Proc. Indian Acad. Sci. (Math. Sci.) Vol. 127, No. 2, April 2017, pp. 203–217.
DOI 10.1007/s12044-017-0331-x

Counting rises and levels in r-color compositions

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MS received 2 April 2015; revised 15 December 2015

Abstract. An r-color composition of a positive integer n is a sequence of positive integers, called *parts*, summing to n in which each part of size r is assigned one of r possible colors. In this paper, we address the problem of counting the r-color compositions having a prescribed number of rises. Formulas for the relevant generating functions are computed which count the compositions in question according to a certain statistic. Furthermore, we find explicit formulas for the total number of rises within all of the r-color compositions of n having a fixed number of parts. A similar treatment is given for the problem of counting the number of levels and a further generalization in terms of rises of a particular type is discussed.

Keywords. Combinatorial statistic; integer compositions; rise; level.

2010 Mathematics Subject Classification. 05A15, 05A19.

1. Introduction

A non-increasing sequence of positive integers, called *parts*, whose sum is a given positive integer is called a *partition*. A *composition* is a partition in which the parts may come in any order, as originally defined by MacMahon [8]. For example, there are five partitions and eight compositions of 4. The partitions are 4, 31, 2^2, 21^2, 1^4 and the compositions are 4, 31, 2^2, 21^2, 13, 121, 1^22, 1^4.

An *r-color partition*, as defined by Agarwal and Andrews [4], is one in which a part of size r can come in r different colors. The color of a part will be indicated by a subscript and so the possible parts of size r will be denoted by $r_1, r_2, \ldots, r_r$. In analogy, Agarwal [1] later defined an *r-color composition* as an ordered r-color partition. For example, there are eight r-color compositions of 3, namely, 3_1, 3_2, 3_3, 2_11_1, 2_21_1, 1_12_1, 1_12_2, $1_11_11_1$. See, e.g., [2, 5, 10, 11] for further properties of r-color compositions.

Let $\mathcal{A}_n$ denote the set of all r-color compositions of n and $\mathcal{A}_{n,m}$ be the subset of $\mathcal{A}_n$ whose members contain exactly m parts. Given $\alpha = ([a_1]_{b_1}, [a_2]_{b_2}, \ldots, [a_m]_{b_m}) \in \mathcal{A}_{n,m}$, we will say that a *rise* occurs (at index i, where $1 \le i \le m-1$) if $a_i < a_{i+1}$ and that a *level* occurs if $a_i = a_{i+1}$. Agarwal [3] raised the question of counting members of $\mathcal{A}_n$ containing a fixed number of rises. Here, we address this problem and consider a further generalization in terms of a statistic defined on $\mathcal{A}_n$. This polynomial extension allows for a simultaneous generalization of results on ordinary and r-color compositions, upon taking $q = 0$ and $q = 1$, respectively. A similar extension may also be given counting members of $\mathcal{A}_{n,m}$ containing a fixed number of levels.

We will compute the generating function counting the number of rises in two different ways. Not only are the recurrences encountered along the way using either approach of independent interest in their own right, but also one obtains an apparently new infinite series identity via a combinatorial approach. Moreover, when one gives a direct analytic proof of this identity, a further simplified form of the generating function emerges. Furthermore, when one computes either the total number of rises or levels within all the members of $\mathcal{A}_{n,m}$ in different ways, two seemingly new somewhat curious binomial identities are found (see Theorems 2.6 and 3.3 below).

The paper is divided as follows. In the next section, we address the problem of counting members of $\mathcal{A}_{n,m}$ according to the number of rises and compute the relevant generating functions as well as provide explicit formulas for the total number of rises within all the members of $\mathcal{A}_{n,m}$. In §3, a comparable treatment is given for levels. In §4, a more refined counting of rises and levels is discussed. In §5, a comparison is made with another possible definition of rise for an r-color composition.

2. Rises in r-color compositions

In this section, we address the question raised at the end of [3] concerning the distribution for the number of rises on r-color compositions and in fact present a q-generalization by studying a statistic on r-color compositions containing a fixed number of rises. Let $\mathcal{A}_{n,m,\ell}$ denote the subset of $\mathcal{A}_{n,m}$ whose members contain exactly ℓ rises. We recall a statistic from [9]. Given $\alpha = ([a_1]_{b_1}, [a_2]_{b_2}, \ldots, [a_m]_{b_m}) \in \mathcal{A}_{n,m}$, let

$$\sigma(\alpha) = \sum_{i=1}^{m}(b_i - 1).$$

For example, if $n = 16$, $m = 5$ and $\alpha = 3_1 + 5_4 + 4_3 + 1_1 + 3_2 \in \mathcal{A}_{16,5}$, then $\sigma(\alpha) = 0 + 3 + 2 + 0 + 1 = 6$.

Given $n \geq m > \ell \geq 0$, let

$$a(n, m, \ell; q) = \sum_{\pi \in \mathcal{A}_{n,m,\ell}} q^{\sigma(\pi)}.$$

In order to find $a(n, m, \ell; q)$, we consider the last letter i within a member of $\mathcal{A}_{n,m,\ell}$ and let

$$a_i(n, m, \ell; q) = \sum_{\pi \in \mathcal{A}_{n,m,\ell}(i)} q^{\sigma(\pi)}, \qquad 1 \leq i \leq n,$$

where $\mathcal{A}_{n,m,\ell}(i)$ denotes the subset of members of $\mathcal{A}_{n,m,\ell}$ whose last letter is i. Note that $a(n, m, \ell; q) = \sum_{i=1}^{n} a_i(n, m, \ell; q)$ and that $a_i(n, m, \ell; q) = 0$ if $i > n - m + 1$ since $\mathcal{A}_{n,m,\ell}(i)$ is empty in that case. Given a positive integer i and an indeterminate q, let $[i]_q = 1 + q + \cdots + q^{i-1}$, with $[0]_q = 0$. Considering whether or not the penultimate letter within a member of $\mathcal{A}_{n,m,\ell}(i)$ is less than i leads to the following recurrence.

Lemma 2.1. *If* $1 \leq i \leq n - 1$, *then*

$$a_i(n, m, \ell; q) = [i]_q \sum_{j=1}^{i-1} a_j(n-i, m-1, \ell-1; q) + [i]_q \sum_{j=i}^{n-1} a_j(n-i, m-1, \ell; q), \tag{1}$$

with $a_n(n, m, \ell; q) = [n]_q$ *if* $m = 1$, $\ell = 0$ *and zero otherwise.*

Denote the generating function for the sequence of polynomials $a(n, m, \ell; q)$ by

$$A(x, y, z, q) = 1 + \sum_{n \geq m > \ell \geq 0} a(n, m, \ell; q) x^n y^m z^\ell.$$

There is the following explicit formula for $A(x, y, z, q)$.

Theorem 2.2. *The generating function* $A(x, y, z, q)$ *is given by*

$$A(x, y, z, q) = \frac{1}{1 - y \sum_{i \geq 1} x^i [i]_q \prod_{j=1}^{i-1} (1 + x^j y(z-1)[j]_q)}. \tag{2}$$

Proof. Given $1 \leq i \leq n$, let $a_i(n, m; z, q) = \sum_{\ell=0}^{m-1} a_i(n, m, \ell; q) z^\ell$. Multiplying both sides of (1) by z^ℓ, and summing over $0 \leq \ell \leq m-1$, gives for $m \geq 2$ the recurrence

$$\begin{aligned} a_i(n, m; z, q) &= z[i]_q \sum_{j=1}^{i-1} a_j(n-i, m-1; z, q) \\ &\quad + [i]_q \sum_{j=i}^{n-i} a_j(n-i, m-1; z, q), \qquad 1 \leq i \leq n-1, \end{aligned} \tag{3}$$

with $a_i(n, 1; z, q) = \delta_{n,i}[n]_q$.

Define $a_i(n; y, z, q) = \sum_{m=1}^{n} a_i(n, m; z, q) y^m$. Then recurrence (3) implies

$$\begin{aligned} a_i(n; y, z, q) - y\delta_{n,i}[n]_q &= yz[i]_q \sum_{j=1}^{i-1} a_j(n-i; y, z, q) \\ &\quad + y[i]_q \sum_{j=i}^{n-i} a_j(n-i; y, z, q), \qquad 1 \leq i \leq n, \end{aligned} \tag{4}$$

upon considering separately the cases when $i = n$ and $1 \leq i \leq n-1$. Dividing both sides of (4) by $[i]_q$, replacing n with $n-1$ and i with $i-1$, and subtracting gives

$$\begin{aligned} a_i(n; y, z, q)/[i]_q &= a_{i-1}(n-1; y, z, q)/[i-1]_q \\ &\quad + y(z-1) a_{i-1}(n-i; y, z, q), \qquad 1 < i \leq n, \end{aligned} \tag{5}$$

with $a_1(n; y, z, q) = y(\delta_{n,1} + \sum_{j=1}^{n-1} a_j(n-1; y, z, q))$ for $n \geq 1$.

Define $A_i(x, y, z, q) = \sum_{n \geq i} a_i(n; y, z, q) x^n$. Then from (5), we obtain

$$A_i(x, y, z, q) = [i]_q (x/[i-1]_q + x^i y(z-1)) A_{i-1}(x, y, z, q), \qquad i \geq 2,$$

with $A_1(x, y, z, q) = xy(1 + \sum_{i \geq 1} A_i(x, y, z, q)) = xyA(x, y, z, q)$. Iterating this last equation gives

$$A_i(x, y, z, q) = x^{i-1}[i]_q \prod_{j=1}^{i-1} (1 + x^j y(z-1)[j]_q) A_1(x, y, z, q), \qquad i \geq 1. \tag{6}$$

Summing (6) over all $i \geq 1$ then yields

$$A(x, y, z, q) - 1 = \sum_{i\geq 1} x^{i-1}[i]_q \prod_{j=1}^{i-1}(1 + x^j y(z-1)[j]_q)xyA(x, y, z, q),$$

and solving for $A(x, y, z, q)$ gives (2). □

Remark. When $q = 0$, only the first color is allowed and r-color compositions reduce to ordinary compositions. Thus, the $q = 0$ case of formula (2) reduces to one given in Theorem 2.1 of [6] for compositions (see also section 4.2 of [7]). Furthermore, when $q = 1$, formula (2) counts r-color compositions of n according to the number of parts and rises.

We now present a second way of obtaining the generating function $A(x, y, z, q)$, as a limiting distribution, though the resulting formula will be different. Let $a^{(d)}(n, m, \ell; q)$ denote the distribution polynomial for the σ statistic restricted to the subset of $\mathcal{A}_{n,m,\ell}$ whose members have parts in $[d] = \{1, 2, \ldots, d\}$, where $d \geq 1$. Let $A^{(d)}(x, y, z, q) = 1 + \sum_{n\geq m>\ell\geq 0} a^{(d)}(n, m, \ell; q)x^n y^m z^\ell$. Then considering whether or not a restricted member of $\mathcal{A}_{n,m,\ell}$ contains a part of size d, and if so conditioning on the left-most occurrence of d, yields the recurrence

$$\begin{aligned} A^{(d)}(x, y, z, q) &= A^{(d-1)}(x, y, z, q) + x^d yz[d]_q(A^{(d-1)}(x, y, z, q) - 1) \\ &\quad \times A^{(d)}(x, y, z, q) + x^d y[d]_q A^{(d)}(x, y, z, q), \end{aligned}$$

i.e.,

$$A^{(d)}(x, y, z, q) = \frac{A^{(d-1)}(x, y, z, q)}{1+x^d y(z-1)[d]_q - x^d yz[d]_q A^{(d-1)}(x, y, z, q)}, \qquad d > 1, \tag{7}$$

with $A^{(1)}(x, y, z, q) = \frac{1}{1-xy}$.

Theorem 2.3. *The generating function $A^{(d)}(x, y, z, q)$ is given by*

$$\begin{aligned} &A^{(d)}(x, y, z, q) \\ &= \frac{1}{\prod_{j=1}^{d}(1+x^j y(z-1)[j]_q) - \sum_{j=1}^{d} x^j yz[j]_q \prod_{i=j+1}^{d}(1+x^i y(z-1)[i]_q)}. \end{aligned} \tag{8}$$

Proof. By (7), we have

$$A^{(d)}(x, y, z, q) = \frac{1}{\frac{1+x^d y(z-1)[d]_q}{A^{(d-1)}(x,y,z,q)} - x^d yz[d]_q}.$$

Formula (8) then follows by induction on d. □

Remark. Taking $x = 1$ in (8), one obtains the generating function counting the number of d-ary r-color words according to the number of rises and the value of the statistic σ (here, the variable y would mark the length of a word).

By taking the limit as $d \to \infty$ in the preceding theorem, we obtain the following alternate formula for $A(x, y, z, q)$.

COROLLARY 2.4

The generating function $A(x, y, z, q)$ is given by

$$A(x, y, z, q) = \frac{1}{\prod_{j\geq 1}(1+x^j y(z-1)[j]_q)\left(1-\sum_{j\geq 1} \frac{x^j yz[j]_q}{\prod_{i=1}^{j}(1+x^i y(z-1)[i]_q)}\right)}. \tag{9}$$

Differentiating $A^{(d)}(x, y, z, q)$ with respect to z, and then substituting $z = 1$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}z} A^{(d)}(x, y, z, q) \mid_{z=1} = \frac{\sum_{j=1}^{d} x^j y[j]_q \sum_{i=j+1}^{d} x^i y[i]_q}{(1 - \sum_{j=1}^{d} x^j y[j]_q)^2},$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}z} A(x, y, z, q) \mid_{z=1} = \frac{x^3 y^2(1 + q - xq - x^3 q^2)}{(1-x(1+y+q) + x^2 q)^2(1 + x)(1 - x^2 q)(1 + xq)}$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}z} A(x, y, z, 1) \mid_{z=1} = \frac{x^3 y^2(2 + x + x^2)}{(1 + x)^3((1 - x)^2 - xy)^2}.$$

Therefore, the coefficient of y^m in $\frac{\mathrm{d}}{\mathrm{d}z} A(x, y, z, 1) \mid_{z=1}$ is given by

$$\frac{(m - 1)(2 + x + x^2)x^{m+1}}{(1 + x)^3(1 - x)^{2m}}.$$

Extracting the coefficient of x^n in the last expression then yields the following result.

COROLLARY 2.5

If $n \geq m \geq 1$, then the total number of rises within all of the r-color compositions of n having m parts is given by

$$(m - 1) \sum_{j=1}^{n-m} (-1)^{j-1}(j^2 + 1)\binom{n + m - j - 1}{2m - 1}.$$

It is also possible to obtain a formula for the total number of rises by a direct combinatorial argument, which yields the following identity that we are unable to find in literature.

Theorem 2.6. *If $n \geq m \geq 3$, then*

$$\sum_{j=1}^{n-m} (-1)^{j-1}(j^2 + 1)\binom{n + m - j - 1}{2m - 1} = \sum_{p=2}^{n-m+2} \binom{n + m - p - 3}{2m - 5} a(p), \tag{10}$$

where

$$a(p) = \begin{cases} \dfrac{p(p-1)(p+1)}{12}, & \text{if } p \text{ is odd;} \\ \dfrac{p(p-2)(2p+1)}{24}, & \text{if } p \text{ is even.} \end{cases}$$

Proof. To show this, we argue that the total number of rises within all the members of $\mathcal{A}_{n,m}$ is given by $(m-1)\sum_{p=2}^{n-m+2}\binom{n-p+m-3}{2m-5}a(p)$, and then compare with Corollary 2.5. Given $1 \le i < j \le n$, by a *rise of type* (i, j) within a member of $\mathcal{A}_{n,m}$, we will mean one involving the parts i and j. We consider rises of type (i, j) in which the i is the b-th part from the left within a composition for some $1 \le b \le m-1$. Since there are ij ways of coloring the parts involved in a rise and since the remaining parts constitute some member of $\mathcal{A}_{n-i-j,m-2}$ regardless of b, it follows that the number of rises of type (i, j) within all of the members of $\mathcal{A}_{n-i-j,m-2}$ is given by $(m-1)ij|\mathcal{A}_{n-i-j,m-2}|$. By Theorem 2.1 of [3], we have

$$|\mathcal{A}_{n-i-j,m-2}| = \binom{n-i-j+m-3}{2m-5}.$$

Summing over all possible i and j and letting $p = i + j$, it follows that the total number of rises within all of the members of $\mathcal{A}_{n,m}$ is given by

$$(m-1)\sum_{p=2}^{n-m+2}\sum_{i=1}^{\lfloor\frac{p-1}{2}\rfloor} i(p-i)\binom{n-p+m-3}{2m-5} = (m-1)\sum_{p=2}^{n-m+2}\binom{n-p+m-3}{2m-5}a(p),$$

as desired, where $a(p) = \sum_{i=1}^{\lfloor\frac{p-1}{2}\rfloor} i(p-i)$ is as given. □

Equating the results in (2) and (9) yields an infinite series identity which we are unable to find in the literature. We have provided in essence a combinatorial proof of this identity as we have shown that both sides give the generating function $A(x, y, z, q)$. One can also give a direct analytic proof as follows. Let $a_j = 1 + x^j y(z-1)[j]_q$ for $j \ge 1$. Then the expression in (2) may be rewritten as

$$\left(1 - \sum_{i\ge1}\frac{a_i-1}{z-1}\prod_{j=1}^{i-1}a_j\right)^{-1} = \left(1 - \frac{1}{z-1}\left(\sum_{i\ge1}\prod_{j=1}^{i}a_j - \sum_{i\ge1}\prod_{j=1}^{i-1}a_j\right)\right)^{-1}$$
$$= \left(1 - \frac{1}{z-1}\left(\prod_{j\ge1}a_j - 1\right)\right)^{-1} = \frac{z-1}{z-\prod_{j\ge1}a_j}.$$

On the other hand, the expression in (9) may be rewritten as

$$\left(\prod_{j\ge1}a_j\left(1 - \frac{z}{z-1}\sum_{j\ge1}\frac{a_j-1}{\prod_{i=1}^{j}a_i}\right)\right)^{-1}$$
$$= \left(\prod_{j\ge1}a_j\left(1 - \frac{z}{z-1}\left(\sum_{j\ge1}\frac{1}{\prod_{i=1}^{j-1}a_i} - \sum_{j\ge1}\frac{1}{\prod_{i=1}^{j}a_i}\right)\right)\right)^{-1}$$

$$= \left(\prod_{j\geq 1} a_j \left(1 - \frac{z}{z-1} \left(1 - \frac{1}{\prod_{i\geq 1} a_i} \right) \right) \right)^{-1}$$

$$= \left(\frac{z}{z-1} - \frac{1}{z-1} \prod_{j\geq 1} a_j \right)^{-1} = \frac{z-1}{z - \prod_{j\geq 1} a_j},$$

as before. From the preceding, we obtain the following result.

Theorem 2.7. *The generating function $A(x, y, z, q)$ is given by*

$$A(x, y, z, q) = \frac{z-1}{z - \prod_{j\geq 1}(1 + x^j y(z-1)[j]_q)}. \tag{11}$$

Recall the notation $(a; b)_k = \prod_{j=0}^{k-1}(1 - ab^j)$, with $(a; b)_\infty$ the limiting product as $k \to \infty$. Note that substituting $q = -1$ into $A(x, y, z, q)$ gives the generating function for the sign-balance of the σ statistic on the set $\mathcal{A}_{n,m,\ell}$. Since $[j]_q \mid_{q=-1} = [j \text{ is odd}]$, we obtain from (11) the following result.

COROLLARY 2.8

The generating function $A(x, y, z, -1)$ is given by

$$A(x, y, z, -1) = \frac{z-1}{z - (-a; x^2)_\infty} = \frac{1}{1 - xy \sum_{j\geq 0} \frac{x^{j(j+1)}}{(x^2;x^2)_{j+1}} a^j}, \tag{12}$$

where $a = xy(z-1)$.

We conclude this section with an additional recurrence satisfied by the polynomials $a_i(n, m, \ell; q)$.

PROPOSITION 2.9

If $n \geq 3$ and $2 \leq i \leq n-1$, then

$$\begin{aligned} a_i(n, m, \ell; q) &= a_{i-1}(n-1, m, \ell; q) - [i-1]_q (a_{i-1}(n-i, m-1, \ell; q) \\ &\quad - a_{i-1}(n-i, m-1, \ell-1; q)) \\ &\quad + q^{i-1} \sum_{j=1}^{i-1} a_j(n-i, m-1, \ell-1; q) \\ &\quad + q^{i-1} \sum_{j=i}^{n-1} a_j(n-i, m-1, \ell; q), \end{aligned} \tag{13}$$

with $a_1(n, m, \ell; q) = a(n-1, m-1, \ell; q)$ and $a_n(n, m, \ell; q) = [n]_q$ if $m = 1, \ell = 0$ and zero otherwise for all $n \geq 2$.

Proof. Note first that $a_1(n, m, \ell; q) = a(n-1, m-1, \ell; q)$ since adding a part of size one to the end does not ever introduce a rise, whereas $a_n(n, m, \ell; q) = [n]_q$ only when $m = 1$ and $\ell = 0$ since the single part in this case may be colored in n ways. To show

(13), we first consider members of $\mathcal{A}_{n,m,\ell}(i)$ that may be obtained by adding 1 to the final part (keeping the color the same) of some member of $\mathcal{A}_{n-1,m,\ell}(i-1)$ whose second-to-last part is not $i-1$. By subtraction, the weight (with respect to σ) of such members of $\mathcal{A}_{n,m,\ell}(i)$ is seen to be $a_{i-1}(n-1,m,\ell;q)-[i-1]_q a_{i-1}(n-i,m-1,\ell;q)$. The weight of the members of $\mathcal{A}_{n,m,\ell}(i)$ whose second-to-last part is $i-1$ and whose final part i is not marked with the i-th color is given by $[i-1]_q a_{i-1}(n-i,m-1,\ell-1;q)$. Finally, the weight of the members of $\mathcal{A}_{n,m,\ell}(i)$ whose final part i is assigned the i-th color is given by $q^{i-1}\sum_{j=1}^{i-1} a_j(n-i,m-1,\ell-1;q)$ if the penultimate part is less than i and given by $q^{i-1}\sum_{j=i}^{n-1} a_j(n-i,m-1,\ell;q)$ if the penultimate part is at least i. □

3. Counting by levels

Let $\mathcal{B}_{n,m,\ell}$ denote the set of n-color compositions of n having m parts and ℓ levels and $\mathcal{B}_{n,m,\ell}(i)$ the subset of $\mathcal{B}_{n,m,\ell}$ whose last letter is i. Let $b_i(n,m,\ell;q)$ denote the distribution polynomial for the σ statistic (marked by q) on the set $\mathcal{B}_{n,m,\ell}(i)$, where $n \geq m > \ell \geq 0$ and $1 \leq i \leq n$. Considering whether or not the next-to-last part within a member of $\mathcal{B}_{n,m,\ell}(i)$ is i leads to the recurrence

$$\begin{aligned} b_i(n,m,\ell;q) &= [i]_q b_i(n-i,m-1,\ell-1;q) \\ &\quad + [i]_q \sum_{j=1, j\neq i}^{n-1} b_j(n-i,m-1,\ell;q), \qquad 1 \leq i \leq n-1, \end{aligned} \tag{14}$$

with $b_n(n,m,\ell;q) = [n]_q$ if $m=1, \ell=0$ and zero otherwise. Let $b(n,m,\ell;q) = \sum_{i=1}^{n} b_i(n,m,\ell;q)$ and

$$B(x,y,z,q) = 1 + \sum_{n\geq m>\ell\geq 0} b(n,m,\ell;q)x^n y^m z^\ell.$$

Theorem 3.1. *The generating function $B(x,y,z,q)$ is given by*

$$B(x,y,z,q) = \frac{1}{1-\sum_{i\geq 1}\frac{x^i y[i]_q}{1-x^i y(z-1)[i]_q}}. \tag{15}$$

Proof. By (14), we have for $1 \leq i \leq n-1$ the recurrence

$$\begin{aligned} b_i(n,m,\ell;q) &= [i]_q b_i(n-i,m-1,\ell-1;q) + [i]_q (b(n-i,m-1,\ell;q) \\ &\quad - b_i(n-i,m-1,\ell;q)). \end{aligned}$$

Given $1 \leq i \leq n$, let $b_i(n,m;z,q) = \sum_{\ell=0}^{m-1} b_i(n,m,\ell;q)z^\ell$ and $b(n,m;z,q) = \sum_{\ell=0}^{m-1} b(n,m,\ell;q)z^\ell$. Then the last recurrence implies

$$b_i(n,m;z,q) = (z-1)[i]_q b_i(n-i,m-1;z,q) + [i]_q b(n-i,m-1;z,q).$$

Define $b_i(n;y,z,q) = \sum_{m=1}^{n} b_i(n,m;z,q)y^m$ and $b(n;y,z,q) = \sum_{m=1}^{n} b(n,m;z,q)\, y^m$. Then the last recurrence yields

$$\begin{aligned} b_i(n;y,z,q) &= y(z-1)[i]_q b_i(n-i;y,z,q) \\ &\quad + y[i]_q b(n-i;y,z,q), \qquad 1 \leq i \leq n-1, \end{aligned}$$

with $b_n(n;y,z,q) = y[n]_q$ for $n \geq 1$.

If $i \geq 1$, then let $B_i(x, y, z, q) = \sum_{n \geq i} b_i(n; y, z, q)x^n$. Note that, by the definitions, we have $B(x, y, z, q) = 1 + \sum_{n \geq 1} b(n; y, z, q)x^n = 1 + \sum_{i \geq 1} B_i(x, y, z, q)$. Multiplying both sides of the last recurrence by x^n and summing over $n \geq i + 1$, we have

$$\begin{aligned} B_i(x, y, z, q) - x^i y[i]_q &= x^i y(z-1)[i]_q B_i(x, y, z, q) \\ &\quad + x^i y[i]_q (B(x, y, z, q) - 1), \end{aligned}$$

which implies

$$B_i(x, y, z, q) = \frac{x^i y[i]_q B(x, y, z, q)}{1 - x^i y(z-1)[i]_q}, \qquad i \geq 1.$$

Summing the preceding equation over $i \geq 1$, one obtains

$$B(x, y, z, q) - 1 = B(x, y, z, q) \sum_{i \geq 1} \frac{x^i y[i]_q}{1 - x^i y(z-1)[i]_q},$$

which gives (15). □

Now we consider the problem of counting the total number of levels within all the members of $\mathcal{A}_{n,m}$. By differentiating with respect to z and substituting $z = 1$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}z} B(x, y, z, q)\bigg|_{z=1} = \frac{\sum_{i \geq 1} (x^i y[i]_q)^2}{\left(1 - \sum_{i \geq 1} x^i y[i]_q\right)^2},$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}z} B(x, y, z, q)\bigg|_{z=1} = \frac{x^2 y^2 (1-x)(1-xq)(1+x^2 q)}{(1 - x(1+y+q) + x^2 q)^2 (1+x)(1+xq)(1-x^2 q)}$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}z} B(x, y, z, 1)\bigg|_{z=1} = \frac{x^2 y^2 (1-x)(1+x^2)}{(1+x)^3((1-x)^2 - xy)^2}.$$

Therefore, the coefficient of y^m in $\frac{\mathrm{d}}{\mathrm{d}z} B(x, y, z, 1)\, |_{z=1}$ is given by

$$\frac{(m-1)(1-x)(1+x^2)x^m}{(1+x)^3(1-x)^{2m}}.$$

Extracting the coefficient of x^n in the last expression then yields the following result.

COROLLARY 3.2

If $n \geq m \geq 1$, then the total number of levels within all of the r-color compositions of n having m parts is given by

$$(m-1)\left(\binom{n+m-1}{2m-1} + 2\sum_{j=1}^{n-m} (-1)^j (j^2+1) \binom{n+m-j-1}{2m-1}\right).$$

One can also compute the total number of levels by a combinatorial argument, which yields the following apparently new binomial identity.

Theorem 3.3. *If $n \geq m \geq 3$, then*

$$\binom{n+m-1}{2m-1} + 2\sum_{j=1}^{n-m}(-1)^j(j^2+1)\binom{n+m-j-1}{2m-1} = \sum_{i=1}^{\lfloor \frac{n-m+2}{2} \rfloor} i^2\binom{n+m-2i-3}{2m-5}. \tag{16}$$

Proof. To show this, we argue that the total number of levels within all the members of $\mathcal{A}_{n,m}$ is given by $(m-1)\sum_{i=1}^{\lfloor \frac{n-m+2}{2} \rfloor} i^2\binom{n-2i+m-3}{2m-5}$, and then compare with Corollary 3.2. Given $1 \leq i \leq n$, by an *i-level* within a member of $\mathcal{A}_{n,m}$, we will mean an occurrence of two consecutive parts of size i. Note that there are i^2 ways of coloring the involved parts and $m-1$ possible positions for an occurrence of an i-level. Thus, the total number of i-levels is given by $(m-1)i^2|\mathcal{A}_{n-2i,m-2}| = (m-1)i^2\binom{n-2i+m-3}{2m-5}$, by Theorem 2.1 of [3]. Summing over all possible i gives the desired formula for the total number of levels. □

We consider now the distribution for the number of levels on compositions having parts in $[d]$. Let $\mathcal{B}^{(d)}_{n,m,\ell}$ denote the subset of $\mathcal{B}_{n,m,\ell}$ whose members have parts in $[d]$. Define $b^{(d)}(n,m,\ell;q)$ to be the distribution polynomial for the σ statistic restricted to $\mathcal{B}^{(d)}_{n,m,\ell}$. Let $B^{(d)}(x,y,z,q) = 1 + \sum_{n \geq m > \ell \geq 0} b^{(d)}(n,m,\ell;q)x^n y^m z^\ell$.

Theorem 3.4. *The generating function $B^{(d)}(x,y,z,q)$ is given by*

$$B^{(d)}(x,y,z,q) = \frac{1}{1 - \sum_{j=1}^{d} \frac{x^j y[j]_q}{1 - x^j y(z-1)[j]_q}}, \qquad d \geq 1. \tag{17}$$

Proof. Let $B^{(d)} = B^{(d)}(x,y,z,q)$ and $\tilde{B}^{(d)} = \tilde{B}^{(d)}(x,y,z,q)$ denote the generating function enumerating the subset of $\mathcal{B}^{(d)}_{n,m,\ell}$ whose members have first part d. Considering whether or not a member of $\mathcal{B}^{(d)}_{n,m,\ell}$ contains a part of size d implies

$$B^{(d)} = B^{(d-1)} + B^{(d-1)}\tilde{B}^{(d)}, \qquad d \geq 1,$$

with $B^{(0)} = 1$. Since compositions enumerated by $\tilde{B}^{(d)}$ are of the form $\pi = d^p\pi'$ for some $p \geq 1$ and π' not starting with d, we also have

$$\tilde{B}^{(d)} = \frac{x^d y[d]_q}{1 - x^d yz[d]_q}(B^{(d)} - \tilde{B}^{(d)}), \qquad d \geq 1.$$

Let $a_j = \frac{x^j y[j]_q}{1 - x^j y(z-1)[j]_q}$. Combining the two prior equations gives

$$B^{(d)} = (1 + a_d B^{(d)})B^{(d-1)},$$

which we rewrite as

$$B^{(d)} = \frac{1}{\frac{1}{B^{(d-1)}} - a_d}.$$

Thus, we have $\frac{1}{B^{(d)}} = \frac{1}{B^{(d-1)}} - a_d$ for $d \geq 1$, which gives $\frac{1}{B^{(d)}} = 1 - \sum_{j=1}^{d} a_j$. This completes the proof. □

Remark. Note that taking $d \to \infty$ in (17) gives (15).

4. Counting by (i, j)-rises

By an (i, j)-*rise*, we will mean a consecutive occurrence of two parts of the form a_i and b_j, where $a < b$ and $1 \leq i \leq a$ and $1 \leq j \leq b$. Let $F_d(x, y) = F_d(x, y; z_{1,1}, z_{1,2}, \ldots, z_{d-1,d})$ be the generating function for the number of r-color compositions of n having m parts all of which belong to $[d]$ according to the number of (i, j)-rises for all i and j (which are marked by the variables $z_{i,j}$). In order to find a recurrence relation for this generating function, we need the following further notation. Let $G_{d,c}(x, y) = G_{d,c}(x, y; z_{1,1}, z_{1,2}, \ldots, z_{d-2,d-1})$ be the generating function for the number of r-color compositions π having exactly m parts all of which belong to $[d-1]$ according to the number of (i, j)-rises in πd_c (again marked by the variables $z_{i,j}$).

Note that each r-color composition π of n with parts in $[d]$ can be written as either π' or as $\pi' d_j \pi''$ for some j, where π' and π'' are r-color compositions with parts in $[d-1]$ and $[d]$, respectively. Thus

$$F_d(x, y) = F_{d-1}(x, y) + x^d y \sum_{j=1}^{d} G_{d,j}(x, y) F_d(x, y),$$

which implies

$$F_d(x, y) = \frac{F_{d-1}(x, y)}{1 - x^d y \sum_{j=1}^{d} G_{d,j}(x, y)}.$$

Let π be any member of $\mathcal{A}_{n,m}$ whose parts belong to $[d]$. Note that every r-color composition $\pi(d+1)_c$ can be written as either $\pi'(d+1)_c$ or as $\pi' d_k \pi''(d+1)_c$, where $1 \leq k \leq d$, π' has parts in $[d-1]$ and π'' is either empty or is non-empty and has parts in $[d]$. Thus

$$\begin{aligned} G_{d+1,c}(x, y) = G_{d,c}(x, y) &+ x^d y \sum_{j=1}^{d} z_{j,c} G_{d,j}(x, y) \\ &+ x^d y \sum_{j=1}^{d} G_{d,j}(x, y)(G_{d+1,c} - 1), \end{aligned}$$

which gives

$$G_{d+1,c}(x, y) = \frac{G_{d,c}(x, y) + x^d y \sum_{j=1}^{d} (z_{j,c} - 1) G_{d,j}(x, y)}{1 - x^d y \sum_{j=1}^{d} G_{d,j}(x, y)}.$$

Combining the previous observations yields the following result.

Theorem 4.1. *Fix $d \geq 1$. Then*

$$F_d(x, y) = \prod_{i=1}^{d} \frac{1}{1 - x^i y \sum_{j=1}^{d} G_{i,j}(x, y)}, \tag{18}$$

where for any c we have that $G_{1,c}(x, y) = 1$ *and*

$$G_{d+1,c}(x, y) = \frac{G_{d,c}(x, y) + x^d y \sum_{j=1}^{d}(z_{j,c} - 1)G_{d,j}(x, y)}{1 - x^d y \sum_{j=1}^{i} G_{d,j}(x, y)}.$$

Example 4.2. Theorem 4.1 for $d = 1$ gives $F_1(x, y) = \frac{1}{1-xyG_{1,1}(x,y)} = \frac{1}{1-xy}$, as expected. Theorem 4.1 for $d = 2$ gives

$$F_2(x, y) = \frac{1}{(1 - xy)(1 - x^2 y(G_{2,1}(x, y) + G_{2,2}(x, y)))},$$

where $G_{2,1}(x, y) = 1 + \frac{xyz_{1,1}}{1-xy}$ and $G_{2,2}(x, y) = 1 + \frac{xyz_{1,2}}{1-xy}$. So

$$F_2(x, y) = \frac{1}{1 - xy - 2x^2 y - x^3 y^2 (z_{1,1} + z_{1,2} - 2)}.$$

By an (i, j)-*level*, we will mean an occurrence of two consecutive parts of the form a_i and a_j, where $1 \le i, j \le a$. Define the generating functions $F(x, y, q) = F(x, y, q; z_{1,1}, z_{1,2}, \ldots)$ (resp. $G_{a_c}(x, y, q) = G_{a_c}(x, y, q; z_{1,1}, z_{1,2}, \ldots))$ for the number of members π (resp. $\pi = \pi' a_c$) of $\mathcal{A}_{n,m}$ counted according to the σ statistic and to the number of (i, j)-levels (marked by the variables $z_{i,j}$). By the definitions, we have the following.

PROPOSITION 4.3

The generating function $F(x, y, q)$ *is given by*

$$F(x, y, q) = 1 + \sum_{a\ge1} \sum_{c=1}^{a} G_{a_c}(x, y, q), \tag{19}$$

where

$$G_{a_c}(x, y, q) = x^a y q^{c-1} F(x, y, q) + x^a y q^{c-1} \sum_{k=1}^{a} (z_{k,c} - 1) G_{a_k}(x, y, q).$$

Upon taking $z_{k,c} = z$ for all k and c in Proposition 4.3, it is possible to show

$$F(x, y, q) = 1 + F(x, y, q) \sum_{a\ge1} \frac{x^a y[a]_q}{1 - x^a y(z - 1)[a]_q},$$

which implies Theorem 3.1.

5. Concluding remarks

Concerning r-color compositions, one may consider other definitions of rise. For example, in Definition 1 of [3] and also in [4], the parts are ordered according to

$$1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 3_3 < \cdots .$$

Given $([a_1]_{b_1}, [a_2]_{b_2}, \ldots) \in \mathcal{A}_n$, one may then define a rise as an index i such that $[a_i]_{b_i} < [a_{i+1}]_{b_{i+1}}$ subject to the above ordering. Let $d(n, m, \ell)$ be the number of members of $\mathcal{A}_{n,m}$ having exactly ℓ rises by this definition and let $D(x, y, z) = 1 + \sum_{n\geq m>\ell\geq 0} d(n, m, r)x^n y^m z^\ell$.

Theorem 5.1. *The generating function $D(x, y, z)$ is given by*

$$D(x, y, z) = \frac{z-1}{z - \prod_{i\geq 1}(1 + x^i y(z-1))^i}. \tag{20}$$

Proof. Given $d \geq 1$, let $D_d = D_d(x, y, z)$ be the generating function counting the subset of $\mathcal{A}_{n,m}$ enumerated by $d(n, m, \ell)$ and having parts in $[d]$. In order to obtain an explicit formula for D_d, we need to introduce the generating function $E_{d,c} = E_{d,c}(x, y, z)$ for the number of members π of $\mathcal{A}_{n,m}$ having parts in $[d]$ enumerated according to the number of rises in the composition $d_c\pi$ (in the new order). We have the following relations involving D_d and $E_{d,c}$:

$$D_d = D_{d-1} + x^d y \sum_{c=1}^{d} E_{d,c} + x^d yz(D_{d-1} - 1) \sum_{c=1}^{d} E_{d,c},$$

where

$$\begin{aligned} E_{d,c} &= D_{d-1} + x^d y \sum_{j=1}^{c} E_{d,j} + x^d yz \sum_{j=c+1}^{d} E_{d,j} + x^d yz(D_{d-1} - 1) \sum_{j=1}^{d} E_{d,j} \\ &= D_d + x^d y(z-1) \sum_{j=c+1}^{d} E_{d,j}. \end{aligned}$$

By induction on c, we have

$$E_{d,c} = (1 + x^d y(z-1))^{d-c} D_d, \qquad 1 \leq c \leq d.$$

Thus,

$$\begin{aligned} D_d &= D_{d-1} + x^d y \sum_{c=0}^{d-1} (1 + x^d y(z-1))^c D_d \\ &\quad + x^d yz(D_{d-1} - 1) \sum_{c=0}^{d-1} (1 + x^d y(z-1))^c D_d, \end{aligned}$$

which implies

$$D_d = \frac{D_{d-1}}{1+x^d y(z-1)\sum_{c=0}^{d-1}(1+x^d y(z-1))^c - x^d yz D_{d-1} \sum_{c=0}^{d-1}(1+x^d y(z-1))^c}.$$

Hence,

$$\frac{1}{D_d} = \frac{A_d}{D_{d-1}} - B_d, \qquad d \geq 1,$$

where $A_d = 1 + x^d y(z-1) \sum_{c=0}^{d-1}(1 + x^d y(z-1))^c$ and $B_d = x^d yz \sum_{c=0}^{d-1}(1 + x^d y$ $(z-1))^c$.

Note that $D_0 = 1$ and $D_1 = \frac{1}{1-xy}$, so $\frac{1}{D_1} = 1 - xy = A_1 - B_1$. Thus, by induction on d, we have

$$\frac{1}{D_d} = \prod_{j=1}^{d} A_j - \sum_{j=1}^{d} B_j \prod_{i=j+1}^{d} A_i,$$

which gives

$$D_d = \frac{1}{\prod_{j=1}^{d} A_j \left(1 - \sum_{j=1}^{d} \frac{B_j}{\prod_{i=1}^{j} A_i}\right)},$$

where $A_d = (1 + x^d y(z-1))^d$ and $B_d = \frac{z}{z-1}((1 + x^d y(z-1))^d - 1)$. Hence,

$$\begin{aligned}
D_d &= \frac{1}{\prod_{j=1}^{d}(1 + x^j y(z-1))^j \left(1 - \frac{z}{z-1} \sum_{j=1}^{d} \frac{(1+x^j y(z-1))^j - 1}{\prod_{i=1}^{j}(1+x^i y(z-1))^i}\right)} \\
&= \frac{1}{\prod_{j=1}^{d}(1+x^j y(z-1))^j \left(1 - \frac{z}{z-1}\left(\sum_{j=1}^{d} \frac{1}{\prod_{i=1}^{j-1}(1+x^i y(z-1))^i} - \sum_{j=1}^{d} \frac{1}{\prod_{i=1}^{j}(1+x^i y(z-1))^i}\right)\right)} \\
&= \frac{z-1}{z - \prod_{i=1}^{d}(1 + x^i y(z-1))^i}.
\end{aligned}$$

Taking the limit as $d \to \infty$ in the last expression gives (20). □

By analogy, one could define a level as an occurrence of a_i-a_i for some a and i. Let $H(x, y, z)$ be the generating function that counts the members of $\mathcal{A}_{n,m}$ according to the number of levels per this definition. Then one can show the following result.

Theorem 5.2. *The generating function $H(x, y, z)$ is given by*

$$H(x, y, z) = \frac{1}{1 - \sum_{i \geq 1} \frac{i x^i y}{1 - x^i y(z-1)}}. \tag{21}$$

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COMMUNICATING EDITOR: Sharad S Sane