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Axioms of spheres in lightlike geometry of submanifolds

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Abstract. We prove that if an indefinite Kaehler manifold $\bar{M}$ with lightlike submanifolds satisfies the axioms of holomorphic $2r$-spheres, axioms of holomorphic $2r$-planes, axioms of transversal r-spheres and axioms of transversal r-planes, then it is an indefinite complex space form.

Keywords. Lightlike submanifolds; indefinite Kaehler manifolds; axioms of holomorphic $2r$-spheres and planes; axioms of transversal r-spheres and planes.

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1. Introduction

The notion of axioms of planes for Riemannian manifolds was originally introduced by Cartan [2]. In [8], Leung and Nomizu generalized the notion of axioms of planes to the axioms of spheres on Riemannian manifolds. In [7], Kumar *et al.* studied the axioms of spheres and planes for indefinite Riemannian manifolds having lightlike submanifolds. Yano and Mogi [10] generalized the concept of axioms of planes for Riemannian manifolds to the axioms of holomorphic planes for Kaehler manifolds. The same conclusion prevails for a Kaehler manifold satisfying the axioms of anti-holomorphic planes, shown by Chen and Ogiue [3]. In [6], Goldberg and Moskal generalized the concept of axioms of holomorphic planes and axioms of anti-holomorphic planes to the axioms of holomorphic spheres and axioms of anti-holomorphic spheres for a Kaehler manifold, respectively and proved as follows.

Theorem 1.1. *Let $\bar{M}$ be a Kaehler manifold of complex dimension $d > 1$ satisfying the axioms of holomorphic $2r$-spheres for some fixed r, $1 \leq r < d$. Then $\bar{M}$ has constant holomorphic curvature.*

Theorem 1.2. *Let $\bar{M}$ be a Kaehler manifold of complex dimension $d > 1$ satisfying the axioms of anti-holomorphic r-spheres for some fixed r, $2 \leq r < d$. Then $\bar{M}$ has constant holomorphic curvature.*

The aim of the present paper is to generalize the axioms of holomorphic spheres and axioms of anti-holomorphic spheres to indefinite Kaehler manifolds having lightlike submanifolds.

2. Preliminaries

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an m-dimensional submanifold of $\bar{M}$ and g the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M then M is called a lightlike submanifold of $\bar{M}$. For a degenerate metric g on M, $TM^{\perp}$ is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $\mathrm{Rad}(T_xM) = T_xM \cap T_xM^{\perp}$ which is known as radical (null) subspace. If the mapping $\mathrm{Rad}(TM) : x \in M \longrightarrow \mathrm{Rad}(T_xM)$ defines a smooth distribution on M of rank $s > 0$, then the submanifold M of $\bar{M}$ is called s-lightlike submanifold and $\mathrm{Rad}(TM)$ is called the radical distribution on M (for details, see [4]).

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $\mathrm{Rad}(TM)$ in TM, that is,

$$TM = \mathrm{Rad}(TM) \perp S(TM). \tag{1}$$

$S(TM^{\perp})$ is a complementary vector subbundle to $\mathrm{Rad}(TM)$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}\mid_M$ and to $\mathrm{Rad}(TM)$ in $S(TM^{\perp})^{\perp}$ respectively. Then, we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}). \tag{2}$$

$$\begin{aligned} T\bar{M}\mid_M &= TM \oplus tr(TM) \\ &= (\mathrm{Rad}(TM) \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \end{aligned} \tag{3}$$

Let u be a local coordinate neighborhood of M and consider the local quasi-orthonormal fields of frames of $\bar{M}$ along M, on u as

$$\{\xi_1, \ldots, \xi_s, W_{s+1}, \ldots, W_n, N_1, \ldots, N_s, X_{s+1}, \ldots, X_m\},$$

where $\{\xi_1, \ldots, \xi_s\}$, $\{N_1, \ldots, N_s\}$ are local lightlike bases of $\Gamma(\mathrm{Rad}(TM)|_u)$, $\Gamma(ltr(TM)|_u)$ and $\{W_{s+1}, \ldots, W_n\}$, $\{X_{s+1}, \ldots, X_m\}$ are local orthonormal bases of $\Gamma(S(TM^{\perp})\mid_u)$ and $\Gamma(S(TM)\mid_u)$ respectively. For this quasi-orthonormal fields of frames, we have

Theorem 2.1 [4]. *Let $(M, g, S(TM), S(TM^{\perp}))$ be an s-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle $ltr(TM)$ of* $\mathrm{Rad}(TM)$ *in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TM)\mid_u)$ consisting of smooth sections $\{N_i\}$ of $S(TM^{\perp})^{\perp}\mid_u$, where u is a coordinate neighborhood of M, such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \tag{4}$$

where $\{\xi_1, \ldots, \xi_s\}$ is a lightlike basis of $\Gamma(\mathrm{Rad}(TM))$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$. Then, according to the decomposition (3), the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \tag{5}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$, where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^{\perp}_X U\}$ belongs to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called a second fundamental form, A_U is a linear operator on M, known as a shape operator.

According to (2), considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively then (5) becomes

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad \bar{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \quad (6)$$

where we put $h^l(X, Y) = L(h(X, Y))$, $h^s(X, Y) = S(h(X, Y))$, $D^l_X U = L(\nabla^{\perp}_X U)$, $D^s_X U = S(\nabla^{\perp}_X U)$.

As h^l and h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued respectively, therefore they are known as the lightlike second fundamental form and the screen second fundamental form on M, respectively. In particular,

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \quad (7)$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \quad (8)$$

where $X \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Using (2)–(3) and (6)–(8), we obtain

$$\bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, W)) = g(A_W X, Y), \quad (9)$$

$$\bar{g}(h^l(X, Y), \xi) + \bar{g}(Y, h^l(X, \xi)) + g(Y, \nabla_X \xi) = 0. \quad (10)$$

Let $\bar{P}$ be a projection of TM on $S(TM)$. Then using the decomposition in (1), we have

$$\nabla_X \bar{P}Y = \nabla^*_X \bar{P}Y + h^*(X, \bar{P}Y), \quad \nabla_X \xi = -A^*_{\xi} X + \nabla^{*t}_X \xi, \quad (11)$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}(TM))$, where $\{\nabla^*_X \bar{P}Y, A^*_{\xi} X\}$ and $\{h^*(X, \bar{P}Y), \nabla^{*t}_X \xi\}$ belongs to $\Gamma(S(TM))$ and $\Gamma(\mathrm{Rad}(TM))$, respectively. Here ∇^* and ∇^{*t}_X are linear connections on $S(TM)$ and $\mathrm{Rad}(TM)$ respectively. Using linear connections, we have the following covariant derivatives;

$$(\nabla_X h^l)(Y, Z) = \nabla^l_X(h^l(Y, Z)) - h^l(\nabla_X Y, Z) - h^l(Y, \nabla_X Z), \quad (12)$$

$$(\nabla_X h^s)(Y, Z) = \nabla^s_X(h^l(Y, Z)) - h^s(\nabla_X Y, Z) - h^s(Y, \nabla_X Z). \quad (13)$$

Denoting $\bar{R}$ and R as the curvature tensor of $\bar{\nabla}$ and ∇, respectively and using (6)–(8) and (12)–(13), we obtain

$$\begin{aligned} \bar{R}(X, Y)Z = {} & R(X, Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y \\ & - A_{h^s(Y,Z)}X + (\nabla_X h^l)(Y, Z) - (\nabla_Y h^l)(X, Z) + (\nabla_X h^s)(Y, Z) \\ & - (\nabla_Y h^s)(X, Z) + D^l(X, h^s(Y, Z)) - D^l(Y, h^s(X, Z)) \\ & + D^s(X, h^l(Y, Z)) - D^s(Y, h^l(X, Z)). \end{aligned} \quad (14)$$

In [1], Barros and Romero defined indefinite Kaehler manifolds as follows.

DEFINITION 2.2

Let $(\bar{M}, \bar{J}, \bar{g})$ be an indefinite almost Hermitian manifold and $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$ with respect to $\bar{g}$. Then $\bar{M}$ is called an indefinite Kaehler manifold if the almost complex structure $\bar{J}$ is parallel with respect to $\bar{\nabla}$, that is, $(\bar{\nabla}_X \bar{J})Y = 0$, for any $X, Y \in \Gamma(T\bar{M})$.

In [9], Sahin introduced transversal lightlike submanifolds of indefinite Kaehler manifolds as follows.

DEFINITION 2.3

Let M be a lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then M is called a transversal lightlike submanifold of $\bar{M}$ if the radical distribution $\mathrm{Rad}(TM)$ is transversal with respect to $\bar{J}$, that is, $\bar{J}(\mathrm{Rad}(TM)) = ltr(TM)$ and the screen distribution $S(TM)$ is transversal with respect to $\bar{J}$, that is, $\bar{J}(S(TM)) \subseteq S(TM^{\perp})$.

Let M be a transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ and let P and Q be the projection morphisms on $S(TM)$ and $\mathrm{Rad}(TM)$, respectively. Then for $X \in \Gamma(TM)$, we have $X = PX + QX$. Applying $\bar{J}$ both sides, we obtain $\bar{J}X = \bar{J}PX + \bar{J}QX$, where $\bar{J}PX \in \Gamma(S(TM^{\perp}))$ and $\bar{J}QX \in \Gamma(ltr(TM))$. Let μ denote the complementary orthogonal distribution to $\bar{J}(S(TM))$ in $S(TM^{\perp})$. Then for $V \in \Gamma(S(TM^{\perp})$, we have $\bar{J}V = BV + CV$, where $BV \in \Gamma(S(TM))$ and $CV \in \Gamma(\mu)$.

DEFINITION 2.4 [5]

A lightlike submanifold M of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is totally umbilical in $\bar{M}$ if there is a smooth transversal vector field $H \in \Gamma(ltr(TM))$ on M, called transversal curvature vector field of M, such that, for all $X, Y \in \Gamma(TM)$,

$$h(X, Y) = Hg(X, Y). \tag{15}$$

Using (5), it is clear that M is totally umbilical if and only if on each coordinate neighborhood u there exists smooth vector fields $H^l \in \Gamma(ltr(TM))$ and $H^s \in \Gamma(S(TM^{\perp}))$, such that

$$h^l(X, Y) = H^l g(X, Y), \quad h^s(X, Y) = H^s g(X, Y). \tag{16}$$

For a totally umbilical lightlike submanifold, using (9), (10) and (16), we have

$$h^l(X, \xi) = 0, \quad h^l(X, \xi) = 0, \quad D^l(X, W) = 0. \tag{17}$$

The transversal curvature vector field H of M is said to be parallel in the transversal vector bundle $tr(TM)$ if $\nabla^{\perp}_X H = 0$, for any $X \in \Gamma(TM)$. Let M be a totally umbilical lightlike submanifold. Then using (5), (7), (8) and (17), the transversal curvature vector field H of M is said to be parallel in the transversal vector bundle $tr(TM)$ if and only if

$$\nabla^l_X H^l = 0, \quad \nabla^s_X H^s + D^s(X, H^l) = 0. \tag{18}$$

Theorem 2.5 [4]. *Let M be a s-lightlike submanifold of semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the induced connection ∇ is a metric connection on M if and only if h^l vanishes identically on M.*

Theorem 2.6 [4]. *Let M be a s-lightlike submanifold of semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the induced connection ∇ on M is a metric connection if and only if one of the following conditions is fulfilled*:

(*i*) A^*_ξ *vanishes on* $\Gamma(TM)$ *for any* $\xi \in \Gamma(\mathrm{Rad}(TM))$.
(*ii*) $\mathrm{Rad}(TM)$ *is a Killing distribution.*
(*iii*) $\mathrm{Rad}(TM)$ *is a parallel distribution with respect to* ∇.

Theorem 2.7. *Let M be a transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ such that $D^s(X, N) \in \Gamma(\mu)$. Then the induced connection ∇ is a metric connection on M.*

Proof. Using the Kaehlerian property of $\bar{M}$, we have

$$\bar{\nabla}_X \xi = -\bar{J}\bar{\nabla}_X \bar{J}\xi \tag{19}$$

for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}(TM))$. Since M is a transversal lightlike submanifold, let $\bar{J}\xi = N$, where $N \in \Gamma(ltr(TM))$. Then (6), (7), (11) and (19) give

$$-A^*_\xi X + \nabla^{*t}_X \xi + h(X, \xi) = \bar{J}A_N X - BD^s(X, N) - CD^s(X, N) - \bar{J}\nabla^l_X N.$$

Taking the components along the screen distribution, we get $A^*_\xi X = BD^s(X, N)$. By hypothesis, $D^s(X, N) \in \Gamma(\mu)$, therefore $BD^s(X, N) = 0$. Then we have $A^*_\xi X = 0$. Hence the assertion follows from Theorem (2.6). □

DEFINITION 2.8 [4]

Let $(\bar{M}, \bar{J}, \bar{g})$ be a real $2m$-dimensional indefinite Kaehler manifold and M be an n-dimensional submanifold of $\bar{M}$. Then M is said to be a CR-lightlike submanifold if $\bar{J}(\mathrm{Rad}(TM))$ is a distribution on M, such that $\mathrm{Rad}(TM) \cap \bar{J}(\mathrm{Rad}(TM)) = 0$ and there exist vector bundles D_0 and D' over M, such that $S(TM) = \{\bar{J}(\mathrm{Rad}(TM)) \oplus D'\} \perp D_0$, $\bar{J}(D_0) = D_0$ and $\bar{J}(D') = L_1 \perp L_2$, where $\Gamma(D_0)$ is a non-degenerate distribution on M. $\Gamma(L_1)$ and $\Gamma(L_2)$ are vector sub-bundles of $\Gamma(ltr(TM))$ and $\Gamma(S(TM^\perp))$, respectively.

Clearly, the tangent bundle of a CR-lightlike submanifold is decomposed as $TM = D \oplus D'$, where $D = \mathrm{Rad}(TM) \perp \bar{J}(\mathrm{Rad}(TM)) \perp D_0$. If $D' = \{0\}$, then M is called holomorphic (or invariant) lightlike submanifold of indefinite Kaehler manifolds.

DEFINITION 2.9

Let M be an holomorphic lightlike submanifold of an indefinite Kaehler manifold then the distribution D is said to define a totally geodesic foliation in M, if and only if, $\nabla_X Y \in \Gamma(D)$, for any $X, Y \in \Gamma(D)$.

Theorem 2.10. *Let M be a totally umbilical holomorphic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ such that the distribution D_0 defines a totally geodesic foliation in M. Then the induced connection ∇ is a metric connection on M.*

Proof. Let $X, Y \in \Gamma(D_0)$, then we have

$$g(\nabla_X \bar{J}X, Y) = \bar{g}(\bar{\nabla}_X \bar{J}X, Y) = -\bar{g}(\bar{\nabla}_X X, \bar{J}Y) = \bar{g}(\bar{J}\nabla_X X, Y).$$

Using the hypothesis that the distribution D_0 defines a totally geodesic foliation in M and the fact that D_0 is non degenerate, we obtain $\nabla_X \bar{J}X = \bar{J}\nabla_X X$, for any $X \in \Gamma(D_0)$.

Now, let M be a totally umbilical holomorphic lightlike submanifold and using the fact that the distribution D_0 defines a totally geodesic foliation in M, we have

$$\bar{g}(\bar{J}\bar{\nabla}_X X, \bar{J}\xi) = \bar{g}(\bar{\nabla}_X \bar{J}X, \bar{J}\xi) = \bar{g}(\nabla_X X, \xi) = 0$$

and

$$\bar{g}(\bar{J}\bar{\nabla}_X X, \bar{J}\xi) = g(X, X)\bar{g}(H^l, \xi),$$

for any $\xi \in \Gamma(\mathrm{Rad}(TM))$ and $X \in \Gamma(D_0)$. Hence, we have $g(X, X)\bar{g}(H^l, \xi) = 0$. Then the non degeneracy of the distribution D_0 implies that $H^l = 0$. Thus, our assertion follows from (16) and Theorem 2.5. □

3. Axioms of spheres and planes

Axioms of holomorphic $2r$-planes: An indefinite Kaehler manifold $\bar{M}$ of complex dimension $d > 1$ satisfies the axioms of holomorphic $2r$-planes if for each $m \in \bar{M}$ and $2r$-dimensional holomorphic subspace S of $T_m(\bar{M})$, $1 \leq r < d$, there exists a totally geodesic holomorphic lightlike submanifold M satisfying $m \in M$ and $T_m(M) = S$.

Axioms of holomorphic $2r$-spheres: An indefinite Kaehler manifold $\bar{M}$ of complex dimension $d > 1$ satisfies the axioms of holomorphic $2r$-spheres if for each $m \in \bar{M}$ and $2r$-dimensional holomorphic subspace S of $T_m(\bar{M})$, $1 \leq r < d$, there exists a totally umbilical holomorphic lightlike submanifold M with parallel transversal curvature vector field satisfying $m \in M$ and $T_m(M) = S$.

Axioms of transversal r-planes: An indefinite Kaehler manifold $\bar{M}$ of complex dimension $d > 1$ satisfies the axioms of transversal r-planes if for each $m \in \bar{M}$ and r-dimensional transversal subspace S of $T_m(\bar{M})$, $1 \leq r < d$, there exists a totally geodesic transversal lightlike submanifold M satisfying $m \in M$ and $T_m(M) = S$.

Axioms of transversal r-spheres: An indefinite Kaehler manifold $\bar{M}$ of complex dimension $d > 1$ satisfies the axioms of transversal r-spheres if for each $m \in \bar{M}$ and r-dimensional transversal subspace S of $T_m(\bar{M})$, $1 \leq r < d$, there exists a totally umbilical transversal lightlike submanifold M with parallel transversal curvature vector field satisfying $m \in M$ and $T_m(M) = S$.

Lemma 3.1. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a totally umbilical holomorphic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ such that the distribution D_0 defines a totally geodesic foliation in M. Then $\nabla^l_X H^l = 0$ if and only if $\nabla_X h^l = 0$ and $\nabla^s_X H^s = 0$ if and only if $\nabla_X h^s = 0$.*

Proof. Let $X, Y, Z \in \Gamma(TM)$, then using (12) and (16), we have

$$\begin{aligned}(\nabla_X h^l)(Y, Z) &= \nabla^l_X(g(Y, Z)H^l) - g(\nabla_X Y, Z)H^l - g(Y, \nabla_X Z)H^l \\ &= X(g(Y, Z)H^l) + g(Y, Z)\nabla^l_X H^l - g(\nabla_X Y, Z)H^l \\ &\quad - g(Y, \nabla_X Z)H^l \\ &= (\nabla_X g)(Y, Z)H^l + g(Y, Z)\nabla^l_X H^l.\end{aligned}$$

By virtue of Theorem 2.10, we have

$$(\nabla_X h^l)(Y, Z) = g(Y, Z)\nabla^l_X H^l, \tag{20}$$

and similarly,

$$(\nabla_X h^s)(Y, Z) = g(Y, Z)\nabla^s_X H^s. \tag{21}$$

Thus, the assertions follow from (20) and (21). □

Similarly, we can prove the following assertions.

Lemma 3.2. *Let $(M, g, S(TM), S(TM^\perp))$ be a totally umbilical transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ such that $D^s(X, N) \in \Gamma(\mu)$. Then $\nabla^l_X H^l = 0$ if and only if $\nabla_X h^l = 0$ and $\nabla^s_X H^s = 0$ if and only if $\nabla_X h^s = 0$.*

Theorem 3.3 [1]. *Let $\bar{M}$ be an indefinite Kaehler manifold with complex dimension ≥ 2. Then $\bar{M}$ is an indefinite complex space form if and only if $\bar{g}(\bar{R}(X, Y)\bar{J}X, X) = 0$, for every orthonormal vectors $X, Y, \bar{J}X \in T\bar{M}$.*

Theorem 3.4. *Let $(M, g, S(TM), S(TM^\perp))$ be a lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of complex dimension $d > 1$, such that the distribution D_0 defines a totally geodesic foliation in M. If $\bar{M}$ satisfies the axiom of holomorphic $2r$-spheres for some fixed r, $1 \leq r < d$, then $\bar{M}$ has a constant holomorphic curvature.*

Proof. Let $X, \bar{J}X, V$ be orthonormal vector fields at arbitrary point $m \in \bar{M}$. Let S be a $2r$-dimensional holomorphic subspace of $T_m(\bar{M})$ containing $X, \bar{J}X$ and transversal to V. By the axiom of holomorphic $2r$-spheres there exists a $2r$-dimensional totally umbilical holomorphic lightlike submanifold M with parallel transversal curvature vector field H such that $T_m(M) = S$. Since the transversal curvature vector field is parallel, that is, $\nabla^\perp_X H = 0$, using (18) and Lemma 3.1, we have $\nabla_X h^l = 0$ and $\nabla_X h^s + D^s(X, H^l) = 0$. Since M is a totally umbilical holomorphic lightlike submanifold such that the distribution D_0 defines a totally geodesic foliation in M, therefore, using Theorems 2.5 and 2.10 with (16), we have $H^l = 0$. Hence we have $\nabla_X h^l = 0$ and $\nabla_X h^s = 0$ or, in particular,

$$\begin{aligned}&(\nabla_X h^l)(\bar{J}X, X) = 0, (\nabla_{\bar{J}X} h^l)(X, X) = 0,\\ &(\nabla_X h^s)(\bar{J}X, X) = 0, (\nabla_{\bar{J}X} h^s)(X, X) = 0.\end{aligned} \tag{22}$$

Using (14), the transversal form of $\bar{R}(X, \bar{J}X)X$ is given by

$$\begin{aligned}(\bar{R}(X, \bar{J}X)X)^N = {} & (\nabla_X h^l)(\bar{J}X, X) - (\nabla_{\bar{J}X} h^l)(X, X) + (\nabla_X h^s)(\bar{J}X, X)\\ & - (\nabla_{\bar{J}X} h^s)(X, X) + D^l(X, h^s(\bar{J}X, X)) - D^l(\bar{J}X, h^s(X, X))\\ & + D^s(X, h^l(\bar{J}X, X)) - D^s(\bar{J}X, h^l(X, X)),\end{aligned} \tag{23}$$

for any $X, \bar{J}X \in \Gamma(TM)$. Since M is totally umbilical, by using Theorem 2.5, (16) and (22) in (23), we obtain $(\bar{R}(X, \bar{J}X)X)^N = 0$. Hence $\bar{g}(\bar{R}(X, \bar{J}X)X, V) = 0$. Thus the assertion follows from Theorem 3.3. □

COROLLARY 3.5

Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of complex dimension $d > 1$, such that the distribution D_0 defines a totally geodesic foliation in M. If $\bar{M}$ satisfies the axiom of holomorphic $2r$-planes for some fixed r, $1 \le r < d$, then $\bar{M}$ has a constant holomorphic curvature.

Theorem 3.6. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of complex dimension $d > 1$, such that $D^s(X, N) \in \Gamma(\mu)$. If $\bar{M}$ satisfies the axiom of transversal r-spheres for some fixed r, $1 \le r < d$, then, $\bar{M}$ has a constant holomorphic curvature.*

Proof. Let $X, Y, \bar{J}X$ be orthonormal vector fields at arbitrary point $m \in \bar{M}$. Let S be a r-dimensional transversal subspace of $T_m(\bar{M})$ containing X, Y and transversal to $\bar{J}X$. By the axioms of transversal r-spheres there exists an r-dimensional totally umbilical transversal lightlike submanifold M with parallel transversal curvature vector field H such that $T_m(M) = S$. Proceeding as in the proof of the Theorem 3.4, by virtue of Lemma 3.2, we obtain

$$\begin{aligned} &(\nabla_X h^l)(Y, X) = 0; (\nabla_Y h^l)(X, X) = 0, \\ &(\nabla_X h^s)(Y, X) = 0; (\nabla_Y h^s)(X, X) = 0, \end{aligned} \tag{24}$$

and hence $(\bar{R}(X, Y)X)^N = 0$, so that $\bar{g}(\bar{R}(X, Y)X, \bar{J}X) = 0$. Thus the assertion follows from Theorem 3.3. □

COROLLARY 3.7

Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of complex dimension $d > 1$, such that $D^s(X, N) \in \Gamma(\mu)$. If $\bar{M}$ satisfies the axioms of transversal r-planes for some fixed r, $1 \le r < d$, then $\bar{M}$ has constant holomorphic curvature.

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