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On (m, n)-absorbing ideals of commutative rings

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Abstract. Let R be a commutative ring with $1 \neq 0$ and $U(R)$ be the set of all unit elements of R. Let m, n be positive integers such that $m > n$. In this article, we study a generalization of n-absorbing ideals. A proper ideal I of R is called an (m, n)-absorbing ideal if whenever $a_1 \cdots a_m \in I$ for $a_1, \ldots, a_m \in R \setminus U(R)$, then there are n of the a_i's whose product is in I. We investigate the stability of (m, n)-absorbing ideals with respect to various ring theoretic constructions and study (m, n)-absorbing ideals in several commutative rings. For example, in a Bézout ring or a Boolean ring, an ideal is an (m, n)-absorbing ideal if and only if it is an n-absorbing ideal, and in an almost Dedekind domain every (m, n)-absorbing ideal is a product of at most $m - 1$ maximal ideals.

Keywords. (m, n)-absorbing ideal; n-absorbing ideal; idealization of a module; prime ideal.

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1. Introduction

We assume, throughout, that R is a commutative with $1 \neq 0$ and $U(R)$ is the set of all unit elements of R. The concept of 2-absorbing ideal of a ring, which is a generalization of a prime ideal, was defined in [4]. In recent years, 2-absorbing ideals have been generalized and studied in several directions (see, for example, [5–7, 9–11]). As in [1], for a positive integer n, a proper ideal I of a ring R is called an n-absorbing ideal if whenever $a_1 \cdots a_{n+1} \in I$ for $a_1, \ldots, a_{n+1} \in R$, then there are n of the a_i's whose product is in I. In this article, we study (m, n)-absorbing ideals in commutative rings with identity, which are a generalization of n-absorbing ideals. Let m, n be positive integers such that $m > n$. A proper ideal I of a commutative ring R is called an (m, n)-absorbing ideal if whenever $a_1 \cdots a_m \in I$ for $a_1, \ldots, a_m \in R \setminus U(R)$, then there are n of the a_i's whose product is in I. Equivalently, a proper ideal I of R is an (m, n)-absorbing ideal if and only if whenever $a_1 \cdots a_s \in I$ for $a_1, \ldots, a_s \in R \setminus U(R)$ with $s \geq m$, then there are n of the a_i's whose product is in I. Thus an $(n+1, n)$-absorbing ideal is just an n-absorbing ideal and, in particular, a (2, 1)-absorbing ideal is just a prime ideal. Also, every (m, n)-absorbing ideal is an $m - 1$-absorbing ideal.

In §2, we give some basic properties of (m, n)-absorbing ideals. For example, we show that for all positive integers $m > n$, every ring R has a minimal (m, n)-absorbing ideal

(Theorem 2.4). Let s, t be positive integers such that $0 \leq s \leq t$. We use the notation $C(t, s)$ to represent $\frac{t!}{s!(t-s)!}$, where $t!$ is the product of positive integers less than or equal to t and $0! = 1$. We prove that if I is an (m, n)-absorbing ideal of R with t minimal prime ideal and s is a positive integer such that $2 \leq s \leq t \leq m-1$, then I has at least $C(t, s)$ minimal s-absorbing ideals (Theorem 2.6), and that if an (m, n)-absorbing ideal I has exactly n minimal prime ideals $P_1, \ldots, P_n$, then $P_1 \cdots P_n \subseteq I$ (Theorem 2.9). In this section, we also investigate when $(I :_R x)$ is an (m, n)-absorbing ideal of R for a proper ideal I of R.

In §3, we study the stability of (m, n)-absorbing ideals with respect to various ring-theoretic constructions. For example, we show that I is an (m, n)-absorbing ideal of a ring R if and only if $I(+)M$ is an (m, n)-absorbing ideal of the idealization $R(+)M$ (Theorem 3.5), and that if m, n are two positive integers such that $n \leq m-2$ and I is an (m, n)-absorbing ideal of R, then $I(+)I$ is an $(m+n-1, 2n)$-absorbing ideal of $R(+)R$ (Theorem 3.5), and if R is a domain, then $0(+)I$ is an $(m+1, n+1)$-absorbing ideal of $R(+)R$. We also determine the (m, n)-absorbing ideals in the direct product of two rings (Theorem 3.4).

In the final section, we study (m, n)-absorbing ideals in several classes of commutative rings. In Theorem 5.3 of [1], it has been shown that every proper ideal of a Noetherian ring is an n-absorbing ideal for some positive integer n. Thus every proper ideal is an $(n+1, n)$-absorbing for some positive integer n (Theorem 4.6). We show that an ideal I of a Bézout ring R is an (m, n)-absorbing ideal if and only if it is an n-absorbing ideal of R (Theorem 4.3). More generally, an ideal of a Prüfer domain is an (m, n)-absorbing ideal for some positive integers m, n if and only if it is a product of prime ideals (Theorem 4.8).

2. Basic properties of (m, n)-absorbing ideals

Let m, n be two positive integers. Recall that a proper ideal I of a ring R is an (m, n)-absorbing ideal if whenever $a_1 \cdots a_m \in I$ for $a_1, \ldots, a_m \in R \setminus U(R)$, then there are n of the a_i's whose products are in I. In this section, we give some basic properties of (m, n)-absorbing ideals. We start with several basic results.

Theorem 2.1. *Let R be a ring and let m, n, s, t be positive integers.*

(1) *A proper ideal I of R is an (m, n)-absorbing ideal of R if and only if whenever $a_1 \cdots a_s \in I$ for $a_1, \ldots, a_s \in R \setminus U(R)$ with $s \geq m$, then there are n of the a_i's whose product is in I.*

(2) *If I is an (m, n)-absorbing ideal of R, then I is an (s, t)-absorbing ideal of R for all $s \geq m, t \geq n$. In particular, I is an $(m-1)$-absorbing ideal of R.*

(3) *If I_i is an (m_i, n_i)-absorbing ideal of R for each $(1 \leq i \leq s)$, then $I_1 \cap \cdots \cap I_s$ is an (m, n)-absorbing ideal of R for $n = n_1 + \cdots + n_s$ and $m = \max\{m_1, \ldots, m_s, n+1\}$. In particular, if I_i is an m_i-absorbing ideal of R for each $1 \leq i \leq s$, then $I_1 \cap \cdots \cap I_s$ is an m-absorbing ideal of R for $m = m_1 + \cdots + m_s$.*

(4) *If I is an (m, n)-absorbing ideal of R, then $\sqrt{I} = \{x \in R \mid x^n \in I\}$. Moreover, if $\lceil \frac{m}{n} \rceil$ is the smallest integer not less than $\frac{m}{n}$ and $s = \max\{\lceil \frac{m}{n} \rceil, n+1\}$, then $\sqrt{I}$ is an (s, n)-absorbing ideal of R.*

Proof.

(1) and (2). The proofs are routine, and thus they are omitted.

(3) Let I_i $(1 \leq i \leq s)$ be an (m_i, n_i)-absorbing ideal of R. Then by (2), I_i is an (m, n_i)-absorbing ideal for $m = \max\{m_1, \ldots, m_s, n+1\}$ and $n = n_1 + \cdots + n_s$. Let $I = I_1 \cap \cdots \cap I_s$ and $a_1 \cdots a_m \in I$ for $a_1, \ldots, a_m \in R \setminus U(R)$; so there are $\Gamma_1, \ldots, \Gamma_s \subseteq \{1, \ldots, m\}$ such that $\prod_{\gamma \in \Gamma_i} a_\gamma \in I_i$ and $|\Gamma_i| = n_i$ for all $1 \leq i \leq s$. Let $\Gamma = \bigcup_{i=1}^{s} \Gamma_i$; so $|\Gamma| \leq n$. Hence I is an (m, n)-absorbing ideal of R, since $\prod_{\gamma \in \Gamma} a_\gamma \in I$. The 'in particular' statement is clear.

(4) Let I be an (m, n)-absorbing ideal of R and $x \in \sqrt{I}$. Then there exists a smallest positive integer t such that $x^t \in I$. Suppose on the contrary, $t > n$. By (2), I is an $(m-1)$-absorbing ideal of R and so, by Theorem 2.1 of [1], $t \leq m - 1$. Thus $x^m \in I$. Now since I is an (m, n)-absorbing ideal of R, $x^n \in I$, which is a contradiction. Hence $t \leq n$ and we have $x^n \in I$.

For the 'moreover' part, let $s = \max\{\lceil \frac{m}{n} \rceil, n + 1\}$ and $a_1 \cdots a_s \in \sqrt{I}$ for $a_1, \ldots, a_s \in R \setminus U(R)$. Then $a_1^n \cdots a_s^n \in I$. Since $s \geq \frac{m}{n}$, I is an (sn, n)-absorbing ideal of R. Thus there is $\{i_1, \ldots, i_n\} \subseteq \{1, \ldots, s\}$ such that $a_{i_1}^{t_1} \cdots a_{i_n}^{t_n} \in I$ for some positive integers $t_1, \ldots, t_n$ with each $0 \leq t_i \leq n$ and $t_1 + \cdots + t_n = n$. Therefore $a_{i_1}^n \cdots a_{i_n}^n \in I$. This implies that $a_{i_1} \cdots a_{i_n} \in \sqrt{I}$ and hence $\sqrt{I}$ is an (s, n)-absorbing ideal of R. □

Let I be a proper ideal of a ring R. If I is an m-absorbing ideal of R for some positive integer m, then define $\nu(I) = \min\{n \mid I \text{ is an } (\omega(I) + 1, n)\text{-absorbing ideal of } R\}$, where $\omega(I) = \min\{n \mid I \text{ is an } n\text{-absorbing ideal of } R\}$; otherwise, set $\nu(I) = \omega(I) = \infty$. It is evident from the definition that $\nu(I) \leq \omega(I)$.

Remark 2.2. Several of the results in Theorem 2.1 may be recast using the ν function. For example, Theorem 2.1(3) becomes $\omega(I_1 \cap \cdots \cap I_s) \leq \max\{\omega(I_1), \ldots, \omega(I_s), \nu(I_1) + \cdots + \nu(I_s)\}$ and $\nu(I_1 \cap \cdots \cap I_s) \leq \nu(I_1) + \cdots + \nu(I_s)$. Theorem 2.1(4) becomes $\omega(\sqrt{I}) \leq \max\{\lceil \frac{\omega(I)+1}{\nu(I)} \rceil - 1, \nu(I)\}$ and $\nu(\sqrt{I}) \leq \nu(I)$.

Theorem 2.3. *Let P be a principal prime ideal of a domain R such that $\omega(P^n) = n$ for some positive integer n. Then $\nu(P^n) = n$. In particular, if M is a principal maximal ideal of a domain R, then $\nu(M^n) = \omega(M^n) = n$ for all positive integer n.*

Proof. Let $P = (x)$. Then $P^{n+1} \subset P^n$. Since $\omega(P^n) = n$, $\nu(P^n) \leq n$ and there are $a_1, \ldots, a_n \in P \setminus P^2$ such that $a_1 \cdots a_n \in P^n$ and no product of $n-1$ of the a_i's is in P^n. If $\nu(P^n) < n$, we must have $a_1^2 a_{i_1} \cdots a_{i_{n-2}} \in P^n$ for some $2 \leq i_j \leq n$. Thus there are $s, r_j \in R$ such that $a_j = r_j x$ for $1 \leq j \leq n$ and $a_1^2 a_{i_1} \cdots a_{i_{n-2}} = r_1^2 r_{i_1} \cdots r_{i_{n-2}} x^{n-1} = sx^n$. Therefore $x^{n-1}(r_1^2 r_{i_1} \cdots r_{i_{n-2}} - sx) = 0$. Hence $r_1^2 r_{i_1} \cdots r_{i_{n-2}} = sx \in P$. Thus $r_j \in P$ for some $1 \leq j \leq n$, a contradiction. Hence $\nu(P^n) = \omega(P^n) = n$.

The 'in particular' statement follows by Lemma 2.8 of [1]. □

Theorem 2.4. *Let I be an ideal of a ring R and m, n be two positive integers such that $m > n$. Then there is an (m, n)-absorbing ideal of R which is minimal among all (m, n)-absorbing ideals of R containing I. In particular, R has a minimal (m, n)-absorbing ideal.*

Proof. Let S be the set of all (m, n)-absorbing ideals of R containing I. Since every maximal ideal of R containing I is an (m, n)-absorbing ideal for all positive integers $m > n$, S is non-empty. It is clear that $(S, \leq)$ is a partially ordered set in which $I_1 \leq I_2$ if and only if $I_1 \supseteq I_2$, for all $I_1, I_2 \in S$. Let $C = \{I_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary non-empty chain of elements of S and set $J = \bigcap_{\lambda\in\Lambda} I_\lambda$. We show that J is an (m, n)-absorbing ideal of R. Since C is non-empty, $J \neq R$. Let $a_1 \cdots a_m \in J$ for some $a_1, \ldots, a_m \in R \setminus U(R)$. Assume that any n-subproduct of the a_i's is not in J. Since C is a chain, there exists $I_\lambda \in C$ such that no n-subproduct of the a_i's is in I_λ, a contradiction. Thus there is an n-subproduct of the a_i's in J. Hence by Zorn's lemma, $(S, \leq)$ has a maximal element, i.e., there is a minimal (m, n)-absorbing ideal of R containing I. □

COROLLARY 2.5

Let I be an ideal of a ring R and n a positive integer. Then there exists a minimal n-absorbing ideal of R containing I.

Theorem 2.6. *Let I be an (m, n)-absorbing ideal of R and s, t be two positive integers such that $2 \leq s \leq t \leq m - 1$. If I has t minimal prime ideals, then I has at least $C(t, s)$ minimal s-absorbing ideals.*

Proof. Let $s = 2$. Suppose that $P_1, \ldots, P_t$ are distinct prime ideals of R minimal over I. Then by Theorem 2.1 of [1], $P_i \cap P_j$ is a 2-absorbing ideal of R containing I, for all $1 \leq i \neq j \leq t$. Thus by Corollary 2.5, there exists a 2-absorbing ideal I_{ij} of R minimal over I and contained in $P_i \cap P_j$ for all $1 \leq i \neq j \leq t$. Since $I \subseteq I_{ij} \subseteq P_i$, P_j and P_i, P_j are minimal over I, by Theorem 2.3 of [4], P_i and P_j are only minimal prime ideals of I_{ij}. Now if $I_{ij} = I_{i'j'}$ for some $1 \leq i, j, i', j' \leq t$ with $\{i, j\} \neq \{i', j'\}$, then P_i, P_j , $P_{i'}$ and $P_{j'}$ are distinct minimal prime ideals of I_{ij}. This is a contradiction with Theorem 2.3 of [4]. Therefore, I_{ij}'s are distinct. Hence there are at least $C(t, 2)$ 2-absorbing ideals of R minimal over I.

The proof for $s > 2$ is similar. □

Lemma 2.7. *Let $P_1, \ldots, P_n$ be incomparable primes ideals of a ring R, and let I be an (m, n)-absorbing ideal of R contained in $P_1 \cap \cdots \cap P_n$. If $a_1^{s_1} \cdots a_n^{s_n} \in I$ for positive integers $s_1, \ldots, s_n$ and $a_i \in P_i \setminus (\bigcup_{j\neq i} P_j)$, then $a_1 \cdots a_n \in I$.*

Proof. Let $s = s_1 + \cdots + s_n$. We can assume that $s \geq m$. By Theorem 2.1(2), I is an (s, n)-absorbing ideal of R. Then we have $a_1^{t_1} \cdots a_n^{t_n} \in I$ for integers $t_1, \ldots, t_n$ with each $0 \leq t_i \leq s_i$ and $t_1 + \cdots + t_n = n$. If $t_i = 0$, say $t_1 = 0$, then $a_2^{t_2} \cdots a_n^{t_n} \in I \subseteq P_1$, a contradiction since $a_i \notin P_1$ for each $2 \leq i \leq n$. Thus $a_1 \cdots a_n \in I$. □

In the following results, we use the notation $P_j \prod_{i\neq j} c_i$ to represent the set of all products of the form $a \prod_{i\neq j} c_i$, where $a \in P_j$.

Lemma 2.8. *Let $n \geq 2$ and I be an (m, n)-absorbing ideal of a ring R such that I has exactly n minimal prime ideals, say $P_1, \ldots, P_n$. If $1 \leq j \leq n$, and for every $i \neq j$ with $1 \leq i \leq n$, $c_i \in P_i \setminus \left(\bigcup_{j\neq i} P_j\right)$, then $P_j \prod_{i\neq j} c_i \in I$.*

Proof. Let $a \in P_j$. If $a \in P_j \setminus \left(\bigcup_{i\neq j} P_i\right)$, then $a \prod_{i\neq j} c_i \in \sqrt{I}$ and by Theorem 2.1(4) and Lemma 2.7, $a \prod_{i\neq j} c_i \in I$. Now suppose that $a \in P_j \cap \left(\bigcup_{i\neq j} P_i\right)$. Thus by Corollary 2.13 of [1], there is an element $d \in P_j \setminus \left(\bigcup_{i\neq j} P_i\right)$ and $b \in R$ such that $bd + a \in P_j \setminus \left(\bigcup_{i\neq j} P_i\right)$. Therefore $(bd + a) \prod_{i\neq j} c_i \in I$. Hence $a \prod_{i\neq j} c_i \in I$. Thus $P_j \prod_{i\neq j} c_i \in I$. □

In the following theorem, we give a sort of consequences whose proofs are similar to those of n-absorbing ideals (see Theorem 2.14, Corollary 2.15, Corollary 2.16 of [1]).

Theorem 2.9. *Let $n \geq 2$ and I be an (m, n)-absorbing ideal of a ring R such that I has exactly n minimal prime ideals, say $P_1, \ldots, P_n$. Then*

(1) *$P_1 \cdots P_n \subseteq I$. Moreover, $\nu(I) = n$.*
(2) *If the P_i's are comaximal, then $I = P_1 \cdots P_n$. Moreover, $\nu(I) = n$. In particular, this holds if either each P_i is maximal, $\dim(R) = 0$, or R is an integral domain with $\dim(R) \leq 1$.*
(3) *$I_{P_i} = {P_i}_{P_i}$ in R_{P_i} for all $1 \leq i \leq n$.*

Recall that a prime ideal P of a ring R is said to be a divided prime ideal if $P \subset xR$ for every $x \in R \setminus P$ [3].

Theorem 2.10.

(1) *Let P be a prime ideal of a ring R, and let I be a P-primary ideal of R such that $P^n \subseteq I$ for some positive integer n (for example, if R is a Noetherian ring). Then I is an (m, n)-absorbing ideal of R for any integer $m \geq n+1$. Moreover, $\nu(I) \leq n$. In particular, if P^n is a P-primary ideal of R, then P^n is an (m, n)-absorbing ideal of R for any integer $m \geq n + 1$ and $\nu(P^n) = n$.*
(2) *Let P be a divided prime ideal of a ring R, and let I be an (m, n)-absorbing ideal of R with $\sqrt{I} = P$. Then I is a P-primary ideal of R.*
(3) *Let $\mathrm{Nil}(R) \subset P$ be divided prime ideals of a ring R. Then P^n is a P-primary ideal of R, and thus P^n is an (m, n)-absorbing ideal of R with $\nu(P^n) \leq n$, for every positive integers m, n such that $m > n$.*

Proof.

(1) By Theorem 3.1 of [1].
(2) If I is an (m, n)-absorbing ideal of R, then I is an $(m-1)$-absorbing ideal of R. Now by Theorem 3.2 of [1], I is a P-primary ideal of R.
(3) By Theorem 3.3 of [1], P^n is a P-primary ideal of R for every positive integer n. Thus P^n is an (m, n)-absorbing ideal of R with $\nu(P^n) \leq n$, for every positive integer $m > n$, by (1). □

Let I be a proper ideal of a ring R. For $x \in R$, let $I_x = \{y \in R \mid yx \in I\} = (I :_R x)$.

Theorem 2.11. *Let I be an (m, n)-absorbing ideal of a ring R, $s = \max\{m-1, n+1\}$ and $x \in R \setminus (U(R) \cup I)$. Then I_x is an (s, n)-absorbing ideal of R containing I. Moreover, $\nu(I_x) \leq \nu(I)$.*

Proof. The proof is routine. □

PROPOSITION 2.12

Let $n \geq 2$ and $I \subset \sqrt{I}$ be an (m, n)-absorbing ideal of a ring R. Suppose that $x \in \sqrt{I} \setminus I$ and let $s \geq 2$ be the least positive integer such that $x^s \in I$. Then $I_{x^{s-1}} = (I :_R x^{s-1})$ is an $(m-s+1, t)$-absorbing ideal of R containing I, where $t = \min\{n, m-s\}$.

Proof. Since I is an (m, n)-absorbing ideal of R, I is an $(m-1)$-absorbing ideal of R. Then by Theorem 3.5 of [1], $I_{x^{s-1}}$ is an $(m-s)$-absorbing ideal of R containing I, i.e., $I_{x^{s-1}}$ is an $(m-s+1, m-s)$-absorbing ideal of R. Now let $n \leq m-s$ and $a_1 \cdots a_{m-s+1} \in I_{x^{s-1}}$ for $a_1, \ldots, a_{m-s+1} \in R \setminus U(R)$. Since $x^{s-1}a_1 \cdots a_{m-s+1} \in I$ and I is an (m, n)-absorbing ideal of R, there are l of the a_i's whose product with a power of x is in I for some $1 \leq l \leq n$. For example, $x^{n-l}a_1 \cdots a_l \in I$. Thus $a_1 \cdots a_l \in I_{x^{s-1}}$ and hence $a_1 \cdots a_n \in I_{x^{s-1}}$. Then $I_{x^{s-1}}$ is an $(m-s+1, n)$-absorbing ideal of R. □

COROLLARY 2.13

Let $m \geq 4$ and $I \subset \sqrt{I}$ be an (m, n)-absorbing ideal of a ring R. Suppose that $x \in \sqrt{I} \setminus I$ and $m-2$ is the least positive integer such that $x^{m-2} \in I$, then $I_{x^{m-3}}$ is a 2-absorbing ideal of R.

3. Extensions of (m, n)-absorbing ideals

The first two theorems below generalize the well-known results about n-absorbing ideals and follow directly from the definitions; so their proofs are omitted.

Theorem 3.1. *Let I be an (m, n)-absorbing ideal of a ring R, and let S be a multiplicatively closed subset of R with $I \cap S = \emptyset$. Then I_S is an (m, n)-absorbing ideal of R_S. Moreover, $\nu(I_S) \leq \nu(I)$.*

Theorem 3.2. *Let $f : R \longrightarrow T$ be a homomorphism of rings.*

(1) *Let J be an (m, n)-absorbing ideal of T. If $f(U(R)) = U(T)$, then $f^{-1}(J)$ is an (m, n)-absorbing ideal of R and $\nu(f^{-1}(J)) \leq \nu(J)$.*
(2) *If J is an (m, n)-absorbing ideal of T, then $f^{-1}(J)$ is an $(m-1)$-absorbing ideal of R.*
(3) *Let f be surjective and I be an ideal of R containing* ker(f). *If I is an (m, n)-absorbing ideal of R, then $f(I)$ is an (m, n)-absorbing ideal of T. Moreover, $\nu(f(I)) \leq \nu(I)$. The converse is true if $f(U(R)) = U(T)$.*
(4) *Let f be surjective and I be an ideal of R containing* ker(f). *If $f(I)$ is an (m, n)-absorbing ideal of R, then I is an $(m-1)$-absorbing ideal of T.*

COROLLARY 3.3

(1) *Let $R \subseteq T$ be an extension of rings and J an (m, n)-absorbing ideal of T. If $U(R) = U(T)$, then $J \cap R$ is an (m, n)-absorbing ideal of R. Moreover, $\nu(J \cap R) \leq \nu(J)$.*
(2) *Let $R \subseteq T$ be an extension of rings and J an (m, n)-absorbing ideal of T. Then $J \cap R$ is an $(m-1)$-absorbing ideal of R.*
(3) *Let $I \subseteq J$ be ideals of a ring R and J an (m, n)-absorbing ideal of R. Then J/I is an (m, n)-absorbing ideal of R/I. Moreover, $\nu(J/I) \leq \nu(J)$. The converse is true if $U(R/I) = \{x + I \mid x \in U(R)\}$. In this case $\nu(J/I) = \nu(J)$.*
(4) *Let $I \subseteq J$ be ideals of a ring R and J/I an (m, n)-absorbing ideal of R. Then J is an $(m-1)$-absorbing ideal of R.*

Theorem 3.4. *Let I_1 be an (m, n)-absorbing ideal of a ring R_1 and I_2 an (s, t)-absorbing ideal of a ring R_2. Then $I_1 \times I_2$ is an (u, w)-absorbing ideal of the ring $R_1 \times R_2$ for*

$u = m+s-1$ *and* $w = \max\{m+t-1, s+n-1\}$. *Moreover*, $\nu(I_1 \times I_2) = \max\{\omega(I_1) + \nu(I_2), \omega(I_2) + \nu(I_1)\}$.

Proof. Let $R = R_1 \times R_2$; so, by Theorem 4.7 of [1], $\omega(I_1 \times I_2) = \omega(I_1) + \omega(I_2)$. We show that $\nu(I_1 \times I_2) = \max\{\omega(I_1) + \nu(I_2), \omega(I_2) + \nu(I_1)\}$. First suppose that $\omega(I_1) = m$, $\nu(I_1) = n < \infty$ and $\omega(I_2) = s$, $\nu(I_2) = t < \infty$. Let $s+n \le m+t$, $w = m+t$. Then there are $a_1, \ldots, a_m \in R_1 \backslash U(R_1)$ and $b_1, \ldots, b_{s+1} \in R_2 \backslash U(R_2)$ such that $a_1 \cdots a_m \in I_1$ and $b_1 \cdots b_{s+1} \in I_2$, but no proper subproduct of a_i's is in I_1 and no $(t-1)$-subproduct of b_i's is in I_2. Then $(a_1, 1) \cdots (a_m, 1)(1, b_1) \cdots (1, b_{s+1}) = (a_1 \cdots a_m, b_1 \cdots b_{s+1}) \in I_1 \times I_2$, but no $(u-1)$-subproduct of them is in $I_1 \times I_2$. Hence $\nu(I_1 \times I_2) \ge u = \max\{\omega(I_1) + \nu(I_2), \omega(I_2) + \nu(I_1)\}$. Next, let $u = m+s-1$ and $(a_1, b_1) \cdots (a_u, b_u) \in I_1 \times I_2$ for $(a_i, b_i) \in R \setminus U(R)$. Then $a_1 \cdots a_u \in I_1$ and $b_1 \cdots b_u \in I_2$. We have the following cases:

Case 1. If at least s of the a_i's are unit, then at least s of the b_i's are non-unit. Thus there are at most $m-1$ of the a_i's whose product is in I_1 and there are at most t of the b_i's whose product is in I_2. Hence there are at most $m+t-1$ of the (a_i, b_i)'s whose product is in $I_1 \times I_2$.

Case 2. If at most $s-1$ of the a_i's are unit, then there are n of the a_i's whose product is in I_1. On the other hand, at most there are $s-1$ of the b_i's whose product is in I_2. Hence there are at most $s+n-1$ of the (a_i, b_i)'s whose product is in $I_1 \times I_2$; so $\nu(I_1 \times I_2) \le u = \max\{\omega(I_1) + \nu(I_2), \omega(I_2) + \nu(I_1)\}$. The above proof also shows that $\nu(I_1 \times I_2)$ is infinite if and only if either $\nu(I_1)$ or $\nu(I_2)$ is infinite. Hence $\nu(I_1 \times I_2) = \max\{\omega(I_1) + \nu(I_2), \omega(I_2) + \nu(I_1)\}$. □

Let R be a ring and M an R-module. Then $R(+)M = R \times M$ is a ring with identity $(1, 0)$ under addition defined by $(r, m)+(s, n) = (r+s, m+n)$ and multiplication defined by $(r, m)(s, n) = (rs, rn + sm)$.

Theorem 3.5. *Let R be a ring, I be an ideal of R and M be an R-module. Then I is an (m, n)-absorbing ideal of R if and only if $I(+)M$ is an (m, n)-absorbing ideal of $R(+)M$; so $\nu(I(+)M) = \nu(I)$.*

Proof. Let $T = R(+)M$ and $(a_1, b_1), \ldots, (a_m, b_m) \in T \setminus U(T)$ such that $(a_1, b_1) \cdots (a_m, b_m) \in I(+)M$. Since $U(T) = U(R(+)M) = U(R)(+)M$, by Theorem 3.7 of [2], $a_1 \cdots a_m \in I \setminus U(R)$. Thus there are n of the a_i's whose product is in I. Therefore there are n of the (a_i, b_i)'s whose product is in $I(+)M$. Hence $I(+)M$ is an (m, n)-absorbing ideal of T. The converse is clear. □

Theorem 3.6. *Let R be a ring and $T = R(+)R$. Let m, n be two positive integers such that $n \le m-2$ and I be an (m, n)-absorbing ideal of R. Then $I(+)I$ is an $(m+n-1, 2n)$-absorbing ideal of T. Moreover, if $\nu(I) < \omega(I)$, then $\omega(I(+)I) \le \omega(I) + \nu(I) - 1$ and $\nu(I(+)I) \le 2\nu(I)$.*

Proof. Let $c_1 = (a_1, b_1), \ldots, c_{m+n-1} = (a_{m+n-1}, b_{m+n-1}) \in T \setminus U(T)$ such that $c_1 \cdots c_{m+n-1} \in I(+)I$. Then $a_1 \cdots a_{m+n-1} \in I \setminus U(R)$ and $\Sigma_{i=1}^{m+n-1} \hat{a}_i b_i \in I$, where $\hat{a}_i = a_1 \cdots a_{i-1} a_{i+1} \cdots a_{m+n-1}$ for $1 \le i \le m+n-1$. Thus there are n of a_i's whose product is in I, say $a_1 \cdots a_n \in I$. Therefore $\hat{a}_i \in I$ for $n+1 \le i \le m+n-1$ and hence

$(b_1a_2\cdots a_n+\cdots+a_1\cdots a_{j-1}b_ja_{j+1}\cdots a_n+\cdots+a_1\cdots a_{n-1}b_n)a_{n+1}\cdots a_{m+n-1}\in I$. Since I is an (m,n)-absorbing ideal of R, either the product of

$$b_1a_2\cdots a_n+\ldots+a_1\cdots a_{j-1}b_ja_{j+1}\cdots a_n+\ldots a_1\cdots a_{n-1}b_n$$

with $n-1$ of the a_i's $(n+1\leq i\leq m+n-1)$ is in I or the product of n of the a_i's $(n+1\leq i\leq m+n-1)$ is in I. In both cases, the product of $c_1\cdots c_n$ with n of the c_i's $(n+1\leq i\leq m+n-1)$ is in $I(+)I$. Hence $I(+)I$ is an $(m+n-1,2n)$-absorbing ideal of T.

For the 'moreover' statement, since I is an $(\omega(I)+1,\nu(I))$-absorbing ideal of R, $I(+)I$ is an $(\omega(I)+\nu(I),2\nu(I))$-absorbing ideal of T. □

Theorem 3.7. *Let D be an integral domain, $R=D(+)D$. Let m,n be two positive integers such that $n\leq m-2$ and I be an (m,n)-absorbing ideal of D that is not an $(m,n-1)$-absorbing ideal of D. Then $0(+)I$ is an $(m+1,n+1)$-absorbing ideal of R that is not an $(m+1,n)$-absorbing ideal of R; so $\nu(0(+)I)=\nu(I)+1$.*

Proof. Since I is an (m,n)-absorbing ideal of D that is not an $(m,n-1)$-absorbing ideal of D, there are $d_1,\ldots,d_m\in D\setminus U(D)$ such that $d_1\cdots d_m\in I$ and no product of $n-1$ of the d_i's is in I. Let $b_1=(d_1,0),\ldots,b_m=(d_m,0)$ and $b_{m+1}=(0,1)$. Then

$$b_1\cdots b_{m+1}=(0,d_1\cdots d_m)\in 0(+)I,$$

and it is clear that the no n-subproduct of the b_i's is in $0(+)I$. Thus $0(+)I$ is not an $(m+1,n)$-absorbing ideal of R. Next we show that $0(+)I$ is an $(m+1,n+1)$-absorbing ideal of R. Let

$$c_1=(a_1,e_1),\ldots,c_{m+1}=(a_{m+1},e_{m+1})\in R\setminus U(R)$$

be such that $c_1\cdots c_{m+1}\in 0(+)I$. Then $a_1\cdots a_{m+1}=0$. It follows that $a_j=0$, for some $1\leq j\leq m+1$, say $a_1=0$. Hence $c_1\cdots c_{m+1}=(0,e_1a_2\cdots a_{m+1})\in 0(+)I$ and thus $e_1a_2\cdots a_{m+1}\in I$. Since $U(R)=U(D)(+)D$ by Theorem 3.7 of [2], $a_i\in D\setminus U(D)$ for all $1\leq i\leq m+1$. If $e_1\in U(D)$, then $a_2\cdots a_{m+1}\in I$. Hence there are n of a_i's whose product is in I; so the product of c_1 with n of the c_i's $(i\neq 1)$ is in $0(+)I$. If $e_1\in D\setminus U(D)$, then either the product of e_1 with $n-1$ of the a_i's is in I or the product of n of the a_i's is in I; so either the product of c_1 with $n-1$ of the c_i's $(i\neq 1)$ is in $0(+)I$ or the product of c_1 with n of the c_i's $(i\neq 1)$ is in $0(+)I$. Thus $0(+)I$ is an $(m+1,n+1)$-absorbing ideal of R, and hence $\nu(0(+)I)=\nu(I)+1$. □

Theorem 3.8. *Let I be an ideal of a ring R. Then*

(1) *If (I,X) is an (m,n)-absorbing ideal of $R[X]$, then I is an (m,n)-absorbing ideal of R. Moreover, $\nu(I)\leq\nu((I,X))$.*
(2) *If I is an (m,n)-absorbing ideal of R, then (I,X) is an $(m-1)$-absorbing ideal of $R[X]$.*

Proof. These follow directly from Corollary 3.3(3), (4) respectively, since $(I,X)/(X)\cong I$ in $R[X]/(X)\cong R$. The 'moreover' statement is clear. □

Theorem 3.9. *Let $T=K+M$ be an integral domain, where K is a field which is a subring of T and M is a nonzero maximal ideal of T. Let D be a subring of K and*

$R = D + M$. Let I be an ideal of D. If $I + M$ is an (m, n)-absorbing ideal of R, then I is an (m, n)-absorbing ideal of D. Moreover, $\nu(I) \leq \nu(I + M)$.

Proof. This follows directly from Corollary 3.3(3) since $(I + M)/M \cong I$ in $R/M \cong D$. The 'moreover' statement is clear. □

4. (m, n)-absorbing ideals in specific rings

In this section, we study (m, n)-absorbing ideals in several special classes of commutative rings. If I is an n-absorbing ideal of R, then I is an (m, n)-absorbing ideal of R for each $m \geq n + 1$, by Theorem 2.1(2).

Theorem 4.1. *Let I be a radical ideal of R. Then I is an (m, n)-absorbing ideal of R if and only if I is an n-absorbing ideal of R. Moreover, $\nu(I) = \omega(I)$. In particular, $I = P_1 \cap \cdots \cap P_n = P_1 \cdots P_n$ where $P_1, \ldots, P_n$ are prime ideals of a ring R that are pairwise comaximal.*

Proof. Let I be an (m, n)-absorbing ideal of R. It suffices to show that I is an $(m - 1, n)$-absorbing ideal of R.

Let I be a radical ideal of R and $a_1 \cdots a_{m-1} \in I$ for $a_1, \ldots, a_{m-1} \in R \setminus U(R)$. Since I is an (m, n)-absorbing ideal of R and $a_1^2 a_2 \cdots a_{m-1} \in I$, either there are n of a_i's whose product is in I or there are $n - 2$ of $a_2, \ldots, a_{m-1}$ whose product with a_1^2 is in I. If there are n of a_i's whose product is in I, then we are done. In the other case, there are $n - 2$ of $a_2, \ldots, a_{m-1}$ whose product with a_1^2 is in I, for example, $a_1^2 a_2 \cdots a_{n-2} \in I$, then $a_1 a_2 \cdots a_{n-2} \in \sqrt{I} = I$. Hence I is an $(m - 1, n)$-absorbing ideal of R. The 'in particular' statement is clear. □

COROLLARY 4.2

Let R be a Boolean ring. Then an ideal I of R is an (m, n)-absorbing ideal of R if and only if I is an n-absorbing ideal of R. Moreover, $\nu(I) = \omega(I)$.

An integral domain R is said to be a valuation domain if either $x \mid y$ or $y \mid x$ (in R) for all $0 \neq x, y \in R$. Every valuation domain R is a Bézout domain, i.e., a domain in which every finitely generated ideal of R is principal. Moreover, every local Bézout domain is a valuation domain, by Proposition 1.5 of [8].

Theorem 4.3. *Let R be a Bézout ring. Then an ideal I is an (m, n)-absorbing ideal if and only if I is an n-absorbing ideal of R. Moreover, $\nu(I) = \omega(I)$. In particular, this holds if either R is a valuation domain or R is a PID.*

Proof. Let I be an (m, n)-absorbing ideal of R. Now, we show that I is an $(m - 1, n)$-absorbing ideal of R. Let $a_1 \cdots a_{m-1} \in I$ for $a_1, \ldots, a_{m-1} \in R \setminus U(R)$. Since R is a Bézout ring, we have $(a_1, \ldots, a_{m-1}) = xR$ for some $x \in R$. Thus for each $1 \leq i \leq m-1$, there exists $y_i \in R$ such that $a_i = xy_i$. Therefore, $a_1 \cdots a_{m-1} = x^{m-1} y_1 \cdots y_{m-1} \in I$. Then either $x^n \in I$ or $y_{i_1} \cdots y_{i_n} \in I$ for some $\{i_1, \ldots, i_n\} \subseteq \{1, \ldots, m - 1\}$ or $x^t y_{i_1} \cdots y_{i_{n-t}} \in I$ for some $\{i_1, \ldots, i_{n-t}\} \subseteq \{1, \ldots, m - 1\}$. If $x^n \in I$ or $y_{i_1} \cdots y_{i_n} \in I$, then $a_{i_1}\cdots a_{i_n} = x^n y_{i_1}\cdots y_{i_n} \in I$ and we are done. If $x^t y_{i_1}\cdots y_{i_{n-t}} \in I$, then $a_{i_1}\cdots a_{i_n} \in I$. This implies that I is an $(m-1,n)$-absorbing ideal of R. The 'in particular' statement follows from Proposition 1.5 of [8]. □

If I is a nonzero fractional ideal of a ring R, then $I^{-1} = \{x \in T(R) \mid xI \subseteq R\}$. An integral domain R is called a Dedekind domain if $II^{-1} = R$ for every nonzero fractional ideal I of R.

Theorem 4.4. *Let R be a Noetherian integral domain. Then the following statements are equivalent*:

(1) *R is a Dedekind domain*;
(2) *If I is an (m,n)-absorbing ideal of R, then $I = M_1\cdots M_t$ for maximal ideals $M_1,\ldots,M_t$ of R with $1 \le t \le m-1$.*

Proof.

(1) ⇒ (2). Let I be an (m,n)-absorbing ideal of a Dedekind domain R. Then I is an $(m-1)$-absorbing ideal of R. Thus by Theorem 5.1 of [1], $I = M_1\cdots M_t$ for some maximal ideals $M_1,\ldots,M_t$ of R with $1 \le t \le m-1$.
(2) ⇒ (1). Let I be an n-absorbing ideal of R. Then I is an $(n+1,n)$-absorbing ideal of R. Thus by the hypothesis, $I = M_1\cdots M_t$ for maximal ideals $M_1,\ldots,M_t$ of R with $1 \le t \le n$. Hence by Theorem 5.1 of [1], R is a Dedekind domain. □

An integral domain R is called an almost Dedekind domain if R_M is a Noetherian valuation domain (DVR) for every maximal ideal M of R.

Theorem 4.5. *Let R be an almost Dedekind domain and m,n be two positive integers such that $m > n$. If a non-zero ideal I of R is an (m,n)-absorbing ideal of R, then $I = M_1\cdots M_t$ for maximal ideals $M_1,\ldots,M_t$ of R with $1 \le t \le m-1$. Conversely, if $I = M_1\cdots M_t$ for maximal ideals $M_1,\ldots,M_t$ of R with $1 \le t \le m-1$, then I is an $(s,m-1)$-absorbing ideal of R for all $s \ge m$.*

Proof. Let I be an (m,n)-absorbing ideal of R. Then I is an $(m-1)$-absorbing ideal of R. Thus by Theorem 5.2 of [1], $I = M_1\cdots M_t$ for maximal ideals $M_1,\ldots,M_t$ of R with $1 \le t \le m-1$. Conversely, if $I = M_1\cdots M_t$ for maximal ideals $M_1,\ldots,M_t$ of R with $1 \le t \le m-1$, then by Theorem 5.2 of [1], I is an $m-1$-absorbing ideal of R and hence I is an $(s,m-1)$-absorbing ideal of R for all $s \ge m$. □

Theorem 4.6. *Let R be a Noetherian ring. Then every proper ideal of R is an (m,n)-absorbing ideal of R for some positive integers m,n.*

Proof. It follows from Theorem 5.3 of [1] and the fact that every (m,n)-absorbing ideal of R is an $(m-1)$-absorbing ideal of R. □

Theorem 4.7. *Let R be a valuation domain, and m,n be two positive integers such that $m > n$ and I be an ideal of R. Then the following are equivalent*:

(1) *I is an (m,n)-absorbing ideal of R*;
(2) *I is a P-primary ideal of R for some prime ideal P of R and $P^n \subseteq I$*;

(3) *$I = P^t$ for some prime ideal $P(= \mathrm{Rad}(I))$ of R and integer t with $1 \leq t \leq n$.*

Proof. By Theorem 4.3, I is an (m, n)-absorbing ideal of R if and only if I is an n-absorbing ideal of R. Now use Theorem 5.5 of [1]. □

An integral domain R is called a Prüfer domain if $II^{-1} = R$ for every nonzero finitely generated fractional ideal I of R. An integral domain R is a Prüfer domain if and only if R_M is a valuation domain for every maximal ideal M of R.

Theorem 4.8. *Let R be a Prüfer domain. Then an ideal I of R is an (m, n)-absorbing ideal of R for some positive integers $m > n$ if and only if I is a product of prime ideals of R.*

Proof. It follows from Theorem 5.7 of [1]. □

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