic}^0(X)/\mathbf{Pic}_{\text{red}}^0(X)$. Then, using that $\text{Ext}^1(B, \mu_m) = 0$ for any abelian variety B (Remark, p. 310 of [5]), we arrive at the exact sequence

$$\begin{aligned} 0 \longrightarrow \text{Hom}(\mu_{p^n}, \mathbf{Pic}_{\text{red}}^0(X)) &\longrightarrow \text{Hom}(\mu_{p^n}, \mathbf{Pic}^0(X)) \\ &\longrightarrow \text{Hom}(\mu_{p^n}, Q) \longrightarrow 0. \end{aligned}$$

Since

$$\begin{aligned} \text{Hom}(\mu_{p^n}, \mathbf{Pic}_{\text{red}}^0(X)) &= \text{Hom}(\mu_{p^n}, \mathbf{Pic}_{\text{red}}^0(X)[p^n]) \\ &\simeq \text{Hom}((\mathbf{Pic}_{\text{red}}^0(X)[p^n])^\vee, \mathbb{Z}/p^n\mathbb{Z}) \\ &\simeq \text{Hom}(A[p^n], \mathbb{Z}/p^n\mathbb{Z}) \end{aligned}$$

(see §2.4 of [16] for the first isomorphism and Ch. 15, p. 134, Theorem 1 of [8] for the second isomorphism), it follows that

$$\begin{aligned} \mathrm{Hom}(\mu_{p^n}, \mathbf{Pic}_{\mathrm{red}}^0(X)) &\simeq \mathrm{Hom}(A[p^n], \mathbb{Z}/p^n\mathbb{Z}) \\ &\simeq \mathrm{Hom}(T_p A, \mathbb{Z}/p^n\mathbb{Z}) \end{aligned}$$

thus completing the proof. \square

4. The main result

Since the F -divided fundamental group $\Pi(A, \alpha(x_0))$ of the Albanese variety A is abelian (p. 707, Theorem 21 of [2]), we have a commutative diagram

$$\begin{array}{ccc} \Pi(X, x_0) & \longrightarrow & \Pi(A, \alpha(x_0)) \\ \downarrow & \nearrow \alpha_{\#} & \\ \Pi(X, x_0)^{\mathrm{ab}} & & \end{array} .$$

Theorem 4.1. *Let $\mathrm{NS}(X)'$ be as in Corollary 2.3 and K as in Proposition 3.1. The homomorphism*

$$\alpha_{\#} : \Pi(X, x_0)^{\mathrm{ab}} \longrightarrow \Pi(A, \alpha(x_0))$$

is surjective, and its kernel is

$$\mathrm{Diag}(\mathrm{NS}(X)') \times K.$$

Proof. We can write $\Pi(X, x_0)^{\mathrm{ab}}$ as a product of $U \times \Delta$, where U is unipotent and Δ is diagonal (§9.5, p. 70, Theorem of [16]). By definition, $\Pi(X, x_0)^{\mathrm{diag}}$ is the largest diagonal quotient of $\Pi(X, x_0)^{\mathrm{ab}}$, and there are no nontrivial homomorphisms from U to Δ (use §8.3, Corollary, p. 65 of [16]). Therefore, we have $\Pi(X, x_0)^{\mathrm{diag}} \simeq \Delta$. The same argument shows, employing Exercise 6, p. 67 of [16], that $\Pi(X, x_0)^{\mathrm{ab,uni}} \simeq U$. As remarked in p. 707, Theorem 21 of [2], we have isomorphisms

$$\begin{aligned} \Pi(A, \alpha(x_0)) &\simeq \mathrm{Diag}(\mathrm{Pic}^0(A)\langle p \rangle) \times \pi_1^{\mathrm{et}}(A, \alpha(x_0))^{\mathrm{uni}} \\ &\simeq \mathrm{Diag}(\mathrm{Pic}^0(A)\langle p \rangle) \times T_p(A). \end{aligned}$$

Since there are no nontrivial homomorphisms from an unipotent (respectively, diagonal) affine group scheme to a diagonal (respectively, unipotent) affine group scheme, the homomorphism $\alpha_{\#}$ in the statement is given by a pair of homomorphisms

$$\alpha_{\#}^{\mathrm{diag}} \times \alpha_{\#}^{\mathrm{uni}} : \Pi(X, x_0)^{\mathrm{diag}} \times \Pi(X, x_0)^{\mathrm{ab,uni}} \longrightarrow \mathrm{Diag}(\mathrm{Pic}^0(A)\langle p \rangle) \times T_p(A).$$

From Corollary 2.4, we know that the homomorphism $\alpha_{\#}^{\mathrm{diag}}$ is surjective with kernel $\mathrm{Diag}(\mathrm{NS}(X)')$. From p. 704, Corollary 16 of [2], we know that $\Pi(X, x_0)^{\mathrm{ab,uni}}$ is pro-etale, that is, $\Pi(X, x_0)^{\mathrm{ab,uni}} \simeq \pi_1^{\mathrm{et}}(X, x_0)^{\mathrm{ab,uni}}$; applying Proposition 3.1, we conclude that $\alpha_{\#}^{\mathrm{uni}}$ is surjective with kernel K . \square

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