Proc. Indian Acad. Sci. (Math. Sci.) Vol. 127, No. 2, April 2017, pp. 263–268.
DOI 10.1007/s12044-017-0329-4

Nonfiniteness of Hilbert–Kunz functions

V TRIVEDI

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400 005, India
E-mail: vija@math.tifr.res.in

MS received 30 September 2015; revised 18 February 2016

Abstract. Here we answer a question of C. Huneke, by giving an example of a family (parametrized by $\mathbb{A}^1_{\overline{\mathbb{F}}_p}$) of one-dimensional reduced Cohen–Macaulay rings such that every member has the same Hilbert–Kunz multiplicity but there are infinitely many Hilbert–Kunz functions in the family.

Keywords. Cohen-Macaulay rings; Hilbert–Kunz functions; Hilbert–Kunz multiplicity.

1991 Mathematics Subject Classification. 13A35, 11T99, 13A02.

Let $(R, \mathbf{m})$ be a Noetherian local ring of prime characteristic $p > 0$ and of dimension d. Then the Hilbert–Kunz function of R with respect to $\mathbf{m}$ is given by

$$\mathrm{HK}(R, \mathbf{m})(q) = \frac{1}{q^d}\ell(R/\mathbf{m}^{[q]}),$$

where $q = p^n$, $\mathbf{m}^{[q]} = n$-th Frobenius power of $\mathbf{m}$ = ideal generated by q-th power of elements of $\mathbf{m}$, and $\ell(R/\mathbf{m}^{[q]})$ denotes the length of the R-module $R/\mathbf{m}^{[q]}$. Moreover, for a pair $(R, \mathbf{m})$ the Hilbert–Kunz multiplicity is defined as

$$e_{\mathrm{HK}}(R, \mathbf{m}) = \lim_{n\to\infty} \frac{\ell(R/\mathbf{m}^{[p^n]})}{p^{nd}}.$$

Monsky [2] proved that this limit exists and

$$\mathrm{HK}(R, \mathbf{m})(q) = e_{\mathrm{HK}}(R, \mathbf{m})q^d + O(q^{d-1}).$$

In case $\dim R = 1$, he proved (Theorem 3.11 of [2]) that there is a periodic function $\Delta : \mathbb{N} \to \mathbb{N}$ such that, for $n >> 0$,

$$\mathrm{HK}(R, \mathbf{m})(p^n) = e_0(R, \mathbf{m})p^n + \Delta(n).$$

It is easy to prove that in the case of $\dim R = 1$, $e_{\mathrm{HK}}(R, \mathbf{m}) = e_0(R, \mathbf{m})$, where $e_0(R, \mathbf{m})$ is the usual Hilbert–Samuel multiplicity of $(R, \mathbf{m})$.

Later, Kreuzer [1] in 2007 proved that, for a standard graded one-dimensional ring defined over a finite field $\mathbb{F}_{p^e}$ the period of Δ divides e.

Now we state the following question of C. Huneke (private communication):

Question. Are there only finitely many Hilbert–Kunz functions for a set of Cohen–Macaulay local rings with a fixed dimension and a fixed Hilbert–Kunz multiplicity e_{HK}? Does this hold at least when $\dim R = 1$?

Recall that in [3], we had proved that if $(R, \mathbf{m})$ is a Cohen–Macaulay local ring of dimension d with Hilbert–Samuel polynomial

$$P_R(n) = e_0(R,\mathbf{m})\binom{n+d-1}{d} - e_1(R,\mathbf{m})\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(R,\mathbf{m}),$$

then the postulation number for the pair $(R, \mathbf{m})$ and all the higher coefficients of $e_1(R, \mathbf{m})$, ..., $e_d(R, \mathbf{m})$ are bounded in terms of the Hilbert–Samuel multiplicity $e_0(R, \mathbf{m})$. In particular, for a given set of Cohen–Macaulay local rings with given dimension and multiplicity (say with respect to maximal ideal), there are only finitely many possible Hilbert–Samuel polynomials and functions.

Note that, since $e_0(R, \mathbf{m})/d! \leq e_{\mathrm{HK}}(R, \mathbf{m})$ and $e_0(R, \mathbf{m})$ is a positive integer, any given set of Cohen–Macaulay rings with fixed e_{HK} will have finitely many possible Hilbert–Samuel multiplicities and hence finitely many Hilbert polynomials and in fact finitely many possible Hilbert functions.

In this note we give an example of a family of one-dimensional reduced Cohen–Macaulay local rings (can also be considered as standard graded rings with irrelevant maximal ideals) $(S_\alpha, \mathbf{m}_\alpha)$ of characteristic p, parametrized by $\alpha \in \mathbb{A}^1_{\bar{\mathbb{F}}_p}$, such that $\mathrm{HK}(S_\alpha, \mathbf{m}_\alpha) = 4q + \Delta_\alpha(n)$, where Δ_α is of period = the degree of $\mathbb{F}_p[\alpha]/\mathbb{F}_p$. In particular, there are infinitely many HK functions with the same e_{HK}.

More specifically we prove the following:

Theorem 1. *Let $k \supseteq \bar{\mathbb{F}}_p$, where $p > 2$. Let $\alpha \in k \setminus \{0, 1, -1\}$, and let*

$$S_\alpha = \frac{k[X,Y]}{(Y(Y^2 - X^2)(Y - \alpha X))} = \frac{k[X,Y]}{(\alpha X^3 Y - X^2 Y^2 - \alpha X Y^3 + Y^4)}.$$

Let $n_0(\alpha)$ = degree of $\mathbb{F}_p[\alpha]/\mathbb{F}_p$. In particular, if α is transcendental then we take $n_0(\alpha) = \infty$. Then

$$\mathrm{HK}(\hat{S}_\alpha, \mathbf{m}_\alpha \hat{S}_\alpha)(q) = \mathrm{HK}(S_\alpha, \mathbf{m}_\alpha)(q) = 4q + \Delta_\alpha(n),$$

where

$$\Delta_\alpha(n) = -4 \quad \text{if } n \text{ is not divisible by } n_0(\alpha) \text{ (or if } n_0(\alpha) = \infty),$$
$$\Delta_\alpha(n) = -3 \quad \text{if } n \text{ is divisible by } n_0(\alpha).$$

Proof. We fix $\alpha \in \bar{\mathbb{F}}_p$ and denote $S_\alpha = R$. Note that the hypothesis $n_0(\alpha)$ = deg of $\mathbb{F}_p[\alpha]/\mathbb{F}_p$ means $n_0(\alpha) \geq 1$ is the least integer, such that $\alpha \in \mathbb{F}_q$ for $q = p^{n_0}$. Since R is a graded ring, we can write $R = \oplus_{m \geq 0} R_m$ in the canonical way and $\mathbf{m} = R_+$ the irrelevant maximal ideal. Then

$$\mathrm{HK}(R, \mathbf{m})(q) = \ell(R_0) + \cdots + \ell(R_{q-1}) + \sum_{m \geq 0} \ell\left(\frac{R_{m+q}}{\mathrm{Im}(R_1^{[q]} \otimes R_m)}\right).$$

Note that $\ell(R_0) = 1$, $\ell(R_1) = 2$, $\ell(R_2) = 3$ and $\ell(R_m) = 4$, for all $m \geq 3$. Now we make the following claim.

Claim.

(1) $\ell\left(\frac{R_q}{\mathrm{Im}(R_1^{[q]} \otimes R_0)}\right) = 2.$

(2) $\ell\left(\frac{R_{q+1}}{\mathrm{Im}(R_1^{[q]} \otimes R_1)}\right) = 0$, if $\alpha \notin \mathbb{F}_q$ and $\ell\left(\frac{R_{q+1}}{\mathrm{Im}(R_1^{[q]} \otimes R_1)}\right) = 1$, if $\alpha \in \mathbb{F}_q$.

(3) $\ell\left(\frac{R_{q+m}}{\mathrm{Im}(R_1^{[q]} \otimes R_m)}\right) = 0$, for all $m \geq 2$.

Proof of the Claim: We note that

$$\ell\left(\frac{R_{q+m}}{\mathrm{Im}(R_1^{[q]} \otimes R_m)}\right) = \ell(R_{q+m}) - (\ell(R_1^{[q]} \otimes R_m) - \ell(\ker \phi_m)),$$

where $\phi_m : R_1^{[q]} \otimes R_m \to R_{q+m}$. Since $\ell(R_{q+m}) = 4$, for $m \geq 0$ (as $q \geq 3$), we have

$$\ell\left(\frac{R_{q+m}}{\mathrm{Im}(R_1^{[q]} \otimes R_m)}\right) = 4 - 2\ell(R_m) + \ell(\ker \phi_m). \tag{1}$$

Part (1). It is easy to check that $\ell(\ker \phi_0) = 0$. Therefore,

$$\ell\left(\frac{R_q}{\mathrm{Im}(R_1^{[q]} \otimes R_0)}\right) = \ell(R_q) - 2\ell(R_0) = 4 - 2 = 2.$$

Part (2). We note that $\ell(R_{q+1}/\mathrm{Im}(R_1^{[q]} \otimes R_1)) = \ell(\ker \phi_1)$ and $R_1^{[q]}$ and R_1 are two dimensional k-vector spaces with bases $\{X^q, Y^q\}$ and $\{X, Y\}$ respectively. Let $A = k[X, Y]$ with standard grading, then we have a commutative diagram of the canonical maps

$$\begin{array}{ccc} A_1^{[q]} \otimes A_1 & \xrightarrow{\psi_1} & A_{q+1} \\ \downarrow & & \downarrow \\ R_1^{[q]} \otimes R_1 & \xrightarrow{\phi_1} & R_{q+1} \end{array},$$

where the first vertical arrow is an isomorphism and the map ψ_1 is injective. Any $f(X, Y) \in R_1^{[q]} \otimes R_1$ can thus be written as

$$f(X, Y) = u_0 X^{q+1} + u_1 X^q Y + u_2 X Y^q + u_3 Y^{q+1}, \text{ where } u_0, u_1, u_2, u_3 \in k.$$

Let $f(X, Y) \in \ker \phi_1$, then $f(X, Y) \in (Y^4 - \alpha X Y^3 - X^2 Y^2 + \alpha X^3 Y)$ in $k[X, Y]$. This implies that $u_0 = 0$ and therefore there exists $a_0, \ldots, a_{q-3} \in k$ such that the identity

$$u_1 X^q + u_2 X Y^{q-1} + u_3 Y^q = (\alpha X^3 - X^2 Y - \alpha X Y^2 + Y^3) \times (a_0 X^{q-3} + a_1 X^{q-4} Y + \cdots + a_{q-3} Y^{q-3})$$

holds in $k[X, Y]$.

For $q = 3$, this equation implies that $f(X, Y) = 0$ and hence $\ell(\ker \phi_1) = 0$. On the other hand, $\alpha \notin \mathbb{F}_3$, by hypothesis. Hence assertion of Part (2) is valid in this case.

Now we assume $q \geq 5$. Comparing the coefficients of the monomials on both sides, one can see that the above identity holds if and only if following set of equalities hold:

$$\begin{aligned}
&u_1 = \alpha a_0,\\
&\alpha a_1 - a_0 = 0,\\
&\alpha a_2 - a_1 - \alpha a_0 = 0,\\
&\alpha a_l = a_{l-1} + \alpha a_{l-2} - a_{l-3} \qquad \text{for } 3 \leq l \leq q-3,\\
&-a_{q-3} - \alpha a_{q-4} + a_{q-5} = 0,\\
&u_2 = -\alpha a_{q-3} + a_{q-4},\\
&u_3 = a_{q-3}.
\end{aligned}$$

Let $\beta = 1/\alpha$. Then the above equalities imply that

$$\begin{aligned}
a_l &= a_0(\beta + \beta^3 + \cdots + \beta^l) && \text{if } l \text{ is odd,}\\
a_l &= a_0(1 + \beta^2 + \beta^4 + \cdots + \beta^l) && \text{if } l \text{ is even,}
\end{aligned} \tag{2}$$

for $0 \leq l \leq q-3$. Moreover, $a_{q-3}\beta = \beta a_{q-5} - a_{q-4} = 0$, by equation (2), which implies that

$$a_0(1 + \beta^2 + \beta^4 + \cdots + \beta^{q-3}) = a_{q-3} = 0.$$

We recall that the splitting polynomial of $\mathbb{F}_q/\mathbb{F}_p$ is

$$x^q - x = x(x-1)(x+1)(x^{q-3} + x^{q-5} + \cdots + x^2 + 1).$$

Since $\beta \neq 0$, 1 or -1, we have $\beta \in \mathbb{F}_q$ if and only of $1 + \beta^2 + \cdots + \beta^{q-3} = 0$. Moreover,

$$u_2 = a_{q-4} = a_0(\beta + \beta^3 + \cdots + \beta^{q-4}) = -\frac{a_0}{\beta} = -a_0\beta^{q-2}.$$

Hence

$$\ker\ \phi_1 = \{f(X,Y) \in k[X,Y] \mid f(X,Y) = a_0\alpha X^q Y - a_0\beta^{q-2} X Y^q,$$
$$\text{where } a_0(1 + \beta^2 + \beta^4 + \cdots + \beta^{q-3}) = 0\},$$

which is the same as

$$\ker \phi_1 = \{f(X,Y) \in k[X,Y] \mid f(X,Y) = a_0 X^q Y - a_0 X Y^q,$$
$$\text{where } a_0(1 + \beta^2 + \beta^4 + \cdots + \beta^{q-3}) = 0\}.$$

Therefore,

$$\begin{aligned}
\ell(\ker \phi_1) = 1 &\iff \beta \in \mathbb{F}_q \iff \alpha \in \mathbb{F}_q,\\
\ell(\ker \phi_1) = 0 &\iff \beta \notin \mathbb{F}_q \iff \alpha \notin \mathbb{F}_q.
\end{aligned}$$

This proves Part (2).

Part (3). Suppose $\ell\left(\frac{R_{q+2}}{\mathrm{Im}(R_1^{[q]} \otimes R_2)}\right) = 0$. Then $R_{q+2} \subseteq \mathrm{Im}(R_1^{[q]} \otimes R_2)$. But for $m \geq 2$, we have

$$R_{q+m} = R_{q+2} \cdot R_{m-2} \ \text{ and } \ \mathrm{Im}(R_1^{[q]} \otimes R_m) = \mathrm{Im}(R_1^{[q]} \otimes R_2 \otimes R_{m-2}) \text{ in } R_{q+m},$$

which implies $R_{q+m} \subseteq \mathrm{Im}(R_1^{[q]} \otimes R_m)$. Hence to prove the assertion of Part (3), it is enough to prove the assertion for $m = 2$. Note that $\ell(R_2) = 3$. Therefore, by eq. (1), it is enough to prove that $\ell(\ker \phi_2) = 2$.

Now we proceed as in Part (2). We have a commutative diagram of the canonical maps

$$\begin{array}{ccc} A_1^{[q]} \otimes A_2 & \xrightarrow{\psi_2} & A_{q+2} \\ \downarrow & & \downarrow \\ R_1^{[q]} \otimes R_2 & \xrightarrow{\phi_2} & R_{q+2} \end{array},$$

where the first vertical arrow is an isomorphism and the map ψ_2 is injective (as $2 < q$). Any $f(X, Y) \in R_1^{[q]} \otimes R_2$ can thus be written as

$$f(X, Y) = u_0 X^{q+2} + u_1 X^{q+1} Y + u_2 X^q Y^2 + u_3 X^2 Y^q + u_4 X Y^{q+1} + u_5 Y^{q+2},$$

where $u_0, u_1, u_2, u_3, u_4, u_5 \in k$. If $f(X, Y) \in \ker \phi_2$, then

$$f(X, Y) \in (Y^4 - \alpha X Y^3 - X^2 Y^2 + \alpha X^3 Y) \text{ in } k[X, Y].$$

This implies $u_0 = 0$ and the equality

$$\begin{aligned} &u_1 X^{q+1} + u_2 X^q Y^1 + u_3 X^2 Y^{q-1} + u_4 X Y^q + u_5 Y^{q+1} \\ &\quad = (\alpha X^3 - X^2 Y - \alpha X Y^2 + Y^3)(a_0 X^{q-2} + a_1 X^{q-3} Y + \cdots + a_{q-2} Y^{q-2}) \end{aligned}$$

holds in $k[X, Y]$.

Comparing the coefficients of the monomials on both sides, one can see that this equality holds if and only if the following set of equalities hold:

$$\begin{aligned} u_1 &= \alpha a_0, \\ u_2 &= \alpha a_1 - a_0, \\ \alpha a_2 &= a_1 + \alpha a_0, \\ \alpha a_l &= a_{l-1} + \alpha a_{l-2} - a_{l-3} \quad \text{for } 3 \leq l \leq q-2, \\ u_3 &= a_{q-2} - \alpha a_{q-3} + a_{q-4}, \\ u_4 &= -a_{q-2} + a_{q-3}, \\ u_5 &= a_{q-2}. \end{aligned}$$

In particular, any given $(a_0, a_1) \in k \times k$ uniquely detemines a $(u_0, u_1, u_2, u_3, u_4, u_5) \in k^5$ such that

$$\begin{aligned} f(X,Y) \;=\; &u_0 X^{q+2} + u_1 X^{q+1} Y + u_2 X^q Y^2 + u_3 X^2 Y^q \\ &+ u_4 X Y^{q+1} + u_5 Y^{q+2} \in \ker \phi_2 \end{aligned}$$

and vice versa. Hence $\ell(\ker\ \phi_2) = 2$. This proves Part (3) of the claim.

Since $\ell(R_0) + \cdots + \ell(R_{q-1}) = \ell(R/\mathbf{m}^q) = 4q - 6$ as $q \geq 3$, the proof of the theorem follows from the above claim. □

Remark. Let $(R, \mathbf{m})$ be a 1-dimensional Noetherian Cohen–Macaulay local ring with Hilbert polynomial $e_0 n - e_1$. Then $\mathrm{HK}(R, \mathbf{m})(q) = e_0 q + \Delta(n)$, where $q = p^n$ and $\Delta(n)$ is bounded in terms of e_0, as $\ell(\mathbf{m}^q/\mathbf{m}^{[q]}) \leq e_1$.

Question. Does Huneke's question have a positive answer if we also restrict the residue field to be a fixed finite field $\mathbb{F}_q$?

Note that, from the above mentioned result of Kreuzer and the remark, the answer is affirmative for dim $R = 1$.

Acknowledgements

The author thanks C. Huneke for asking the question mentioned in this paper. She would also like to thank the referee for pointing out some corrections in the paper.

References

[1] Kreuzer M, Computing Hilbert–Kunz functions of 1-dimensional graded rings, *Univ. Iagel. Acta Math.* **45** (2007) 81–95

[2] Monsky P, The Hilbert-Kunz function, *Math. Ann.* **263** (1983) 43–49

[3] Srinivas V and Trivedi V, On the Hilbert function of a Cohen–Macaulay local ring, *J. Algebraic Geom.* **6(4)** (1997) 733–751

COMMUNICATING EDITOR: T R Ramadas