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Eigenfunction statistics for Anderson model with Hölder continuous single site potential

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Abstract. We consider random Schrödinger operators on $\ell^2(\mathbb{Z}^d)$ with α-Hölder continuous ($0 < \alpha \leq 1$) single site distribution. In localized regime, we study the distribution of eigenfunctions in space and energy simultaneously. In a certain scaling limit, we prove limit points are Poisson.

Keywords. Anderson model; Hölder continuous measure; Poisson statistics.

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1. Introduction

The random Schrödinger operators $\{H^\omega\}_{\omega\in\Omega}$ on $\ell^2(\mathbb{Z}^d)$ is given by

$$H^\omega = \Delta + V^\omega, \qquad \omega \in \Omega, \tag{1.1}$$

where Δ is discrete Laplacian defined by

$$(\Delta u)(n) = \sum_{|m-n|=1} u(m), \qquad \forall n \in \mathbb{Z}^d u \in \ell^2(\mathbb{Z}^d)$$

and random potential V^ω is defined by

$$V^\omega = \sum_{n\in\mathbb{Z}^d} \omega_n |, \delta_n\rangle\langle\delta_n|, \tag{1.2}$$

where $\{\delta_n\}_{n\in\mathbb{Z}^d}$ is the standard basis for $\ell^2(\mathbb{Z}^d)$ and $\{\omega_n\}_{n\in\mathbb{Z}^d}$ are real valued iid random variables with common probability distribution μ with compact support. The probability space $(\mathbb{R}^{\mathbb{Z}^d}, \mathcal{B}_{\mathbb{R}^{\mathbb{Z}^d}}, \otimes_{\mathbb{Z}^d}\mu)$ is constructed via Kolmogorov theorem and will be denoted by $(\Omega, \mathcal{B}, \mathbb{P})$, and $\omega_n : \Omega \to \mathbb{R}$ is the projection on the n-th coordinate.

For any bounded set $B \subset \mathbb{R}^d$, we consider the orthogonal projection χ_B onto $\ell^2(B\cap\mathbb{Z}^d)$ and define the matrices

$$\begin{aligned} H_B^\omega &= (\langle\delta_n, H^\omega\delta_m\rangle)_{n,m\in B}, \\ G^B(z; n, m) &= \langle\delta_n, (H_B^\omega - z)^{-1}\delta_m\rangle, \\ G^B(z) &= (H_B^\omega - z)^{-1}. \end{aligned} \tag{1.3}$$

Note that H_B^ω is the matrix

$$\chi_B H^\omega \chi_B : \ell^2(B) \to \ell^2(B) \quad \text{a.e. } \omega.$$

Let $E_{H_B^\omega}(\cdot)$ be the spectral projection of H_B^ω.

Set the resolvent operator and its matrix elements (Green's function) as

$$G(z) = (H^\omega - z)^{-1}, G(z; n, m) = \langle \delta_n, (H^\omega - z)^{-1}\delta_m \rangle \qquad z \in \mathbb{C}^+.$$

Throughout this article we will be assuming the following two conditions:

(a) The single site distribution μ is uniformly α-Hölder continuous for some $0 < \alpha \leq 1$.
(b) For any $0 < s < 1$, there exists $r, C > 0$ such that for any $\Lambda \subseteq \mathbb{Z}^d$,

$$\sup_{\substack{z \in \mathbb{C}^+ \\ \text{Re}(z) \in [a,b]}} \mathbb{E}[|G^\Lambda(z; n, m)|^s] \leq C e^{-r|n-m|} \tag{1.4}$$

for any $n, m \in \Lambda$.

When the energy E lies in $[a, b]$, then we say that E is in localized regime. Using the resolvent identity, we have

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} G^\Lambda(z; n, m) = G(z; n, m) \qquad \text{a.e. } \omega$$

for $z \in \mathbb{C}^+$, so (1.4) holds for $\mathbb{E}[|G(z; n, m)|^s]$ with the same constant C, r. The condition (b) was established by Carmona *et al.* [3] and, Aizenman and Molchanov [1] at high disorder for α-Hölder continuous single site distribution. Refer to inequalities (2.10), (3.19) and (3.20) of [1] for more details.

It was shown by Krishna [14] and Combes *et al.* [4, 5] that whenever the single site distribution is uniformly α-Hölder continuous for $0 < \alpha \leq 1$, the density of states measure (DOS) is also uniformly α-Hölder continuous.

Before describing our main result, we need some notations in place. Let ν be the density of states measure for the operator H^ω. Define the fractional derivatives:

$$d_\nu^\alpha(x) = \lim_{\epsilon \to 0} \frac{\nu(x-\epsilon, x+\epsilon)}{(2\epsilon)^\alpha} \quad \text{and} \quad D_\nu^\alpha(x) = \overline{\lim_{\epsilon \to 0}} \frac{\nu(x-\epsilon, x+\epsilon)}{(2\epsilon)^\alpha}. \tag{1.5}$$

Let $g : \mathbb{R}^d \longrightarrow \mathbb{R}^+$ and $f : \mathbb{R} \longrightarrow \mathbb{R}^+$ be compactly supported continuous functions. The multiplication operator M_g is defined by

$$(M_g u) = g(n)u(n) \qquad \forall\, n \in \mathbb{Z}^d, u \in \ell^2(\mathbb{Z}^d).$$

For a self-adjoint operator H on $\ell^2(\mathbb{Z}^d)$ with pure point spectrum (i.e. $\sigma(H) = \sigma_{pp}(H)$), let $\{E_j\}_j$ be the eigenvalues (repeated according to multiplicity) of H and ψ_j be the normalized eigenfunction corresponding to eigenvalue E_j. Then

$$\text{Tr}(M_g f(H)) = \sum_j \sum_{n \in \mathbb{Z}^d} f(E_j) g(n) |\psi_j(n)|^2. \tag{1.6}$$

Following the above notation, one can define the measure ξ^ω for the operator H^ω (described by (1.1)) by

$$\int_{\mathbb{R}\times\mathbb{R}^d} f(E, x) \mathrm{d}\xi^\omega(E, x) = \sum_j \sum_{n \in \mathbb{Z}^d} f(E_j, n) |\psi_j(n)|^2 \qquad \forall\, f \in C_c(\mathbb{R}\times\mathbb{R}^d). \tag{1.7}$$

Following the notation from physics literature, the above measure can be described as

$$d\xi^{\omega}(E,x)=\sum_{j}\sum_{n}|\psi_j(n)|^2\delta(E-E_j)dE\delta(x-n)dx, \tag{1.8}$$

where $\delta(x)$ is the Dirac-delta distribution. So for any Borel set $I\in\mathcal{B}_{\mathbb{R}}$ and $Q\in\mathcal{B}_{\mathbb{R}^d}$, we have

$$\xi^{\omega}(I\times Q)=\mathrm{Tr}(\chi_Q E_{H^{\omega}}(I)\chi_Q). \tag{1.9}$$

Killip and Nakano [12] (also see [17, 18]) studied eigenfunction statistics for discrete Anderson model with bounded density. There they studied the sequence of random measures given by

$$\int f(E,x)d\theta^{\omega}_{L,\lambda}(E,x)=\int_{\mathbb{R}\times\mathbb{R}^d} f\left(L^d(E-\lambda),\frac{x}{L}\right)d\xi^{\omega}(E,x) \qquad \forall f\in C_c(\mathbb{R}\times\mathbb{R}^d)$$

and proved its convergence to Poisson point process whenever λ lies in localized regime. We are interested in a similar object, and hence we will study the limit of *random measures* $\xi^{\omega}_{L,\lambda}$ defined by

$$\int_{\mathbb{R}\times\mathbb{R}^d} f(E,x)d\xi^{\omega}_{L,\lambda}(E,x):=\int_{\mathbb{R}\times\mathbb{R}^d} f\left(\beta_L(E-\lambda),\frac{x}{L}\right)d\xi^{\omega}(E,x) \qquad \forall f\in C_c(\mathbb{R}\times\mathbb{R}^d), \tag{1.10}$$

where $\lambda\in[a,b]$ satisfies (1.4); equivalently

$$\xi^{\omega}_{L,\lambda}(I\times Q)=\mathrm{Tr}(\chi_{LQ}E_{H^{\omega}}(\lambda+\beta_L^{-1}I)\chi_{LQ}),\quad I\in\mathcal{B}_{\mathbb{R}},\ Q\in\mathcal{B}_{\mathbb{R}^d}, \tag{1.11}$$

with $\beta_L=L^{d/\alpha}$. β_L is chosen based on the work of Dolai and Krishna [7]. They used β_L as scaling factor for eigenvalue statistics and showed the convergence to a Poisson random variable. Our main result is the following theorem.

Theorem 1.1. *Let $\xi^{\omega}_{L,\lambda}$ be defined by (1.10) where H^{ω} is given by (1.1) and μ is uniformly α-Hölder continuous measure and λ is in localized regime. Let $I\subset\mathbb{R}$ be a bounded symmetric interval and $Q\subset\mathbb{R}^d$ be rectangles with sides parallel to the axes. Then there exists a subsequence $\{L_n\}$ such that the sequence of random variables $\{\xi^{\omega}_{L_n,\lambda}(I\times Q)\}$ converge in distribution to a Poisson random variable with parameter $|I|^{\alpha}D^{\alpha}_{\nu}(\lambda)|Q|=\gamma_{\lambda}(I\times Q)$, whenever $0<D^{\alpha}_{\nu}(\lambda)<\infty$.*

Remark 1.2. For sequence $\{\xi^{\omega}_{L_n,\lambda}(I\times Q)\}_n$ to converge, the sequence $\{L_n\}_n$ depends only on I and λ but not on Q. This is because

$$\begin{aligned}
\gamma_{\lambda}(I\times Q) &= \lim_{n\to\infty}\mathbb{E}\left[\xi^{\omega}_{\lambda,L_n}(I\times Q)\right]\\
&= \lim_{n\to\infty}\mathbb{E}\left[\sum_{m\in LQ\cap\mathbb{Z}^d}\langle\delta_m,E_{H^{\omega}}(\lambda+\beta_L^{-1}I)\delta_m\rangle\right]\\
&= \lim_{n\to\infty}(|Q|L_n^d+o(L_n^{d-1}))\mathbb{E}[\langle\delta_0,E_{H^{\omega}}(\lambda+\beta_{L_n}^{-1}I)\delta_0\rangle]\\
&= |Q|\lim_{n\to\infty}L_n^d\mathbb{E}[\langle\delta_0,E_{H^{\omega}}(\lambda+\beta_{L_n}^{-1}I)\delta_0 2\rangle]\\
&= |Q|\lim_{n\to\infty}L_n^d\nu(\lambda+\beta_{L_n}^{-1}I)=|Q||I|^{\alpha}D^{\alpha}_{\nu}(\lambda)
\end{aligned}$$

and the limit is obtained through Lemma 2.6.

The term D_ν^α is defined using symmetric intervals. In general, left and right α-derivatives do not coincide with symmetric α-derivative, while in case of usual derivative, all three are the same. It is also hard to determine the set $\{x : D_\nu^\alpha(x) > 0\}$, so we have kept $D_\nu^\alpha(\lambda) > 0$ in the hypothesis and considered the case of symmetric intervals only.

To compute the limit of $\xi_{L,\lambda}^\omega$ as a random measure over a subsequence $\{L_n\}_n$, we should be able to compute $\lim_{n\to\infty}\mathbb{E}[\xi_{L_n,\lambda}^\omega(I \times Q)]$ for any bounded interval I. Even if we consider $\xi_{L,\lambda}^\omega$ as random measure on the Borel σ-algebra $\mathscr{B}$ generated by $\{(-b, b) \setminus (-a, a) : 0 < a < b < \infty\}$, we have to take different subsequences for different $I \in \mathscr{B}$. On other hand, if $d_\nu^\alpha(\lambda)$ exists, then

$$\lim_{L\to\infty} \mathbb{E}[\xi_{L,\lambda}^\omega(I \times Q)] = \alpha 2^{\alpha-1} d_\nu^\alpha(\lambda)|Q| \int_I x^{\alpha-1}\mathrm{d}x,$$

where I is a generator of $\mathscr{B}$. In this case, one can prove convergence as a random measures. As a special case, we can consider $\{\xi_{L_n,\lambda}^\omega(I \times \cdot)\}_n$ as a random measure for fixed interval. Then we have as follows:

COROLLARY 1.3

For a fixed symmetric bounded interval $I \subset \mathbb{R}$, we consider the random measure $\{\xi_{L,\lambda}^\omega(I \times \cdot)\}$ on $\mathbb{R}^d$. There exists a subsequence $\{L_n\}$ such that $\{\xi_{L_n,\lambda}^\omega(I \times \cdot)\}$ converges weakly to a Poisson point process with intensity measure $|I|^\alpha D_\nu^\alpha(\lambda)\mathrm{d}x$, where $\mathrm{d}x$ is the Lebesgue measure on $\mathbb{R}^d$.

Using (iv) in Theorem 16.16 of [10], the above corollary is immediate once we have Theorem 1.1.

Eigenvalue statistics for one dimension was studied by Molchanov [16], and later for higher dimension by Minami [15]. In the region of fractional localization (where (1.4) holds), they showed that the statistics is Poisson. Subsequently the Poisson statistics was shown for the trees by Aizenman and Warzel in [2] and recently Poisson statistics was obtained by Geisinger [8] for regular graphs. In recent results, Germinet and Klopp [9] extended the results of [12].

Recently, Kotani and Nakano [13] investigated the statistics for one-dimensional decaying random Schrödinger operators on $L^2(\mathbb{R})$. An analogue of Minami's work [15] was done by Dolai and Krishna [7] with α-Hölder continuous single site distribution. In [6], Dolai and Krishna considered the Anderson model with decaying random potentials and showed that the statistics inside $[-2d, 2d]$ in dimension $d \geq 3$ is independent of the randomness and agrees with that of the free part Δ.

2. Preliminaries

2.1 *Notations*

Given L large enough, define l_L such that $l_L \approx L^a$ for some $0 < a < 1$. Define the boxes

$$B_p(L) = \{x \in \mathbb{Z}^d : p_j l_L \leq x_j < (p_j + 1)l_L, \text{ for } i = 1, 2, \ldots, d\}, \quad p \in \mathbb{Z}^d. \tag{2.1}$$

Let $H^{\omega}_{B_p(L)}$ denote the restriction of H^{ω} to $B_p(L)$. For λ in localized regime, define the random measure $\eta^{\omega}_{p,\lambda}$ associated with $H^{\omega}_{B_p(L)}$ by

$$\int_{\mathbb{R}\times\mathbb{R}^d} f(E,x)\mathrm{d}\eta^{\omega}_{p,\lambda}(E,x) \;=\; \sum_j f\left(\beta_L(E_j-\lambda), \frac{pl_L}{L}\right), \qquad f \in C_c(\mathbb{R}\times\mathbb{R}^d), \tag{2.2}$$

where $\{E_j\}_j$ are the eigenvalues of $H^{\omega}_{B_p(L)}$. Equivalently,

$$\eta^{\omega}_{p,\lambda}(I\times Q) \;=\; \mathrm{Tr}(E_{H^{\omega}_{B_p(L)}}(\lambda+\beta_L^{-1}I))\chi_{LQ}(pl_L), \qquad I \in \mathcal{B}_{\mathbb{R}},\ Q \in \mathcal{B}_{\mathbb{R}^d}. \tag{2.3}$$

Since $H^{\omega}_{B_p(L)}$ is a finite matrix, for $|I| < \infty$, we have

$$\eta^{\omega}_{p,\lambda}(I\times Q) < \infty$$

and its range is non-negative integers, so is a point process.

Related to $B_p(L)$, we will need boundary and interior as defined:

$$\partial B_p(L) = \{x \in B_p(L) : \exists\, x' \in \mathbb{Z}^d \setminus B_p(L) \text{ such that } |x-x'| = 1\},$$

$$\mathrm{int}(B_p(L)) = \{x \in B_p(L) : \mathrm{dist}\,(x, \partial B_p(L)) > N_L\},$$

where $\{N_L\}_L$ is a increasing sequence of positive integer such that $N_L \approx \gamma \ln L$; we will specify γ later. Observe that

$$|B_p(L) \setminus \mathrm{int}(B_p(L))| = O(l_L^{d-1}\ln L), \qquad N_L \approx \gamma \ln L. \tag{2.4}$$

Let $C_p(L)$ be the cube in $\mathbb{R}^d$ corresponding to $B_p(L)$ defined by

$$C_p(L) = \left\{x \in \mathbb{R}^d : \frac{p_j l_L}{L} \le x_j < \frac{(p_j+1)l_L}{L}, \text{ for } i = 1,2,\ldots,d\right\}, \qquad p \in \mathbb{Z}^d.$$

Hence $B_p(L) = LC_p(L) \cap \mathbb{Z}^d$. Observe that $\mathbb{Z}^d$ (respectively $\mathbb{R}^d$) can be expressed as a disjoint union of $B_p(L)$ (respectively $C_p(L)$).

For a Borel set Q of finite diameter (i.e. $\sup\{|x-y|, x, y \in Q\} < \infty$), let $\Gamma_{L,Q} \subset \mathbb{Z}^d$ be such that

$$C_p(L) \cap Q \neq \phi \qquad \forall p \in \Gamma_{L,Q} \quad \text{and} \quad Q = \bigcup_{p\in\Gamma_{L,Q}} C_p(L) \cap Q. \tag{2.5}$$

The measures $\{\eta^{\omega}_{p,\lambda}\}_{p\in\Gamma_{L,Q}}$ are statistically independent and

$$|\Gamma_{L,Q}| \le \left(\frac{L}{l_L}\right)^d |Q|. \tag{2.6}$$

In the following, whenever we write sum over p, we mean that the sum is taken over $\Gamma_{L,Q}$.

2.2 *Important results*

We will need Wegner and Minami type estimates given in Combes *et al.* [5]. Hence following the notations, set $S_\mu(s) = \sup_{a\in\mathbb{R}} \mu[a, a+s]$ for probability measure μ and define

$$Q_\mu(s) = \begin{cases} \|\rho\|_\infty s, & \text{if } \mu \text{ has bounded density,} \\ 8S_\mu(s), & \text{otherwise.} \end{cases} \tag{2.7}$$

If μ is uniformly α-Hölder continuous with $0 < \alpha \leq 1$, then $S_\mu(s) \leq Us^\alpha$ for small $s > 0$ and some constant U. The following estimates will be used:

Lemma 2.1. *For all bounded interval $I \subset \mathbb{R}$ and any finite volume $\Lambda \subset \mathbb{Z}^d$, we have*

$$\mathbb{E}\left(\langle \delta_n, E_{H^\omega}(I) \quad \delta_n\rangle\right) \leq Q_\mu(|I|) \quad \forall n \in \mathbb{Z}^d, \tag{2.8}$$

$$\mathbb{E}(\mathrm{Tr}(E_{H^\omega_\Lambda}(I))) \leq Q_\mu(|I|)|\Lambda|, \tag{2.9}$$

$$\mathbb{E}(\mathrm{Tr}(E_{H^\omega_\Lambda}(I))(\mathrm{Tr}(E_{H^\omega_\Lambda}(I)) - 1)) \leq (Q_\mu(|I|)|\Lambda|)^2. \tag{2.10}$$

The proof can be found in Combes *et al.* (inequality (2.2) of [5] for (2.8)), (Theorem 2.3 of [5]) for inequality (2.9) and (Theorem 2.1 of [5]) for inequality (2.10) (also see [19]).

The following corollary is immediate from the above lemma.

COROLLARY 2.2

Consider ν the DOS of the operators H^ω satisfying condition (a). Then for any $\psi \in C_c(\mathbb{R})$ and $n \in \mathbb{Z}^d$, we have

$$\int_{\mathbb{R}} \psi(x)\mathrm{d}\nu(x) = \mathbb{E}\big(\langle \delta_n, \psi(H^\omega)\delta_n\rangle\big) \leq \|\psi\|_\infty Q_\mu(|s_\psi|), \quad s_\psi = \mathrm{supp}\ \psi, \tag{2.11}$$

$$\mathbb{E}(\mathrm{Tr}(\psi(H^\omega_\Lambda))) \leq \|\psi\|_\infty Q_\mu(|s_\psi|)|\Lambda|. \tag{2.12}$$

PROPOSITION 2.3

For any $f \in C_c(\mathbb{R} \times \mathbb{R}^d)$, we have

$$\mathbb{E}\left\{\left|\int f(E,x)\mathrm{d}\xi^\omega_{L,\lambda}(E,x) - \sum_p \int f(E,x)\mathrm{d}\eta^\omega_{p,\lambda}(E,x)\right|\right\} \longrightarrow 0 \ \text{ as } \ L \to \infty. \tag{2.13}$$

Remark 2.4. In (2.13) the sum is over $\Gamma_{L,\mathrm{supp}'(f)}$ (here $\mathrm{supp}'(f) = \{x \in \mathbb{R}^d : f(E,x) \neq 0\ \exists E \in \mathbb{R}\}$), but one can choose $\{p \in \mathbb{Z}^d : |p|_\infty < L^2\}$, and this will make it independent of f. This is because for large enough L one has $\Gamma_{L,\mathrm{supp}'(f)} \subset \{p \in \mathbb{Z}^d : |p|_\infty < L^2\}$.

Proof. The functions of the form $f(E,x) = g(x)\mathrm{Im}\frac{1}{E-z}$ for $g \in C_c(\mathbb{R}^d)$ can approximate any function in $C_c(\mathbb{R}^{1+d})$ in the L^∞ norm. So using the notation $z_L = \lambda + \beta_L^{-1} z$ and $G^{B_p}(z_L; n,n) = G^{B_p(L)}(z_L; n,n)$, we obtain

$$\left|\int f \, d\xi^\omega_{L,\lambda} - \sum_p \int f \, d\eta^\omega_{p,\lambda}\right|$$
$$= \frac{1}{\beta_L}\left|\sum_{n\in LQ} g_L(n)\mathrm{Im}\, G(z_L; n,n) - \sum_{p\in\Gamma_{L,Q}}\sum_{n\in B_p(L)} g_L(pl_L)\mathrm{Im}\, G^{B_p}(z_L; n,n)\right|$$
$$\leq \frac{\|g\|_\infty}{\beta_L}\sum_{p\in\Gamma_{L,Q}}\sum_{n\in B_p(L)} |\mathrm{Im}\, G(z_L;n,n) - \mathrm{Im}\, G^{B_p}(z_L;n,n)|$$
$$+ \frac{\sup_{|x-y|_\infty < \frac{l_L}{L}} |g(x)-g(y)|}{\beta_L}\sum_{p\in\Gamma_{L,Q}}\sum_{n\in B_p(L)} |\mathrm{Im}\, G^{B_p}(z_L;n,n)|. \tag{2.14}$$

Here we are adding and subtracting $g_L(n)\left(= g(\frac{n}{L})\right)$ from the second term of (2.14).

For $n \in \mathrm{int}(B_p)$ and $z \in \mathbb{C}^+$, we have the perturbation formula

$$G(z_L; n,n) - G^{B_p}(z_L;n,n) = \sum_{(m,k)\in\partial B_p(L)} G(z_L; n,k) G^{B_p}(z_L; m,n), \tag{2.15}$$

$(m,k) \in \partial B_p(L)$ means $m \in \partial B_p(L)$, $k \in \mathbb{Z}^d \setminus B_p(L)$ such that $|m-k| = 1$. Following the steps from [15], we use (2.15) in (2.14) and we get

$$\frac{\|g\|_\infty}{\beta_L}\sum_{p\in\Gamma_{L,Q}}\sum_{n\in B_p(L)} |\mathrm{Im}\, G(z_L;n,n) - \mathrm{Im}\, G^{B_p}(z_L;n,n)|$$
$$\leq \frac{\|g\|_\infty}{\beta_L}\sum_{p\in\Gamma_{L,Q}}\sum_{n\in B_p\setminus\mathrm{int}(B_p)} [\mathrm{Im}\, G(z_L;n,n) + \mathrm{Im}\, G^{B_p}(z_L;n,n)]$$
$$+ \frac{\|g\|_\infty}{\beta_L}\sum_{p\in\Gamma_{L,Q}}\sum_{n\in\mathrm{int}(B_p(L))}\sum_{(m,k)\in\partial B_p(L)} |G(z_L;n,k)G^{B_p}(z_L;m,n)|$$
$$= A_L + B_L. \tag{2.16}$$

For B_L, we have

$$B_L = \frac{\|g\|_\infty}{\beta_L}\sum_{p\in\Gamma_{L,Q}}\sum_{n\in\mathrm{int}(B_p(L))}\sum_{(m,k)\in\partial B_p(L)} |G(z_L;n,k)G^{B_p}(z_L;m,n)|$$
$$= \frac{\|g\|_\infty}{\beta_L}\sum_{p\in\Gamma_{L,Q}}\sum_{n\in\mathrm{int}(B_p(L))}\sum_{(m,k)\in\partial B_p(L)} |G(z_L;n,k)||G^{B_p}(z_L;m,n)|^s |G^{B_p}(z_L;m,n)|^{1-s}. \tag{2.17}$$

Now $(m,k) \in \partial B_p(L)$ and $n \in \mathrm{int}(B_p(L))$, so we have $|n-k| > N_L$. Using the exponential decay of Green's function given in (1.4), we have

$$\mathbb{E}(|G^{B_p}(z_L;n,k)|^s) \le C\mathrm{e}^{-rN_L}. \tag{2.18}$$

We also have

$$|G(z_L;n,k)| \le \frac{1}{|\mathrm{Im}\ z_L|} \quad \text{and} \quad |G^{B_p}(z_L;m,n)|^{1-s} \le \frac{1}{|\mathrm{Im}\ z_L|^{1-s}}.$$

So using the above together with (2.18) in (2.17), we get

$$\mathbb{E}(B_L) \le \frac{C\|g\|_\infty}{\beta_L|\mathrm{Im}\ z_L|^{2-s}}|\Gamma_{L,Q}|l_L^d l_L^{d-1} N_L \mathrm{e}^{-rN_L}. \tag{2.19}$$

We have $l_L \simeq L^a (0<a<1), \Gamma_{L,Q} = O\left(\frac{L}{l_L}\right)^d$, $\mathrm{Im}\ z_L = \beta_L^{-1}\tau, \tau > 0$ taking $z = \sigma + i\tau$ and $\beta_L = L^{d/\alpha}$. Choose γ so that

$$\gamma > \frac{1}{r}\left[(1-s)\frac{d}{\alpha} + d + (d-1)a\right]$$

in definition of N_L in (2.4). Then from (2.19), we get

$$\mathbb{E}(B_L) = O(\gamma L^{-\delta}\ln L), \quad \text{where } \delta = r\gamma - [(1-s)d/\alpha + d + (d-1)a] > 0. \tag{2.20}$$

From A.9 of [5] we have, for any $k>0$,

$$\mathrm{Im}\ z\mathbb{E}[\mathrm{Im}\ G^\Lambda(z;n,n)] \le \pi\left(1+\frac{k}{2}\right)S_\mu\left(\frac{2\ \mathrm{Im}\ z}{k}\right). \tag{2.21}$$

Since $\mathrm{Im} z_L = \beta_L^{-1}\mathrm{Im}\ z$ with $\mathrm{Im}\ z > 0$, so using $S_\mu(s) \le Us^\alpha$ (α-Hölder continuity of μ), we get

$$\begin{aligned} \frac{1}{\beta_L}\mathbb{E}[\mathrm{Im}\ G^\Lambda(z_L;n,n)] &\le \frac{1}{\mathrm{Im}\ z}\pi\left(1+\frac{k}{2}\right)S_\mu\left(\frac{2\ \mathrm{Im}\ z_L}{k}\right), \Lambda = C_p, \Lambda_L \\ &\le C\left(\frac{2\beta_L^{-1}\mathrm{Im}\ z}{k}\right)^\alpha \\ &\le CL^{-d}(\text{since } \beta_L = L^{d/\alpha}). \end{aligned} \tag{2.22}$$

From (2.16) and (2.4), we have

$$\begin{aligned} \mathbb{E}(A_L) &\le 2C\frac{\|g\|_\infty}{\beta_L}|\Gamma_{L,Q}||B_p(L)\setminus \mathrm{int}\ B_p(L)|N_L L^{-d} \\ &\approx C\left(\frac{L}{l_L}\right)^d \gamma l_L^{d-1}\ L^{-d}\ln L \\ &= O(L^{-a}\ln L), l_l = L^a, 0<a<1. \end{aligned} \tag{2.23}$$

Combining (2.20) and (2.23) gives

$$\mathbb{E}(A_L) + \mathbb{E}(B_L) \xrightarrow{L\to\infty} 0 \tag{2.24}$$

Using (2.22) we also obtain

$$\mathbb{E}\left[\frac{\sup_{|x-y|_\infty<\frac{l_L}{L}}|g(x)-g(y)|}{\beta_L}\sum_{p\in\Gamma_{L,Q}}\sum_{n\in B_p(L)}|\mathrm{Im}\,G^{B_p}(z_L;n,n)|\right]$$
$$=\sup_{|x-y|_\infty<\frac{l_L}{L}}|g(x)-g(y)|\sum_{p\in\Gamma_{L,Q}}\sum_{n\in B_p(L)}\beta_L^{-1}\mathbb{E}[\mathrm{Im}\,G^{B_p}(z_L;n,n)]$$
$$\leq CL^{-d}(|Q|L^d+O(L^{d-1}))\sup_{|x-y|_\infty<\frac{l_L}{L}}|g(x)-g(y)|\xrightarrow{L\to\infty}0,$$

where the last convergence is because of the uniform continuity of g. The above convergence together with (2.24) and (2.16) completes the proof. □

Using $\eta^\omega_{p,\lambda}$ one can note that any limit point of $\xi^\omega_{L,\lambda}$ is a limit point of the point process defined by

$$\eta^\omega_{L,\lambda}:=\sum_p \eta^\omega_{p,\lambda}. \tag{2.25}$$

COROLLARY 2.5

For bounded interval $I\subset\mathbb{R}$ and bounded Borel set $Q\subset\mathbb{R}^d$, we have

$$\mathbb{E}(|\xi^\omega_{L,\lambda}(I\times Q)-\eta^\omega_{L,\lambda}(I\times Q)|)\to 0\quad \text{as}\quad L\to\infty. \tag{2.26}$$

Proof. The function $f=\chi_I(E)\chi_Q(x)$ can be approximated using compactly supported continuous function, so using equation (2.13) we have the result. □

Lemma 2.6. *Given $D^\alpha_\nu(\lambda)>0$ and a symmetric bounded interval $I\subset\mathbb{R}$, there exists a sequence $\{L_n\}_n$ such that*

$$\lim_{n\to\infty}L_n^d\nu(\lambda+\beta_{L_n}^{-1}I)=|I|^\alpha D^\alpha_\nu(\lambda). \tag{2.27}$$

Proof. We have

$$0<D^\alpha_\nu(\lambda)=\overline{\lim_{\epsilon\to 0}}\frac{\nu(\lambda-\epsilon,\lambda+\epsilon)}{(2\epsilon)^\alpha}<\infty.$$

Choose $\beta_{L+1}^{-1}<\epsilon\leq\beta_L^{-1}$, then for the interval $I=[-c,c]$ (for $c>0$) we have

$$\lambda+\epsilon I\subseteq\lambda+\beta_L^{-1}I\ \Rightarrow\ \nu(\lambda+\epsilon I)\leq\nu(\lambda+\beta_L^{-1}I).$$

Using $\beta^\alpha_{L+1}\epsilon^\alpha\geq 1$,

$$\frac{\beta_L^\alpha\nu(\lambda+\beta_L^{-1}I)}{|I|^\alpha}\geq\left(\frac{\beta_L}{\beta_{L+1}}\right)^\alpha\frac{\nu(\lambda+\epsilon I)}{(\epsilon|I|)^\alpha}$$
$$=\left(\frac{\beta_L}{\beta_{L+1}}\right)^\alpha\frac{\nu(\lambda-c\epsilon,\lambda+c\epsilon)}{(\epsilon|I|)^\alpha}, \tag{2.28}$$

From above, we get

$$\sup_{L\geq M}\frac{\beta_L^{\alpha}\nu(\lambda+\beta_L^{-1}I)}{|I|^{\alpha}} \geq \left(\frac{1}{1+\frac{1}{M}}\right)^d \sup_{\epsilon\in(\beta_{L+1}^{-1},\beta_L^{-1}],L\geq M}\frac{\nu(\lambda+\epsilon I)}{(\epsilon|I|)^{\alpha}}$$

$$\geq \left(\frac{1}{1+\frac{1}{M}}\right)^d \sup_{\epsilon\in(0,\beta_M^{-1}]}\frac{\nu(\lambda+\epsilon I)}{(\epsilon|I|)^{\alpha}}. \tag{2.29}$$

Here we used the fact that

$$\bigcup_{L\geq M}(\beta_{L+1}^{-1},\beta_L^{-1}]=(0,\beta_M^{-1}] \quad \text{and} \quad \left(\frac{\beta_L}{\beta_{L+1}}\right)^{\alpha}\geq\left(\frac{1}{1+\frac{1}{M}}\right)^d, \text{ for } L\geq M.$$

Taking limit $M\to\infty$ in (2.29) and using the definition of limsup, we get

$$\overline{\lim_{L\to\infty}}\frac{\beta_L^{\alpha}\nu(\lambda+\beta_L^{-1}I)}{|I|^{\alpha}}\geq D_{\nu}^{\alpha}(\lambda). \tag{2.30}$$

Similarly starting with $\epsilon\in(\beta_{L+1}^{-1},\beta_L^{-1}]$ we get the inequality

$$\frac{\beta_{L+1}^{\alpha}\nu(E+\beta_{L+1}^{-1}I)}{|I|^{\alpha}}\leq\left(\frac{\beta_{L+1}}{\beta_L}\right)^{\alpha}\frac{\nu(\lambda+\epsilon I)}{(\epsilon|I|)^{\alpha}}$$

and proceed as in the above argument, with upper bounds, to get

$$\overline{\lim_{L\to\infty}}\frac{\beta_L^{\alpha}\nu(E+\beta_L^{-1}I)}{|I|^{\alpha}}\leq D_{\nu}^{\alpha}(\lambda). \tag{2.31}$$

Using inequalities (2.30) and (2.31), we get

$$\overline{\lim_{L\to\infty}}\frac{\beta_L^{\alpha}\nu(\lambda+\beta_L^{-1}I)}{|I|^{\alpha}}=D_{\lambda}^{\alpha}(\lambda).$$

Now using the fact $\beta_L=L^{d/\alpha}$, we have

$$\overline{\lim_{L\to\infty}}L^d\nu(\lambda+\beta_L^{-1}I)=|I|^{\alpha}D_{\nu}^{\alpha}(\lambda). \tag{2.32}$$

The above equation implies that there exists a subsequence $\{L_n\}$ such that

$$\lim_{n\to\infty}L_n^d\nu(\lambda+\beta_{L_n}^{-1}I)=|I|^{\alpha}D_{\nu}^{\alpha}(\lambda).$$

□

3. Proof of Theorem 1.1

We have

$$\mathbb{E}|\mathrm{e}^{it\xi_{L,\lambda}^{\omega}(I\times Q)}-\mathrm{e}^{it\eta_{L,\lambda}^{\omega}(I\times Q)}|\leq|t|\mathbb{E}(|\xi_{L,\lambda}^{\omega}(I\times Q)-\eta_{L,\lambda}^{\omega}(I\times Q)|). \tag{3.1}$$

We are using the following fact

$$|\mathrm{e}^{itx}-\mathrm{e}^{ity}|^2 = 2(1-\cos t(x-y)) = 4\sin^2\frac{t(x-y)}{2} \le |t(x-y)|^2.$$

From (2.25) and (2.5) we have

$$\begin{aligned} \mathbb{E}[\mathrm{e}^{it\eta^\omega_{L,\lambda}(I\times Q)}] &= \mathbb{E}[\mathrm{e}^{it\sum_{p\in\Gamma_{L,Q}}\eta^\omega_{p,\lambda}(I\times Q)}] \\ &= \mathbb{E}[\mathrm{e}^{it\eta^\omega_{p,\lambda}(I\times Q)}]^{|\Gamma_{L,Q}|}. \end{aligned} \tag{3.2}$$

By definition of $\eta^\omega_{p,\lambda}$, we have

$$\begin{aligned} \mathbb{E}[\mathrm{e}^{it\eta^\omega_{p,\lambda}(I\times Q)}] &= \sum_{k=0}^\infty \mathrm{e}^{itk}\mathbb{P}(\eta^\omega_{p,\lambda}(I\times Q)=k) \\ &= 1+\mathbb{E}[\eta^\omega_{p,\lambda}(I\times Q)](\mathrm{e}^{it}-1)+R_L, \end{aligned} \tag{3.3}$$

where R_L is given by

$$\begin{aligned} R_L &= \sum_{k=0}^\infty \mathrm{e}^{itk}\mathbb{P}(\eta^\omega_{p,\lambda}(I\times Q)=k) - 1 - \mathbb{E}[\eta^\omega_{p,\lambda}(I\times Q)](\mathrm{e}^{it}-1) \\ &= \sum_{k=0}^\infty \mathrm{e}^{itk}\mathbb{P}(\eta^\omega_{p,\lambda}(I\times Q)=k) - \sum_{k=0}^\infty \mathbb{P}(\eta^\omega_{p,\lambda}(I\times Q)=k) \\ &\quad -(\mathrm{e}^{it}-1)\sum_{k=0}^\infty k\mathbb{P}(\eta^\omega_{p,\lambda}(I\times Q)=k) \\ &= \sum_{k=2}^\infty (\mathrm{e}^{itk}-k\mathrm{e}^{it}+k-1)\mathbb{P}(\eta^\omega_{p,\lambda}(I\times Q)=k). \end{aligned} \tag{3.4}$$

Set $I_{L,\lambda}=\lambda+\beta_L^{-1}I$ and using $|\mathrm{e}^{itk}-k\mathrm{e}^{it}+k-1|\le 2k$ for $k\ge 2$, we get

$$\begin{aligned} |R_L| &\le \sum_{k=2}^\infty |\mathrm{e}^{itk}-k\mathrm{e}^{it}+k-1|\mathbb{P}(\eta^\omega_{p,\lambda}(I\times Q)=k) \\ &= 2\sum_{k=2}^\infty k\mathbb{P}(\eta^\omega_{p,\lambda}(I\times Q)=k) \\ &\le 2\sum_{k=2}^\infty k(k-1)\mathbb{P}(\eta^\omega_{p,\lambda}(I\times Q)=k) \\ &= 2\mathbb{E}[(\eta^\omega_{p,\lambda}(I\times Q)(\eta^\omega_{p,\lambda}(I\times Q)-1)] \\ &\le 2\mathbb{E}[(\eta^\omega_{p,\lambda}(I\times \mathbb{R}^d)(\eta^\omega_{p,\lambda}(I\times \mathbb{R}^d)-1)] \\ &= 2\mathbb{E}[\mathrm{Tr}(E_{H^\omega_{B_p(L)}}(I_{L,\lambda}))(\mathrm{Tr}(E_{H^\omega_{B_p(L)}}(I_{L,\lambda}))-1)] \\ &\le 2(Q_\mu(|I_{L,\lambda}|)|B_p(L)|)^2 \quad \text{(using 2.10)} \\ &\le 2(|I_{L,\lambda}|^\alpha l_L^d)^2 \\ &= O(L^{-2d}l_L^{2d}). \end{aligned} \tag{3.5}$$

Using $|\Gamma_{L,Q}| \simeq O\left(\left(\frac{L}{l_L}\right)^d\right)$ (see (2.6)) in the above equation, we get

$$|\Gamma_{L,Q}||R_L| \leq O\left(\frac{l_L^d}{L^d}\right) \to 0 \quad \text{as } L \to \infty. \tag{3.6}$$

Combining the above equation with (3.3), (3.2) and (3.1) will give

$$\begin{aligned}\lim_{L\to\infty} \mathbb{E}[\mathrm{e}^{it\xi^\omega_{L,\lambda}(I\times Q)}] &= \lim_{L\to\infty} \mathbb{E}[\mathrm{e}^{it\eta^\omega_{L,\lambda}(I\times Q)}] \\ &= \lim_{L\to\infty}\left(1+\frac{|\Gamma_{L,Q}|[\mathbb{E}(\eta^\omega_{p,\lambda}(I\times Q))(\mathrm{e}^{it}-1)+R_L]}{|\Gamma_{L,Q}|}\right)^{|\Gamma_{L,Q}|} \\ &= \lim_{L\to\infty}\left(1+\frac{|\Gamma_{L,Q}|\mathbb{E}(\eta^\omega_{p,\lambda}(I\times Q))(\mathrm{e}^{it}-1)}{|\Gamma_{L,Q}|}\right)^{|\Gamma_{L,Q}|}. \end{aligned} \tag{3.7}$$

To compute the limit, we use the subsequence so that (2.27) holds. Using that subsequence, we get

$$\begin{aligned}\lim_{n\to\infty} |\Gamma_{L_n,Q}|\mathbb{E}(\eta^\omega_{p,\lambda}(I\times Q)) &= \lim_{n\to\infty}\sum_{p\in\Gamma_{L_n,Q}} \mathbb{E}(\eta^\omega_{p,\lambda}(I\times Q)) \\ &= \lim_{n\to\infty} \mathbb{E}(\eta^\omega_{L_n,\lambda}(I\times Q) \quad \text{(using (2.25))} \\ &= \lim_{n\to\infty} \mathbb{E}(\xi^\omega_{L_n,\lambda}(I\times Q) \quad \text{(using 2.26)} \\ &= \lim_{n\to\infty}\sum_{n\in L_nQ} \mathbb{E}(\langle\delta_n, E_{H^\omega}(\lambda+\beta_{L_n}^{-1}I)\delta_n\rangle) \\ &= |Q| \lim_{L_n\to\infty} L_n^d\nu(\lambda+\beta_{L_n}^{-1}I) \\ &= |I|^\alpha D_\nu^\alpha(\lambda)|Q| \qquad \text{(using 2.27).}\end{aligned} \tag{3.8}$$

Using the above equation in (3.7) together with the fact $\left(1+\frac{z_n}{n}\right)^n \to \mathrm{e}^z$, whenever $z_n \to z$ as $n \to \infty$ gives

$$\mathbb{E}[\mathrm{e}^{it\xi^\omega_{L_n,\lambda}(I\times Q)}] \xrightarrow{n\to\infty} \mathrm{e}^{|I|^\alpha D_\nu^\alpha(\lambda)|Q|(\mathrm{e}^{it}-1)}$$

which shows that $\{\xi^\omega_{L_n,\lambda}(I\times Q)\}$ converges in distribution to a Poisson random variable with parameter $|I|^\alpha D_\nu^\alpha(\lambda)|Q|$.

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