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Strictly convex functions on complete Finsler manifolds

YOE ITOKAWA[1], KATSUHIRO SHIOHAMA[3] and BANKTESHWAR TIWARI[2,*]

[1]Department of Information and Communication Engineering, Fukuoka Institute of Technology, Fukuoka 811–0295, Japan
[2]DST-CIMS, Institute of Science, Banaras Hindu University, Varanasi 221 005, India
[3]Institute of Information Science, Fukuoka Institute of Technology, Fukuoka 811–0295, Japan
[*]Corresponding author.
E-mail: itokawa@fit.ac.jp; k-shiohama@fit.ac.jp; banktesht@gmail.com

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Abstract. The purpose of the present paper is to investigate the influence of strictly convex functions on the metric structures of complete Finsler manifolds. More precisely we discuss the properties of the group of isometries and the exponential maps on a complete Finsler manifold admitting strictly convex functions.

Keywords. Finsler manifolds; convex functions; isometries; geodesics.

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1. Introduction

Let (M, F) be an n-dimensional complete Finsler manifold. The well known Hopf–Rinow theorem states that M is complete if and only if the exponential map $\exp_p$ at some point $p \in M$ (and hence at every point on M) is defined on the whole tangent space M_p to M at that point. This is equivalent to state that (M, F) is geodesically complete with respect to *(forward) geodesics* at every point on M. Throughout this article, we assume that (M, F) *is geodesically complete with respect to* (*forward*) *geodesics*. A function $\phi : (M, F) \to \mathbf{R}$ is said to be *convex* if and only if along every geodesic segment $\gamma : [a, b] \to (M, F)$, the restriction $\phi \circ \gamma : [a, b] \to \mathbf{R}$ is convex:

$$\varphi \circ \gamma((1-\lambda)s + \lambda t) \leq (1-\lambda)\varphi \circ \gamma(s) + \lambda\varphi \circ \gamma(t), \tag{1.1}$$

for all $0 \leq \lambda \leq 1$ and for all $a \leq s < t \leq b$. If the inequality in (1.1) is strict for all γ and for all $\lambda \in (0, 1)$, φ is called *strictly convex*, and also *strongly convex* if the second order difference quotient is bounded away from zero on every compact set. A convex function φ is said to be *locally non-constant* if it is not constant on any open set of M. Clearly, every strictly convex function is automatically locally non-constant. We call convex function $\varphi : M \to \mathbf{R}$ *exhaustion* if $\varphi^{-1}(-\infty, a]$ is compact for all $a \in \varphi(M)$ and the exponential map $\exp_p : M_p \to (M, F)$ *proper* if $\exp_p^{-1}(K)$ is compact for all compact set $K \subset M$.

Lemma 3.1 in [11] implies that if a convex function has a compact level $\varphi^{-1}(\{a\})$ for some $a \in \phi(M)$, then so is $\varphi^{-1}(\{b\})$ for all $b \geq a$. Recall that if a convex function on a complete Riemannian manifold has a compact level, then so are all the other levels (see [5]). However it is not certain if $\varphi^{-1}(\{b\})$ is compact for any $b \in \varphi(M)$.

The influence of the existence of convex functions on the metric and topology of underlying manifolds has been investigated in [5] and [6]. In the pioneering works [7] and [4], they have proved that a Busemann function on a complete and non-compact Riemannian manifold (M, g) is strongly convex if its sectional curvature is positive (see [7]) and convex if its sectional curvature is non-negative (see [4]). In particular, every super Busemann function is convex exhaustion if its sectional curvature is non-negative, and the minimum set of a super Busemann function contains a soul of M.

Clearly, a complete simply connected Riemannian manifold H of non-positive sectional curvature, called Hadamard manifold, has the property that the distance function to an arbitrary fixed point is strongly convex exhaustion. Also, the exponential map $\exp_p : H_p \to H$ is proper for every point $p \in H$. The same property holds on a complete non-compact Riemannian manifold (M, g) of positive sectional curvature (see [7]). Further, the classical Cartan theorem states that every compact subgroup G of the group of isometries $I(H)$ of H has a common fixed point. The Cartan theorem has been improved by Yamaguchi [12] on complete Riemannian manifolds admitting strictly convex exhaustion functions. Cartan theorem follows from a simple fact that the distance function to every point on H is strictly convex.

In Theorem B of [12], the group $I(M, g)$ of isometries on a complete Riemannian manifold (M, g), admitting a strictly convex proper function without minimum, is compact. The key point is the strictly increasing property of the diameter function of a strictly convex proper function (without minimum). Here the diameter function $\delta : \phi(M) \to \mathbf{R}$ of φ is defined by

$$\delta(t) := \sup\{d(x, y) \,|\, x, y \in \varphi^{-1}(t)\}.$$

However, it is not certain if the diameter function of a strictly convex exhaustion function has the same property on Finsler manifolds.

The purpose of the present paper is to investigate the influence of strictly convex functions on the metric structures of complete Finsler manifolds. Among many open problems, we discuss the properties of the group of isometries and the exponential maps on (M, F). In a recent work [11], it is proved that if a complete Finsler manifold (M, F) admits a convex function $\varphi : (M, F) \to \mathbf{R}$, all of whose level sets are compact, then there exists an $(n-1)$-manifold N which is homeomorphic to all the levels lying above the infimum and $M \setminus \{x \in M \,|, \varphi(x) = \inf_M \varphi\}$ is homeomorphic to $N \times \mathbf{R}$. If the minimum set is non-empty, then M is homeomorphic to the normal bundle in M over the minimum set $M \setminus \{x \in M|\varphi(x) = \inf_M \varphi\}$ (see Theorem 1.3 of [11]). In the present paper, more precisely, we prove as follows.

Theorem 1.1 (cf. Theorem B of [12]). *Let $\varphi : (M, F) \to \mathbf{R}$ be a strictly convex exhaustion function. Then every compact subgroup G of the group of isometries $I(M, F)$ of (M, F) has a common fixed point.*

We next discuss the exponential maps of (M, F) admitting strictly convex function.

Theorem 1.2 (cf. Theorem A of [5]). *Let $\varphi : (M, F) \to \mathbf{R}$ be a strictly convex exhaustion function. Then the exponential map $\exp_p : M_p \to (M, F)$, at each point $p \in M$ is proper.*

We shall state open problems related to the convex functions. Let (M, F) be a complete noncompact Finsler manifold.

Open problems

(1) Assume that a locally nonconstant convex function on (M, F) has a compact level. Then, is its diameter function monotone increasing?
(2) Assume that (M, F) admits everywhere nonnegative flag sectional curvature. Is a Busemann function on (M, F) convex?
(3) If the flag Ricci curvature is everywhere nonnegative on (M, F), then is it true that a Busemann function is subharmonic?

The basic facts on the geometry of geodesics, including G-surfaces and Alexandrov surfaces, are referred to [2] and [3]. Relevant results in which we expect further developments have been obtained in [8–12].

2. Preliminaries

Let us denote a Finsler manifold by (M, F), where M is an n-dimensional smooth manifold equipped with a function F, called fundamental function, defined on the tangent bundle TM of manifold M to $[0, \infty)$ such that

(1)F is smooth on $TM - \{0\}$ (regularity).
(2) F is positively homogeneus of degree one, i.e., $F(x, \lambda y) = \lambda F(x, y), \lambda > 0, (x, y) \in TM$ (positive homogeneity).
(3) The Hessian matrix $(g_{ij}) := (\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j})$ is positive definite on every fiber of $TM - \{0\}$ (strong convexity).

The Finsler structure is called absolutely homogeneous if $F(x, y) = F(x, -y)$, i.e., the condition (2) mentioned above can be rewritten as $F(x, \lambda y) = |\lambda| F(x, y)$, for any $\lambda \in R$. The length $L(c)$ of a curve $c : I \to (M, F)$ defined on the interval $I = [a, b]$ is given as

$$L(c) = \int_I F\left(c(t), \frac{\mathrm{d}c(t)}{\mathrm{d}t}\right) \mathrm{d}t.$$

The intrinsic distance function $d : M \times M \to \mathbb{R}$ on (M, F) is defined by

$$d(p, q) := \inf\{L(c) | c : [a, b] \to (M, F), c(0) = p, c(b) = q\}.$$

Therefore the positive homogeneity of F implies that $L(c)$ is not necessarily equal to that of its reversed curve c^{-1}, defined as $c^{-1}(t) := c(a + b - t)$, $a \le t \le b$ and hence, in general, we have

$$d(p, q) \neq d(q, p), \qquad p, q \in M.$$

We denote by Σ_p, the indicatrix of the Finsler manifold at point p, i.e.,

$$\Sigma_p = \left\{v \in M_p | F(p, v) = 1\right\}$$

and by Σ_K, the indicatrix bundle for subset K of M, i.e., $\Sigma_K = \cup_{p \in K} \Sigma_p$. For a vector $u \in M_p$, we denote by $\gamma_u(t)$, $t \ge 0$; the geodesic with initial conditions $\gamma_u(0) = p$, $\dot{\gamma}_u(0) = u$. At each point $p \in M$, the exponential map $\exp_p : M_p \to M$ is defined by $\exp_p(u) := \gamma_u(1)$, whenever it is defined. The well known Hopf–Rinow theorem of Finsler geometry states that (M, F) is complete if and only if the exponential map $\exp_p$ at some point $p \in M$ is defined on the whole tangent space M_p to M at that point [1]. We restrict to consider that (M, F) is geodesically complete with respect to forward geodesics. The exponential map $\exp_p$ of class C^1 at the origin, whereas C^∞ away from the origin. Moreover the exponential map $\exp_p$ is C^2 at origin if and only if (M, F) is a Berwald space. In this case, $\exp_p$ is C^∞ on M_p.

The following two propositions have already been established on Riemannian manifolds, without assuming the continuity of convex functions (see [5]). They also have been established on Finsler manifolds (see [11]).

PROPOSITION 2.1 [5, 11]

Let $\varphi : (M, F) \to \mathbf{R}$ be a convex function and $a > \inf_M \varphi$. Then the a-level set $M_a^a(\varphi)$ is a topological submanifold of dimension $n-1$.

PROPOSITION 2.2 [5, 11]

A convex function on (M, F) is locally Lipschitz.

3. Proof of Theorem 1.1

Let $\varphi : M \to \mathbb{R}$ be a strictly convex exhaustion function on a complete Finsler manifold (M, F). Let G be a compact subgroup of the group of isometry of (M, F). Denote by $M^a(\phi)$ $= \phi^{-1}(-\infty, a]$. Since ϕ is exhaustion, $M^a(\phi)$ is compact for all $a \in \phi(M)$. Let μ denote the Haar measure on G normalized by $\int_G \mathrm{d}\mu = 1$. We define a function ψ on (M, F) by

$$\psi(x) = \int_G \varphi(gx)\mathrm{d}\mu(g). \tag{3.1}$$

For every element g of G, $\varphi \circ g$ is also strictly convex, and so is ψ. Now if we would show that ψ also has a minimum, by strict convexity of the function ψ, it has a unique minimum point. Further since ψ is G-invariant, this minimum point of ψ is a fixed point of G.

We now claim that ψ is exhaustion. We show, for any $a \in \mathbb{R}$, there is a $b \in \mathbb{R}$ such that $M^a(\psi) \subset M^b(\phi)$.

If possible, let the above assertion not be true. Then $M^a(\psi)$ is non compact and hence there is an infinite sequence $\{x_j\} \subset M^a(\psi)$, such that $\lim \phi(x_j) \to \infty$. It follows from the definition of ψ that for each j there is an element $g_j \in G$ such that $\phi(g_j x_j) \leq a$. Thus it turns out that $G \cdot M^a(\phi)$ is unbounded. This contradicts the compactness of G and $M^a(\phi)$.

4. Proof of Theorem 1.2

Let $\varphi : M \to \mathbb{R}$ be a strictly convex exhaustion function on a complete Finsler manifold (M, F). Suppose, there is a compact subset K of M and a point $p \in M$ such that $\exp_p^{-1}(K)$ is not compact. We then find a sequence of vectors $\{v_j\}$ in M_p such that

(i) $F(p, v_j) \to \infty$ as $j \to \infty$,
(ii) $\exp_p(v_j) \in K$ for all j.

Then there exists an element $\xi \in \Sigma_p$ such that $\lim_{j\to\infty} \xi_j = \xi$ (if necessary, we may take a converging subsequence ξ_{j_k}), where $F(p, v_j)\xi_j = v_j$ for every j, $\xi_j \in \Sigma_p$. The existence of the limit follows from compactness of the set Σ_p.

Let $\gamma_j : [0, l_j] \to M$ be geodesics on (M, F) with initial data $\gamma_j(0) = p$, $\dot{\gamma}_j(0) = \xi_j$ and $l_j = F(p, v_j)$. Denote $\gamma_j(l_j)$ by q_j. Since K is compact, both $\{\gamma_j(l_j)\} \subset K$ and

$\{\dot{\gamma}_{\xi_j}(l_j)\} \subset \Sigma_K$ contain converging subsequences. Let $q \in K$ and $\eta \in \Sigma_q$ be their limits. A sequence $\{\gamma_j^{-1} : [0, l_j] \to (M, F)\}$ of the reversed geodesics of $\{\gamma_j\}$ has a limit, say $\sigma : [0, \infty) \to (M, F)$ such that $\sigma(0) = q$ and $\dot{\sigma}(0) = -\eta$. Since $\varphi o \sigma : [0, \in fty) \to \mathbb{R}$ is strictly convex and $\varphi \circ \sigma(s) = \lim_{j\to\infty} \varphi \circ \gamma_j(l_j - s)$ holds for all $s \geq 0$, $\varphi \circ \sigma$ is bounded above by $\Lambda := \max\{\varphi(x) | x \in \{p\} \cup K\}$. Therefore $\varphi \circ \sigma$ is either strictly increasing or constant. Suppose $\varphi \circ \sigma$ is strictly increasing. Then the slope inequality would imply that $\lim_{s\to\infty} \varphi \circ \sigma(s) = \infty$, a contradiction. Thus $\exp_p$ is proper for each $p \in M$.

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