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Unitary representations of the fundamental group of orbifolds

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Abstract. Let X be a smooth complex projective variety of dimension n and $\mathcal{L}$ an ample line bundle on it. There is a well known bijective correspondence between the isomorphism classes of polystable vector bundles E on X with $c_1(E) = 0 = c_2(E) \cdot c_1(\mathcal{L})^{n-2}$ and the equivalence classes of unitary representations of $\pi_1(X)$. We show that this bijective correspondence extends to smooth orbifolds.

Keywords. Orbifolds; polystable vector bundle; fundamental group; unitary representation

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1. Introduction

Let $Y/\mathbb{C}$ be a connected smooth projective curve, and let $\mathcal{E} \longrightarrow Y$ be a polystable vector bundle of degree zero. A theorem of Narasimhan and Seshadri, [12], says that $\mathcal{E}$ is necessarily given by a unitary representation of the fundamental group of Y. Let $X/\mathbb{C}$ be a smooth projective variety of dimension n. Fix an ample line bundle $\mathcal{L}$ on X in order to define the degree of torsionfree coherent sheaves on X. The following generalization of the Narasimhan–Seshadri theorem holds.

Theorem 1.1 [3, 11]. *Let $\mathcal{E}$ be a vector bundle on X such that $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) \cdot c_1(\mathcal{L})^{n-2} = 0$. Then $\mathcal{E}$ is polystable if and only if it is given by a unitary representation of the* (*topological*) *fundamental group of X.*

Some clarifications on Theorem 1.1 are necessary.

(1) Polystability refers to Mumford–Takemoto (slope) polystability.
(2) All Chern classes in Theorem 1.1 are topological, taking values in rational cohomological classes.
(3) By p. 231, Proposition 1 of [3], a stable vector bundle on X admits a Hermitian–Yang–Mills connection. Since a polystable vector bundle $\mathcal{E}$ on X is a direct sum of stable vector bundles of same degree/rank quotient, the Hermitian–Yang–Mills connection on the stable direct summands together produce an Hermitian–Yang–Mills connection on $\mathcal{E}$. If we have $c_1(\mathcal{E}) = 0 = c_2(\mathcal{E}) \cdot c_1(\mathcal{L})^{n-2}$, then any Hermitian–Yang–Mills connection on $\mathcal{E}$ is flat (p. 115, Theorem 4.11 of [6]). Therefore, if $\mathcal{E}$ is polystable, and $c_1(\mathcal{E}) = 0 = c_2(\mathcal{E}) \cdot c_1(\mathcal{L})^{n-2}$, then $c_i(\mathcal{E}) = 0$ for all $i \geq 1$.

Our aim here is to generalize the above theorem to projective orbifolds. By an *orbifold* over a field k we always mean a smooth separated Deligne–Mumford stack which is of finite type over k and whose isotropy groups at all generic points are trivial. Thus an orbifold has a dense open substack which is a scheme. We call a Deligne–Mumford stack projective if the coarse moduli space for it is a projective variety.

Stability or polystability would always mean Mumford–Takemoto (slope) stability or polystability.

Theorem 1.2. *Let $X/\mathbb{C}$ be an irreducible projective orbifold of dimension n, and let $\mathcal{L}$ be an ample line bundle on X. Let $\mathcal{E} \longrightarrow X$ be a vector bundle on X such that $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) \cdot c_1(\mathcal{L})^{n-2} = 0$. Then $\mathcal{E}$ is polystable (with respect to $\mathcal{L}$) if and only if it is obtained by a unitary representation of $\pi_1^{\text{top}}(X, x)$ for a (equivalently, any) closed point $x \in X$.*

Here $\pi_1^{\text{top}}(X, x)$ denotes the topological fundamental group of the underlying complex analytical stack. See Definition 4.1 for ample line bundles on a stack. All Chern classes in Theorem 1.2 are topological, taking values in rational cohomological classes.

Theorem 1.2 follows quite easily from Theorem 1.1 in the special case when X is a global quotient of a smooth variety by a finite group. The main work needed to prove Theorem 1.2 involves deducing the general case from this special case.

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2. Rigidification

Let k be a field. Throughout this section, we work with Deligne–Mumford stacks over k, and we assume that all these stacks are of finite type and separated over k. An algebraic stack over k is called a *quotient stack* if it can be expressed as the quotient of an affine scheme by an action of a linear algebraic group. As shown in Proposition 5.2 of [8], Zariski locally, such stacks can be expressed as a quotient of an affine variety by a finite group.

We make a blanket assumption that all stacks considered here are quotient stacks.

The inertia stack of a stack Y, viewed as a sheaf of groups on the big étale site of Y, will be denoted by I_Y.

Recall that the rigidification of a Deligne–Mumford stack is a process to get rid of the given stabilizers in a 'minimal' manner (see section 5.1 of [1]). Throughout this paper, we fix the following definition of rigidification. It should be clarified that this definition of rigidification differs from the one in section 5.1 of [1].

DEFINITION 2.1

Let $f : Y \longrightarrow X$ be a 1-morphism of Deligne–Mumford stacks over k. This f is called a *rigidification* if for any atlas $U \longrightarrow X$, where U is a scheme, the coarse moduli space for $Y \times_X U$ is U, and the natural projection $Y \times_X U \longrightarrow U$ coincides with the morphism to the coarse moduli space.

Remark 2.2. In the above definition, the condition holds for one atlas $U \longrightarrow X$ if and only if it holds for all atlases. To see this, suppose for one atlas $U \longrightarrow X$ the projection $Y \times_X U \longrightarrow U$ gives the coarse moduli space for $Y \times_X U$. Let $V \longrightarrow X$ be any other atlas. By Lemma 2.3, the projection

$$Y \times_X (U \times_X V) \longrightarrow U \times_X V$$

gives the coarse moduli space. Note that the projections

$$U \times_X V \longrightarrow V \quad \text{and} \quad U \times_X V \longrightarrow U$$

are étale covers. Thus, using Lemma 2.3 once again, we see that the projection $Y \times_X V \longrightarrow V$ gives the coarse moduli space for $Y \times_X V$, because its base change by $U \times_X V \longrightarrow V$ gives the coarse moduli space.

Since the morphism to the coarse moduli space of a separated Deligne–Mumford stack is always proper, it follows that a rigidification morphism is a proper morphism.

A direct consequence of Definition 2.1 is that the canonical morphism from a Deligne–Mumford stack to its coarse moduli space is a rigidification. It may be mentioned that Proposition 2.8 provides additional examples of rigidification.

Lemma 2.3. *Let $\mathcal{X}$ be a Deligne–Mumford stack over k, and let $q : \mathcal{X} \longrightarrow V$ be a morphism to a scheme V/k. Let $W \longrightarrow V$ be an étale cover* (*i.e.*, *surjective étale map*). *Then q gives a coarse moduli space for $\mathcal{X}$ if and only if the projection $\mathcal{X} \times_V W \longrightarrow W$ gives a coarse moduli space for $\mathcal{X} \times_V W$.*

Proof. Let $f : \mathcal{X} \longrightarrow X$ be the map to the coarse moduli space of $\mathcal{X}$. Since V is a scheme, by the definition of a coarse moduli space, the map $\mathcal{X} \longrightarrow V$ factors as

$$\mathcal{X} \stackrel{f}{\longrightarrow} X \longrightarrow V\,.$$

Taking fiber product with $W \longrightarrow V$, we also have

$$\mathcal{X} \times_V W \stackrel{f \times \mathrm{Id}_W}{\longrightarrow} X \times_V W \longrightarrow W\,.$$

Since the formation of a coarse moduli space commutes with étale base change, the above map

$$\mathcal{X} \times_V W \stackrel{f \times \mathrm{Id}_W}{\longrightarrow} X \times_V W$$

is a coarse moduli space. The lemma follows from the fact that $X \longrightarrow V$ is an isomorphism if and only if the above projection $X \times_V W \longrightarrow W$ is an isomorphism. □

Lemma 2.4. *Let $g : Y_1 \longrightarrow Y_2$ be a* 1*-morphism of Deligne–Mumford stacks over k. Then for $i = 1, 2$, there exists*

(*i*) *affine schemes U_i and linear algebraic groups G_i/k acting on U_i,*
(*ii*) *a group homomorphism $\phi : G_1 \longrightarrow G_2$, and*
(*iii*) *a k-morphism $f : U_1 \longrightarrow U_2$ which is equivariant with respect to ϕ,*

such that $Y_i = [U_i/G_i]$, and the following diagram is commutative:

$$\begin{array}{ccc} U_1 & \stackrel{f}{\longrightarrow} & U_2 \\ \downarrow & & \downarrow \\ Y_1 = [U_1/G_1] & \longrightarrow & [U_2/G_2] = Y_2 \end{array}\,.$$

Proof. Take any $i \in \{1, 2\}$. Since Y_i is a quotient stack, we can write $Y_i = [V_i/H_i]$, where V_i is an affine scheme and H_i is a linear algebraic group acting on V_i. Now we let

$$\tilde{V}_1 = V_1 \times_{Y_2} V_2 = V_1 \times_{Y_1} (V_2 \times_{Y_2} Y_1) .$$

Then clearly

$$[\tilde{V}_1/(H_1 \times H_2)] = [V_1/H_1] .$$

The projection

$$\tilde{V}_1 \longrightarrow V_1$$

is affine, because it is the base extension of the affine morphism $V_2 \longrightarrow Y_2$.

Since V_1 is an affine variety, and the projection $\tilde{V}_1 \longrightarrow V_1$ is affine, we conclude that $\tilde{V}_1$ is also an affine variety. Furthermore, we have a natural map

$$h : \tilde{V}_1 \longrightarrow V_2$$

which is equivariant for the actions of $H_1 \times H_2$ on $\tilde{V}_1$ and V_2, with $H_1 \times H_2$ acting on V_2 through the projection $H_1 \times H_2 \longrightarrow H_2$. Now, in view of the identification $[\tilde{V}_1/(H_1 \times H_2)] = [V_1/H_1]$, the proof is completed by setting $G_1 = H_1 \times H_2$, $G_2 = H_2$, $U_1 = \tilde{V}_1$ and $U_2 = V_2$. □

The following lemma is a generalization of the universal property of coarse moduli spaces.

Lemma 2.5. *Let $g : Y_1 \longrightarrow Y_2$ be as in Lemma* 2.4. *Assume that for every point p, the homomorphism induced by f from the isotropy group of Y_1 at p to that of Y_2 at $f(p)$ is trivial. Then f factors through the coarse moduli space of Y_1.*

Proof. Let U_i, G_i, ϕ, f be as in Lemma 2.4. Let $K = \text{kernel}(\phi)$. The assumption that the homomorphism induced by f on all isotropy groups is trivial is equivalent to the statement that the stabilizer of every point in U_1 is contained in K.

We recall that for G acting on a scheme U, the geometric quotient $U//G$ is defined as the coarse moduli space of the quotient stack $[U/G]$, and hence $U//G$ is an algebraic space in general.

We now claim that the geometric quotient $V = U_1//K$ exists as an algebraic space. In order to prove this, first consider the stack $[U_1/K]$. It is a Deligne–Mumford stack, because the isotropy groups of the action of G_1 (and hence of K) on U_1 are finite étale over k. Thus V, which is nothing but the coarse moduli space of $[U_1/K]$, exists as an algebraic space by a theorem of Keel and Mori (Theorem 1.1 of [5]). This proves the above claim.

Next, we will show that the action of G_1/K on V is free. To prove this, take $\tilde{x} \in V$ and $\bar{g} \in G_1/K$ such that $\bar{g}(\tilde{x}) = \tilde{x}$. Choose a point $x \in U_1$ lying over $\tilde{x}$, and also fix a lift $g \in G_1$ of $\bar{g}$. Since V is a geometric quotient of U_1 by K, points of V correspond to orbits in U_1 for the action of K. Thus, there exists an element $a \in K$ such that

$$gx = ax .$$

Alternatively, $a^{-1}g$ is in the stabilizer of the point x and hence it is contained in K. Thus $g \in K$, which implies that $\bar{g} = eK$. This proves the above assertion that the action of G_1/K on V is free.

Consequently, the quotient stack $[V/(G_1/K)]$ is actually an algebraic space and hence is its own coarse moduli space. Therefore, $[V/(G_1/K)]$ coincides with the geometric quotient $V//(G_1/K)$. We claim that $[V/(G_1/K)]$ coincides with the geometric quotient $U_1//G_1$. To prove this, we need to show that the natural map

$$U_1//G_1 \longrightarrow (U_1//K)//(G_1/K)$$

is an isomorphism. But this follows from Lemma 2.6, and the claim is proved.

Since the map $f : U_1 \longrightarrow U_2$ intertwines the given action of K on U_1 and the trivial action of K on U_2, it factors through V. Thus we have an induced map $V \longrightarrow U_2$ which is equivariant with respect to the action of G_1/K on V and the action of G_2 on U_2, using the homomorphism ϕ in Lemma 2.4. This in turn induces a map $[V/(G_1/K)] \longrightarrow [U_2/G_2]$. But as explained above, $[V/(G_1/K)]$ is the coarse moduli space of Y_1. This finishes the proof. □

Lemma 2.6. *Let G be a linear algebraic group acting on U and K be a closed normal subgroup of G. Then there is a canonical isomorphism*

$$U//G \longrightarrow (U//K)//(G/K)\,.$$

Proof. By virtue of being a coarse moduli space, the map $U \xrightarrow{q} U//G$ to this geometric quotient has the following universal property: any map $U \longrightarrow T$ which is G invariant (i.e., G equivariant with respect to the trivial action of G on T) factors uniquely through q. Since the natural map

$$\alpha : U \longrightarrow (U//K)//(G/K)$$

is G invariant, we have an induced map

$$\psi : U//G \longrightarrow (U//K)//(G/K)\,.$$

Conversely, since the map $U \longrightarrow U//G$ is G-invariant, it is also K-invariant. Hence we have an induced map

$$U//K \longrightarrow U//G$$

which is easily seen to be G/K invariant. Thus we have an induced map

$$\phi : (U//K)//(G/K) \longrightarrow U//G\,.$$

In order to check $\phi \circ \psi : U//G \longrightarrow U//G$ is the identity map, we note that the following diagram commutes:

$$\begin{array}{ccc} & U & \\ {}^{q}\swarrow & & \searrow^{q} \\ U//G & \xrightarrow{\phi\circ\psi} & U//G \end{array}\,.$$

But by universal property of q, there can only be one map $U//G \longrightarrow U//G$ which makes the above diagram commutative. Thus $\phi \circ \psi$ is necessarily the identity map on $U//G$. A similar argument shows that the other composite $\psi \circ \phi$ is the identity map of $(U//K)//(G/K)$. □

Lemma 2.7. *Let $f : Y \longrightarrow X$ be a 1-morphism of separated Deligne–Mumford stacks over k. Let Σ denote the set of all points in Y where the induced map on isotropy groups is an isomorphism. Then Σ is an open subset of Y.*

Proof. Since the Deligne–Mumford stacks are separated, the inertia stacks I_Y and I_X are finite over Y and X respectively. Thus the kernel and co-kernel of the induced map $I_Y \longrightarrow f^{-1}(I_X)$ are finite over X. Hence the set of all points where both the kernel and co-kernel are trivial is an open subset of Y. □

PROPOSITION 2.8

Let U/k be a k-scheme, and let G be a finite group acting on U. Let K be a normal subgroup of G, and let V denote the geometric quotient of U by K. Then the following two assertions hold:

(*i*) *The natural map $f : [U/G] \longrightarrow [V/(G/K)]$ is a rigidification.*
(*ii*) *If U is the spectrum of a local ring, then any rigidification of $[U/G]$ is of this type for some normal subgroup K of G.*

Proof. We will first prove (i). Since the quotient map $V \longrightarrow [V/(G/K)]$ is étale, and $[U/K] \longrightarrow V$ gives the coarse moduli space, in order to prove (i) it suffices to show that the following diagram is Cartesian:

$$\begin{array}{ccc} [U/K] & \xrightarrow{f'} & V \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ [U/G] & \xrightarrow{f} & [V/(G/K)] \end{array} . \tag{1}$$

We first recall the description of the 1-morphism f. The k-groupoid underlying the stack $[U/G]$ is the category of all triples of the form

$$\big(Z\,,\, P \xrightarrow{h} Z\,,\, P \xrightarrow{\phi} U\big),$$

where Z is a k scheme, h defines a principal G-bundle on Z and ϕ is a G-equivariant morphism. Given such a triple, one has the corresponding induced map

$$\phi' : P' = P/K \longrightarrow V,$$

where $P/K \longrightarrow Z$ is now a principal G/K-bundle. One thus gets a triple $(Z\,,\, P' \xrightarrow{h'} Z\,,\, \phi')$ which defines an object in the groupoid underlying $[V/(G/K)]$. This assignment

$$(Z, P, \phi) \longrightarrow (Z\,,\, P'\,,\, \phi')$$

defines the 1-morphism f.

To prove that the diagram (1) is Cartesian, we need to exhibit a 1-isomorphism

$$[U/K] \longrightarrow [U/G] \times_{[V/(G/K)]} V.$$

By the construction of fibre product of groupoids, the groupoid underlying

$$[U/G] \times_{[V/(G/K)]} V$$

is the category of all

$$(Z, P \xrightarrow{h} Z, \phi, Z \xrightarrow{\gamma} V, \theta), \tag{2}$$

where $(Z, P \xrightarrow{h} Z, \phi)$ is an object, say α, of the groupoid underlying $[U/G]$, while γ defines an object, say β, of the groupoid underlying V, and θ is an isomorphism $f(\alpha) \longrightarrow g'(\beta)$. Note that by the above description of f, the image $f(\alpha)$ is nothing but the object defined by the triple

$$(Z, P', \phi'),$$

where P' is the principal G/K-bundle $P/K \longrightarrow Z$ and $\phi' : P' \longrightarrow V$ is the induced morphism. Moreover, $g'(\beta)$ is the triple

$$(Z, Z \times (G/K), \tilde{\gamma}),$$

where $Z \times G/K \longrightarrow Z$ is the trivial principal G/K-bundle, and

$$\gamma : Z \times G/K \longrightarrow V, \quad (z, g) \longmapsto g \cdot \gamma(z)$$

is the G/K equivariant map.

Thus, giving the isomorphism

$$\theta : (Z, P', \phi') \longrightarrow (Z, Z \times (G/K), \tilde{\gamma})$$

in (2) is equivalent to giving a trivialization (or a section) of $P' \longrightarrow Z$ and imposing the condition that the induced map ϕ' coincides with $\tilde{\gamma}$. Note that $\gamma : Z \longrightarrow V$ could have been any arbitrary k-morphism, to start with. Consequently, the category underlying the groupoid $[U/G] \times_{[V/(G/K)]} V$ coincides with the category of all quadruples

$$(Z, P \xrightarrow{h} Z, \phi, s),$$

where Z, P and ϕ are as above, and s is a trivialization of the principal G/K-bundle $h' : P/K \longrightarrow Z$. But giving a trivialization of the principal G/K-bundle h' is equivalent to giving a reduction of the structure group of h to K. Therefore, the above category of quadruples is equivalent to the category of all triples

$$(Z, Q \xrightarrow{q} Z, \psi),$$

where $q : Q \longrightarrow Z$ is a principal K-bundle and $\psi : Q \longrightarrow U$ is a K-equivariant morphism. One now sees that this is also the groupoid underlying $[U/K]$. This completes the proof of (i).

To prove (ii), let U be a spectrum of a local ring, and let G be a finite group acting on U. Define $Y := [U/G]$. Let

$$f : Y \longrightarrow X$$

be any rigidification map. Let K be the kernel of the homomorphism $I_p \longrightarrow I_{f(p)}$, where I_p and $I_{f(p)}$ are the isotropy groups for the unique closed point p of Y and $f(p)$ of X respectively. Consider the G-invariant map

$$g : U \longrightarrow X$$

induced by f. Let V be the geometric quotient of U by K.

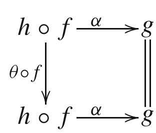
We will first show that g factors through V. Indeed, the induced map from the stack, namely $[U/K] \longrightarrow X$, is trivial on all isotropy groups. Hence by Lemma 2.5, the map $[U/K] \longrightarrow X$ factors through the coarse moduli space, which is V.

Since g is G-invariant, the induced map $V \longrightarrow X$ is G/K invariant, and hence it produces a map

$$[V/(G/K)] \longrightarrow X\,. \tag{3}$$

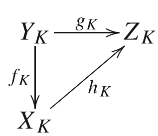
Clearly, this map is also a rigidification, but it is also an isomorphism on the isotropy group at the closed point of the domain. Hence by Lemma 2.7, the map in (3) is an isomorphism on the isotropy group at all points. Therefore, the map in (3) must be an isomorphism. □

Lemma 2.9. *Let $f : Y \longrightarrow X$ be a rigidification, and let $g : Y \longrightarrow Z$ be any* 1-*morphism. Suppose g factors through f, meaning there exists a* 1-*morphism $h : X \longrightarrow Z$ and a* 2-*isomorphism $\alpha : h \circ f \Rightarrow g$. Then the pair $(h\,,\alpha)$ does not have any* 2-*automorphisms*, *i.e.*, *if $\theta : h \Rightarrow h$ makes the diagram*



commutative, *then θ is identity.*

Proof. For an algebraically closed field K over k, let X_K denote the category of K-points of X. Thus f, g and h induce functors



and a natural equivalence $\alpha_K : h_K \circ f_K \Rightarrow g_K$. In order to prove that θ is identity, it is enough to show that θ_K is identity for every algebraically closed field K over k. However, from the definition of rigidification it follows that the functor f_K is full and essentially surjective. It is then a simple exercise in category theory to show that θ_K is identity. □

Let $f : Y \longrightarrow X$ be a rigidification. Let I_Y (respectively, I_X) be the inertia sheaf of Y (respectively, X). Define

$$K^f := \ker(I_Y \longrightarrow f^{-1}(I_X))\,.$$

PROPOSITION 2.10

Given any 1-*morphism $g : Y \longrightarrow Z$ such that K^f is contained in the kernel of $I_Y \longrightarrow g^{-1}(I_Z)$*, *there exists a morphism $h : X \longrightarrow Z$*, *and a* 2-*isomorphism $\alpha : h \circ f \Rightarrow g$. Further*, *$(h\,,\alpha)$ is uniquely determined up to a unique* 2-*isomorphism* (*see* (2.9)).

Proof. Uniqueness of the 2-isomorphism follows from Lemma 2.9. In view of the uniqueness, it is enough to prove that the proposition is Zariski locally around any point $p \in Y$. Thus by Lemma 2.4, we may reduce to the case where $Y = [U_1/G_1]$, $Z = [U_2/G_2]$, and g is induced by a map $h : U_1 \longrightarrow U_2$ which is equivariant with respect to a given group homomorphism $\phi : G_1 \longrightarrow G_2$. Moreover, we may assume that G_1 is precisely the isotropy group at the point p, and $K^f \subset \ker(\phi)$ is a normal subgroup of G_1.

In view of Proposition 2.8(ii), by further taking a smaller Zariski neighborhood of p, we may assume that the rigidification $f : Y \longrightarrow X$ is of the form $[V/(G_1/K^f)]$, where V is the geometric quotient of U_1 by K^f. Note that although Proposition 2.8(ii) is stated in the case where U_1 is the spectrum of a local ring, we use the standard limiting argument to draw a conclusion about a small enough Zariski neighborhood of U_1 (and hence also of Y). Since $K^f \subset \ker(\phi)$, it follows that h induces a map $V \longrightarrow U_2$ which is equivariant with respect to the actions of G_1/K^f and G_2 on V and U_2 respectively. Thus it induces a unique map $[V/(G_1/K^f)] \longrightarrow Z$. This proves the proposition. □

We have the following corollary of Proposition 2.10.

COROLLARY 2.11

A rigidification $f : Y \longrightarrow X$ is determined uniquely by

$$\ker(I_Y \longrightarrow f^{-1}(I_X)) .$$

3. Rigidification of complex Deligne–Mumford stacks

We continue using the notation and assumptions of §2, but now restrict ourselves to the case $k = \mathbb{C}$. Although our main interest is algebraic stacks, we will have to consider complex analytic Deligne–Mumford stacks over $\mathbb{C}$ which arise as infinite covering spaces of algebraic stacks. These complex analytic stacks will have the property of being, locally in the complex analytic topology, quotient of an analytic variety by a finite group. In this section, we only consider complex analytic stacks which have this additional property. We have the analogous definition and results for complex analytic Deligne–Mumford stacks.

DEFINITION 3.1

Let Y be a complex analytic Deligne–Mumford stack. Then a rigidification of Y is a 1-morphism $f : Y \longrightarrow X$ which, locally on X in the analytic topology, is a coarse moduli space.

The proof of the following universal property is similar to that of Proposition 2.10, because of the assumption that all complex analytic stacks considered here are locally quotient spaces for finite group actions.

PROPOSITION 3.2

Let $f : Y \longrightarrow X$ be a rigidification of a complex analytic stack Y. Let K^f be the kernel of the homomorphism $I_Y \longrightarrow f^{-1}(I_X)$. Then given any 1-morphism $g : Y \longrightarrow Z$ such that K^f is in the kernel of

$$I_Y \longrightarrow g^{-1}(I_Z) ,$$

the 1-morphism g factors uniquely (up to a 2-morphism) through f. In particular, the rigidification f is itself uniquely determined by the sheaf K^f.

Lemma 3.3. *Let Y be a Deligne–Mumford stack over $\mathbb{C}$, and let $h : Z \longrightarrow Y$ be a (possibly infinite) covering map. Let $f : Y \longrightarrow X$ be a rigidification. Define*

$$K^f := \ker(I_Y \longrightarrow f^{-1}(I_X)) .$$

Assume that $h^{-1}(K^f)$ lies in the image of the homomorphism $I_Z \longrightarrow h^{-1}(I_Y)$. Then there exists a unique (up to isomorphism) covering map $X' \longrightarrow X$ such that $Z \cong X' \times_X Y$.

Proof. Since h is representable, the corresponding homomorphism $I_Z \longrightarrow h^{-1}(I_Y)$ is injective. We will show that there exists a rigidification $g : Z \longrightarrow Z'$ such that

$$h^{-1}(K^f) = \ker(I_Z \longrightarrow g^{-1}(I_{Z'})) .$$

By the uniqueness part in Proposition 3.2, we may construct the rigidification analytically locally on Z. Due to Proposition 2.8, we may assume that $Y = [U/G]$, where G is a finite group acting on an affine variety U, and there exists a normal subgroup $H \subset G$ such that $X = [V/(G/H)]$ with V being the geometric quotient of U by H. Note that since the map h is representable,

$$\tilde{U} = U \times_Y Z \longrightarrow U$$

is representable, and hence $\tilde{U}$ is an analytic space. Moreover, it has an induced action of G. Now it is easy to see that $Z' = [\tilde{V}/(G/H)]$, where $\tilde{V}$ is the geometric quotient of $\tilde{U}$ by H, which satisfies the claim. Finally, $X' = Z'$ satisfies the condition in the lemma. □

Let Y be a complex analytic Deligne–Mumford stack over $\mathbb{C}$, and let y be a $\mathbb{C}$-point of Y. Let $\pi_1^{\text{top}}(Y, y)$ denote the topological fundamental group of Y. For our purpose, this is the fundamental group defined using Galois theory of covering stacks of Y (see Section 3 of [15]). Let G_y be the isotropy group of the stack Y at the point y. The stack $[\text{Spec } (\mathbb{C})/G_y]$ will be denoted by BG_y. It is well known that $[\text{Spec } (\mathbb{C})/G_y]$ is the classifying space of the group G_y. Since y is a complex point, the residual gerbe at y is neutral (as $\mathbb{C}$ is algebraically closed!). Thus there is a canonical morphism

$$\eta_y : BG_y := [\text{Spec } (\mathbb{C})/G_y] \longrightarrow Y . \tag{4}$$

This induces a homomorphism between the fundamental groups

$$\pi_1^{\text{top}}(BG_y, p) \longrightarrow \pi_1^{\text{top}}(Y, y) ,$$

where p is the unique (up to a 2-isomorphism) complex point of BG_y. However,

$$\pi_1^{\text{top}}(BG_y, p) = G_y .$$

We thus get a canonical map

$$\phi_y : G_y \longrightarrow \pi_1^{\text{top}}(Y, y)$$

(see also Section 7 of [14]). If y' is any other complex point of Y, then there is a natural class of isomorphisms

$$\pi_1^{\mathrm{top}}(Y, y) \longrightarrow \pi_1^{\mathrm{top}}(Y, y')$$

such that any two of them differ by inner automorphism. Therefore, composing $\phi_{y'}$ with such an isomorphism we get a homomorphism

$$\phi_{yy'} : G_{y'} \longrightarrow \pi_1^{\mathrm{top}}(Y, y)$$

which is unique up to an inner automorphism of $\pi_1^{\mathrm{top}}(Y, y)$.

We now set up some more notations. Let $f : Y \longrightarrow X$ be a rigidification of Deligne–Mumford stacks over $\mathbb{C}$. Consider K^f defined as in the statement of Lemma 3.3. For any $\mathbb{C}$-point p of Y, let $K_p^f \subset G_p$ be the stalk of K^f at p. Fix a point $y \in Y(\mathbb{C})$, and let $x = f(y)$. Let $N^{\mathrm{top}}(f)$ denote the normal subgroup of $\pi_1^{\mathrm{top}}(Y, y)$ generated by all $\phi_{yy'}(K_{y'}^f)$, where y' runs over $\mathbb{C}$-points of Y. Note that since $\phi_{yy'}$ is unique up to conjugation, the subgroup $N^{\mathrm{top}}(f)$ is uniquely defined.

Lemma 3.4. *Let $h : Z \longrightarrow Y$ be a connected* (*not necessarily finite*) *Galois étale cover with Galois group G. Let $z \in Z$ be a $\mathbb{C}$-point which lies over y. Then the following two conditions are equivalent*:

(1) *$N^{\mathrm{top}}(f)$ is contained in the kernel of the homomorphism $\pi_1^{\mathrm{top}}(Y, y) \longrightarrow G$.*
(2) *$h^{-1}(K^f)$ lies in the image of the homomorphism $I_Z \longrightarrow h^{-1}(I_Y)$.*

Proof. Let z' be any $\mathbb{C}$-point of Z, and let $y' = h(z')$. We have a 1-morphism

$$BG_{y'} \longrightarrow Y$$

constructed as in (4). Since $h : Z \longrightarrow Y$ is a Galois étale cover with Galois group G, it follows that

$$Z_{y'} = Z \times_Y BG_{y'} \longrightarrow BG_{y'}$$

is also a Galois étale cover (not necessarily connected). Here, by a non-connected Galois étale cover, we mean a cover such that the group of deck transformations acts transitively on all complex points of the preimage of a complex point. Thus a non-connected Galois étale cover is nothing but a disjoint union of copies of a connected Galois étale cover. Since all connected Galois étale covers of $BG_{y'}$ are of the form BH for a normal subgroup $H \subset G_{y'}$, we see that there exists a unique normal subgroup $H \subset G_{y'}$ such that $Z_{y'}$ is a disjoint union of copies of BH.

(1) $\Longrightarrow$ (2): Statement (1) in the lemma is equivalent to the assertion that $K_{y'}^f$ is contained in the kernel of the composite map

$$G_{y'} = \pi_1^{\mathrm{top}}(BG_{y'}, y') \xrightarrow{\phi_{yy'}} \pi_1^{\mathrm{top}}(Y, y) \longrightarrow G \tag{5}$$

for all such z'. But the kernel of the composite map in (5) is precisely the subgroup H, mentioned above. Thus statement (1) is equivalent to the assertion that $K_{y'}^f \subset H$. Now, since z' was any arbitrary point of Z, to prove (1) $\Longrightarrow$ (2), it is enough to show that the subgroup $H \subset G_{y'}$ is contained in the image of $G_{z'} \to G_{y'}$.

We claim that

$$H = \text{image}\,(G_{z'} \longrightarrow G_{y'}).$$

In order to see this, let $z_1, \ldots, z_r$ be distinct r points of Z lying over y'. Then

$$Z \times_Y BG_{y'} \cong \coprod BG_{z_i}\,.$$

But $Z \times_Y BG_{y'}$ was isomorphic to disjoint union of copies of BH as noted above. This proves the above claim.
(2) $\implies$ (1): Since z' is an arbitrary point of Z, and since

$$H = \text{image}\,(G_{z'} \longrightarrow G_{y'})\,,$$

it follows that statement (2) is equivalent to saying

$$K^f_{y'} \subset H\,.$$

It was already observed above that this is equivalent to statement (1). □

The following theorem is a straightforward generalization of a result in [14].

Theorem 3.5. *Let $f : Y \longrightarrow X$ be a proper morphism of Deligne–Mumford stacks over $\mathbb{C}$. Let y be a $\mathbb{C}$-point of Y, and let $x = f(y)$. Assume that there exists a dense open substack $U \subset X$ such that the codimension of $X\setminus U$ is at least two, and the induced morphism*

$$f|_U : f^{-1}(U) \longrightarrow U$$

is a rigidification. Then the following sequence of groups is exact

$$1 \longrightarrow N^{\text{top}}(f) \longrightarrow \pi_1^{\text{top}}(Y, y) \longrightarrow \pi_1^{\text{top}}(X, x) \longrightarrow 1\,.$$

Proof. Let us first consider the case where f is a rigidification. The kernel of the homomorphism $I_Y \longrightarrow f^{-1}(I_X)$ will be denoted by K^f. Let $Et(Y)$ denote the category of covering spaces of Y. Let $\tilde{\mathcal{C}}$ denote the full subcategory of objects $Z \stackrel{h}{\longrightarrow} Y$ in $Et(Y)$ such that $h^{-1}(K^f)$ is contained in the image of the homomorphism $I_Z \longrightarrow h^{-1}(I_Y)$. In view of the definition of $N^{\text{top}}(f)$ and Lemma 3.4, we need to show that the category $\tilde{\mathcal{C}}$ is equivalent to the category $Et(X)$ of coverings of X.

If $X' \longrightarrow X$ is an étale covering, then it is easy to see that $X' \times_X Y \longrightarrow Y$ is in $\tilde{\mathcal{C}}$.

Conversely, let $Z \longrightarrow Y$ be an étale covering in $\tilde{\mathcal{C}}$. Then by Lemma 3.3, there is an étale covering $X' \longrightarrow X$ such that $Z = X' \times_X Y$. This proves the theorem when f is a rigidification.

Now consider the general case where $f : Y \longrightarrow X$ is a rigidification outside a closed substack of codimension at least two. As above, let $Et(Y)$ denote the category of all coverings of Y, and let $\tilde{\mathcal{C}}$ be the full subcategory consisting of all coverings $Z \longrightarrow Y$ such that $h^{-1}(K^f)$ is contained in the image of the homomorphism $I_Y \longrightarrow f^{-1}(I_X)$. Let $U \subset X$ be the open subset of X whose complement has codimension at least two and such that the map $f|_{f^{-1}(U)} : f^{-1}(U) \longrightarrow U$ is a rigidification. Let $\tilde{\mathcal{C}}_U \subset Et(f^{-1}(U))$

denote the full subcategory consisting of all coverings $Z \longrightarrow f^{-1}(U)$ satisfying the condition that

$$h^{-1}(K|_{f^{-1}(U)}) \subset \text{image}(I_Z \longrightarrow I_{f^{-1}(U)})$$

(just like $\tilde{\mathcal{C}} \subset Et(Y)$).

We need to show that $\tilde{\mathcal{C}}$ is equivalent to $Et(X)$. It is clear that for any covering $X' \longrightarrow X$ in $Et(X)$, the fiber product $X' \times_X Y \longrightarrow Y$ is an object of $\tilde{\mathcal{C}}$. To complete the proof, it suffices to show that any $h : Z \longrightarrow Y$ in $\tilde{\mathcal{C}}$ comes from an object in $Et(X)$.

Let $h_U : Z_U \longrightarrow f^{-1}(U)$ denote the restriction of h to $f^{-1}(U)$. Since $f|_{f^{-1}(U)} : f^{-1}(U) \longrightarrow U$ is a rigidification, by the above special case we know that there exists an object $X'_U \longrightarrow U$ in $Et(U)$ such that $Z_U = X'_U \times_U f^{-1}(U)$. Moreover, since the complement of $U \subset X$ has codimension at least two, the morphism $X'_U \longrightarrow U$ extends to a covering $X' \longrightarrow X$. Now to finish the proof, we observe that the two covering spaces $Z \longrightarrow Y$ and $X' \times_X Y \longrightarrow Y$ agree on the dense open subset $f^{-1}(U)$ and hence are isomorphic. □

Theorem 3.6. *Let $X/\mathbb{C}$ be any quasi-projective orbifold. Then there exists a proper* 1-*morphism $\phi : Y \longrightarrow X$ such that*

(1) *Y is an orbifold which is a finite global quotient, and*
(2) *there exists a dense open subset $V \subset X$ such that $\phi^{-1}(V) \longrightarrow V$ is a rigidification morphism, and the complement $X \setminus V$ has codimension at least two.*

Remark 3.7. Recall that throughout, we assumed that all stacks are quotient stacks. Hence, the stack X in the above theorem is a quotient of a scheme by a linear algebraic group. However, the notion of a quotient stack is different from that of a finite global quotient, i.e., quotient of a scheme by a *finite group*. The above theorem allows us to relate an arbitrary X to a stack which is a finite global quotient.

Proof of Theorem 3.6. Let $q : X \longrightarrow X'$ be the coarse moduli space of X. Let $D \subset X'$ be a (reduced) divisor such that q is an isomorphism over $X' \setminus D$. Since X' is normal, there exists an open subset $V \subset X'$ such that

(1) $D \cap V$ is smooth, and
(2) $X' \setminus V$ has codimension at least two.

Let m be a positive integer such that for every point p of X, the order of the isotropy subgroup at p divides m. Let $\mathcal{L}$ be a sufficiently ample line bundle on X', and let D' be the zero locus of a section of $\mathcal{L}^m \otimes \mathcal{O}_{X'}(-D)$ such that $(D+D') \cap V$ is smooth. Note that since X' is not necessarily smooth, the divisor D may not be Cartier. Hence the reflexive sheaf $\mathcal{O}_{X'}(-D)$ may not be a line bundle. We replace D by $D + D'$ without loss of generality and assume that there exists a line bundle $\mathcal{L}$ on X' such that $\mathcal{L}^m \cong \mathcal{O}_{X'}(D)$.

Let $T \subset \mathcal{L}$ be the cyclic covering of X' defined by the section D of $\mathcal{L}^m$. There is a natural action of $G := \mathbb{Z}/m\mathbb{Z}$ on T given by the action of $\mathbb{G}_m$ on $\mathcal{L}$; by the choice of the integer m, we have a rational map $T \longrightarrow X$ defined over all codimension one points of X.

Let Γ be the closure of the graph of the rational map $T \longrightarrow X$. We note that Γ is a Deligne–Mumford stack (possibly singular). Since the algorithm for resolution of singularities in p. 782, Theorem 1.0.3 of [16] commutes with étale base change, it also applies to Deligne–Mumford stacks (see Remark 3.8). We thus use p. 782, Theorem 1.0.3 of [16] and functorially resolve the singularities of Γ to obtain a smooth Deligne–Mumford stack T' and a proper birational map $T' \longrightarrow \Gamma$. By functoriality of the resolution, there is a natural action of G on T' such that the map $T' \longrightarrow \Gamma$ is G-equivariant. Also, by construction, the induced G-equivariant map $T' \longrightarrow X$ is a morphism. Thus we have an induced morphism $\phi : Y \longrightarrow X$, where $Y = [T'/G]$. It is now straight-forward to check that the map ϕ satisfies the required conditions in the theorem. □

Remark 3.8 (*Resolution of singularities for stacks*). Let X be an algebraic stack of finite type over a field of characteristic zero. Fix an algorithm for resolution of singularities as given in p. 782, Theorem 1.0.3 of [16] which commutes with smooth base change. We claim that this algorithm also allows us to resolve singularities of X, i.e., to obtain a representable proper birational morphism $X' \longrightarrow X$ with X'/k smooth. We argue for X a quotient stack, which is our only case of interest. Let $X = [U/G]$. As proved in Proposition 3.9.1 of [7], the above algorithm gives us a resolution of singularities $U' \longrightarrow U$ which is G-equivariant. Here U' is smooth and so is the quotient stack $X' = [U'/G]$. The morphism $[U'/G] \longrightarrow [U/G]$ now satisfies our requirement. The general case (which we do not need) can be handled by mimicking Proposition 3.9.1 of [7] in the situation where the groupoid $G \times U \rightrightarrows U$ is replaced by the groupoid $R \rightrightarrows U$, where $U \longrightarrow X$ is an atlas and $R = U \times_X U$.

4. Polystable vector bundles on orbifolds

In this section, we prove Theorem 1.2. Throughout this section, we work over the field $k = \mathbb{C}$.

The coarse moduli space of a normal Deligne–Mumford stack is normal. In order to see this, we may use Proposition 5.2 of [8] and assume that the stack is a quotient of an affine scheme by a finite group. In this case, the statement just boils down to the assertion that if a finite group G acts on a normal domain A, then the ring of invariants A^G is normal. It is known that the quotient of a normal variety by an action of a finite group is also normal (p. 64, Theorem 16.1.1 of [2]).

DEFINITION 4.1

Let Z be a projective Deligne–Mumford stack over $\mathbb{C}$. A line bundle $\mathcal{L}$ on Z is called ample if some power of $\mathcal{L}$ descends to an ample line bundle on the coarse moduli space of Z (see also Theorem 2.1 of [10].)

Given a very ample line bundle L_0 on a normal projective variety M_0, the *degree* $\text{degree}_{L_0}(F_0)$ of a torsionfree coherent sheaf F_0 on M_0 with respect to L_0 is defined to be the degree of the restriction of F_0 to the general complete intersection curve obtained by intersecting hyperplanes on M_0 from the complete linear system for L_0.

Fix an ample line bundle $\mathcal{L}$ on a projective Deligne–Mumford stack Z. Let Z_0 be the coarse moduli space for Z. The degree of a torsionfree coherent sheaf F on Z with respect to $\mathcal{L}$ is defined to be

$$\text{degree}(F) = \text{degree}_{\mathcal{L}}(F) := \frac{1}{m^{d-1}n}\text{degree}_{\mathcal{L}^m}((\det F)^{\otimes n}),$$

where $d = \dim Z_0$, the integer m is such that $\mathcal{L}^m$ descends to a very ample line bundle on Z_0, and n is such that $(\det F)^{\otimes n}$ descends to Z_0. It is easy to see that degree(F) is independent of the choices of m and n, but it clearly depends on $\mathcal{L}$.

A torsionfree coherent sheaf F on Z is called *stable* (respectively, *semistable*) if for all coherent subsheaves $F' \subset F$ with $0 < \operatorname{rank}(F') < \operatorname{rank}(F)$,

$$\frac{\operatorname{degree}(F')}{\operatorname{rank}(F')} < \frac{\operatorname{degree}(F)}{\operatorname{rank}(F)} \quad \left(\text{respectively, } \frac{\operatorname{degree}(F')}{\operatorname{rank}(F')} \leq \frac{\operatorname{degree}(F)}{\operatorname{rank}(F)}\right).$$

A semistable sheaf is called *polystable* if it is a direct sum of stable sheaves.

4.1 *Normalization*

Let X be an integral variety, and let Spec $(L) \longrightarrow X$ be a dominant morphism, where L is a field which is a finite extension of the function field of X. Then one has the notion of normalization of X in L. One can easily extend this notion to the case when L is replaced by a product of fields where the morphism is now dominant when restricted to each of the component fields. Now, if $\tilde{X}$ denotes the normalization of X in L, and $U \longrightarrow X$ is an étale morphism, then $U \times_X$ Spec (L) is a product of fields with the map

$$U \times_X \operatorname{Spec}\ (L) \longrightarrow U$$

being dominant on each component of the domain. Let $\tilde{U}$ denote the normalization of U in $U \times_X$ Spec (L). Then the following diagram is Cartesian:

$$\begin{array}{ccc} \tilde{U} & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array} \quad .$$

In other words, normalization commutes with étale (or even smooth) base change.

Thus, one can extend this notion to Deligne–Mumford stacks. In other words, if X is a Deligne–Mumford stack, and Spec $(L) \longrightarrow X$ is a dominant morphism from a field, one can define the normalization $\tilde{X}$ of X in L.

Concretely, to define $\tilde{X}$, one first chooses an étale atlas $U \longrightarrow X$. Define $R := U \times_X U$, and denote by $\tilde{U}$ (respectively, $\tilde{R}$) the normalization of U (respectively, R) in $U \times_X$ Spec (L) (respectively, $R \times_X$ Spec (L)). Since $R \rightrightarrows U$ is an étale groupoid, so is $\tilde{R} \rightrightarrows \tilde{U}$. One then defines $\tilde{X}$ as the quotient $[\tilde{U}/\tilde{R}]$. The morphism Spec $(L) \longrightarrow X$ factors through $\tilde{X} \longrightarrow X$. Using the above construction, one now shows that $\tilde{X}$ is the unique normal stack with Spec (L) as the generic point (in particular, it is generically a scheme) such that $\tilde{X} \longrightarrow X$ is a finite representable morphism. We leave these details to the reader, since the details can be reduced to simple exercises dealing with normalization of varieties. This shows that the construction of normalization is independent of the atlas.

The above construction can be seen more clearly when X is the quotient of a scheme U by a finite group G. In this case, one simply has a natural G action on $\tilde{U}$, and hence one defines $\tilde{X}$ as $[\tilde{U}/G]$.

4.2 *Proof of Theorem* 1.2

We first establish some ingredients of the proof.

Lemma 4.2. *Let $X/\mathbb{C}$ be a normal orbifold, and let K be its function field. Let $f : Y \longrightarrow X$ be a dominant finite morphism, with Y normal, such that the corresponding function field extension is Galois. Let G be this Galois group. Let V be a vector bundle on X, and let W be a G-invariant reflexive subsheaf of f^*V. Then there is a closed subset $S \subset Y$ of codimension at least two such that $W|_{Y\setminus S}$ is a pullback of a unique subsheaf of $V|_{X\setminus f(S)}$.*

Proof. First assume that X is a variety. Outside a closed subset $S \subset Y$ of codimension at least two, the map f is flat, and W is locally free. The isotropies for the action of G on Y act trivially on the fibers of f^*V, and hence on the fibers of W. Therefore, W descends outside $f(S)$ to a unique subbundle (see Theorem 1.2 of [13] for descent of sheaves).

In the general case, let $U \longrightarrow X$ be an atlas. From the uniqueness, it is enough to prove the lemma by replacing X and Y with U and $Y \times_X U$ respectively, and also replacing the sheaves by the corresponding pullbacks. Then it is reduced to the above case. □

Lemma 4.3. *Let $X/\mathbb{C}$ be a projective normal orbifold, and let $\mathcal{L}$ be an ample line bundle on X. Let $f : Y \longrightarrow X$ be a dominant finite morphism, where $Y/\mathbb{C}$ is a normal projective variety. Let $\mathcal{E}$ be a vector bundle on X. Then the following three assertions hold*:

(*i*) *The vector bundle $\mathcal{E}$ is semistable with respect to $\mathcal{L}$ if and only if $f^*\mathcal{E}$ is semistable with respect to $f^*\mathcal{L}$.*
(*ii*) *If $\mathcal{E}$ is polystable with respect to $\mathcal{L}$, then $f^*\mathcal{E}$ is polystable with respect to $f^*\mathcal{L}$.*
(*iii*) *Assume that $f^*\mathcal{E}$ has a unitary flat connection. Then $\mathcal{E}$ is polystable.*

Proof. After one understands the notion of normalization of a Deligne–Mumford stack in a field (see Section 4.1), the proof is similar to the case when X is a variety instead of an orbifold. If $f^*\mathcal{E}$ is semistable with respect to $f^*\mathcal{L}$, then clearly $\mathcal{E}$ is semistable with respect to $\mathcal{L}$. To prove the converse, assume that $\mathcal{E}$ is semistable with respect to $\mathcal{L}$. We may replace Y by its normalization in the Galois closure of the function field of Y over that of X. If $f^*\mathcal{E}$ is not semistable, take the first term F in the Harder–Narasimhan filtration of $f^*\mathcal{E}$. Since F descends to give a subsheaf of $\mathcal{E}$ on an open subset whose complement has codimension at least two (see Lemma 4.2), this contradicts semistability of $\mathcal{E}$. Therefore, $f^*\mathcal{E}$ is semistable.

To prove the second statement, assume that $\mathcal{E}$ is polystable with respect to $\mathcal{L}$. Hence $f^*\mathcal{E}$ is semistable with respect to $f^*\mathcal{L}$ (by part (i)). Consider the Socle $F \subset f^*\mathcal{E}$; it is the unique maximal polystable subsheaf with $\text{degree}(F)/\text{rank}(F) = \text{degree}(f^*\mathcal{E})/\text{rank}(f^*\mathcal{E})$ (see p. 23, Lemma 1.5.5 of [4]). From the uniqueness of F, it follows that F is a pullback of a subsheaf F' of $\mathcal{E}$ outside codimension two (Lemma 4.2). Since $\mathcal{E}$ is polystable, $F' \subset \mathcal{E}$ has a direct summand F''. If $F \neq f^*\mathcal{E}$, the direct sum of F with the Socle of f^*F'' contradicts the maximality of F. Hence $F = f^*\mathcal{E}$, implying that $f^*\mathcal{E}$ is polystable.

To prove the last statement, assume that $f^*\mathcal{E}$ has a unitary flat connection ∇. This implies that $f^*\mathcal{E}$ is polystable (pp. 177–178, Theorem 8.3 of [6]), in particular, $f^*\mathcal{E}$ is semistable. Therefore, $\mathcal{E}$ is semistable by the first statement of the lemma. We have $\text{degree}(\mathcal{E}) = 0$ because $\text{degree}(f^*\mathcal{E}) = 0$.

Assume that the semistable vector bundle $\mathcal{E}$ is not polystable. Let

$$F' \subset \mathcal{E}$$

be the Socle of it. We note that f^*F' is polystable by the second statement. Also, $\text{degree}(f^*F') = 0$ because $\text{degree}(F') = \text{degree}(\mathcal{E}) = 0$. From this, it follows that

f^*F' is preserved by the unitary flat connection on $f^*\mathcal{E}$ (see the Gauss–Codazzi equation in p. 23 of [6]). Let

$$W := (f^*F')^{\perp} \subset f^*\mathcal{E}$$

be the orthogonal complement. It should be clarified that although the flat unitary connection on a polystable vector bundle is unique, the Hermitian structure is not unique. On the other hand, the orthogonal complement of a subbundle preserved by the flat unitary connection is independent of the choice of the Hermitian structure.

Since W is preserved by the unitary flat connection, it follows that W is a coherent subsheaf of $f^*\mathcal{E}$. As before, W descends to X. If W'' is a nonzero polystable subsheaf of degree zero of the descent of W, then $F' \oplus W''$ is a polystable subsheaf of $\mathcal{E}$ of degree zero. But this contradicts the assumption that F' is the maximal polystable subsheaf of $\mathcal{E}$. Hence $\mathcal{E}$ is polystable. □

Let X and $\mathcal{L}$ be as above. Let d be the dimension of X. All Chern classes considered here are topological, taking values in rational cohomology classes.

PROPOSITION 4.4

Let $f : Y \longrightarrow X$ be any dominant finite morphism of projective orbifolds. Fix an ample line bundle $\mathcal{L}$ on X. So $f^*\mathcal{L}$ is ample on Y. Let $\mathcal{E}$ be a polystable vector bundle on X with respect to $\mathcal{L}$ such that $c_1(\mathcal{E}) = 0$, and $c_2(\mathcal{E}) \cdot c_1(\mathcal{L})^{d-2} = 0$. Then $f^*\mathcal{E}$ is a polystable vector bundle on Y with respect to $f^*\mathcal{L}$. Also, $c_1(f^*\mathcal{E}) = 0$ and $c_2(f^*\mathcal{E}) \cdot c_1(f^*\mathcal{L})^{d-2} = 0$.

Proof. Clearly, $c_1(f^*\mathcal{E}) = 0$ and $c_2(f^*\mathcal{E}) \cdot c_1(f^*\mathcal{L})^{d-2} = 0$, because these conditions hold for $\mathcal{E}$ and $\mathcal{L}$.

By [9], we can find a diagram

$$\begin{array}{ccc} Y' & \xrightarrow{h} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array},$$

where Y' and X' are smooth projective varieties, and the horizontal morphisms are finite and dominant. By Lemma 4.3, the pullback $g^*\mathcal{E}$ is polystable, and also, $c_1(g^*\mathcal{E}) = 0$ and $c_2(g^*\mathcal{E}) \cdot c_1(g^*\mathcal{L})^{d-2} = 0$; note that $g^*\mathcal{L}$ is ample because g is finite. Thus, $g^*\mathcal{E}$ is given by a unitary representation of the fundamental group of X' (see Theorem 1.1 and the third comment following it). Therefore, $f'^*g^*\mathcal{E}$ is also given by a unitary representation of the fundamental group of Y', and hence it is polystable. Thus by Lemma 4.3(iii), the pullback $f^*\mathcal{E}$ is polystable. □

Lemma 4.5. *Let X be a finite global quotient. Then Theorem* 1.2 *holds for X.*

Proof. Take $\mathcal{E} \longrightarrow X$ as in Theorem 1.2. First, assume that $\mathcal{E}$ is given by a unitary representation ρ of $\pi_1^{\text{top}}(X, x)$. To prove that $\mathcal{E}$ is polystable, first note that for any vector bundle F on X given by a unitary representation of $\pi_1^{\text{top}}(X, x)$, we have $c_1(F) = 0 = c_2(F)$.

By hypothesis, there exists a smooth projective variety Y and a finite group G acting on Y such that $X = [Y/G]$. Let

$$f : Y \longrightarrow X$$

be the natural map. Since $f^*\mathcal{E}$ is given by the unitary representation $f^*\rho$, it follows that $\mathcal{E}$ is polystable (see Lemma 4.3(iii)).

To prove the converse, assume that $\mathcal{E} \longrightarrow X$ is a polystable vector bundle with $c_1(\mathcal{E}) = 0 = c_2(\mathcal{E}) \cdot c_1(\mathcal{L})^{n-2}$. Take Y as above such that $X = [Y/G]$. Then $f^*\mathcal{E}$ is a polystable vector bundle on Y with $c_1(f^*\mathcal{E}) = 0$ and $c_2(f^*\mathcal{E}) \cdot c_1(f^*\mathcal{L})^{n-2} = 0$, where f is the quotient map. Let

$$\pi_Y : U_Y \longrightarrow Y$$

denote the universal covering space of Y. Then $\pi_Y^* f^*\mathcal{E}$ is a trivial vector bundle on U_Y with a $\pi_1^{\text{top}}(Y, y)$-invariant unitary flat metric, say $\langle -, - \rangle$. Note that U_Y is also the covering space for X. Moreover, if $x = f(y)$, then $\pi_1^{\text{top}}(Y, y)$ is naturally a finite index subgroup of $\pi_1^{\text{top}}(X, x)$. Let $\{g_i\}$ denote its finite coset representatives. It is then clear that the "averaged" metric

$$(v_1, v_2) := \sum_i g_i^* \langle v_1, v_2 \rangle$$

is a $\pi_1^{\text{top}}(X, x)$ invariant unitary flat metric on U_Y. □

We now reduce the general case of Theorem 1.2 to the above special case using Theorem 3.6.

Proof of Theorem 1.2. As in Lemma 4.5, if $\mathcal{E}$ is given by a unitary representation of $\pi_1^{\text{top}}(X, x)$, then $\mathcal{E}$ is polystable, and $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) \cdot c_1(\mathcal{L})^{n-2} = 0$.

Assume that $\mathcal{E}$ is polystable, and $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) \cdot c_1(\mathcal{L})^{n-2} = 0$. Let $\phi : Y \longrightarrow X$ be a morphism which is guaranteed by Theorem 3.6. Fix a point y_0 of Y. Then, since Y is a finite global quotient, by Lemma 4.5 and Proposition 4.4, the pullback $\phi^*\mathcal{E}$ is given by a unitary representation of $\pi_1^{\text{top}}(Y, y_0)$.

However, for any point $y \in Y$, the kernel of $G_y \longrightarrow G_{\phi(y)}$ acts trivially on the fiber of $\phi^*\mathcal{E}$ at y. Thus this kernel is also contained in the kernel of the unitary representation. Then by Theorem 3.5, the unitary representation of $\pi_1^{\text{top}}(Y, y_0)$ which defines $\phi^*\mathcal{E}$ actually factors through $\pi_1^{\text{top}}(X, \phi(y_0))$. This completes the proof of Theorem 1.2. □

As a consequence of Theorem 1.2, we obtain the following generalization of Theorem 1.1 to certain types of singular varieties.

COROLLARY 4.6

Let $X/\mathbb{C}$ be a projective variety with at worst quotient singularities. Assume that X is the coarse moduli space of a projective orbifold $\tilde{X}/\mathbb{C}$. Then Theorem 1.1 *holds for X.*

Proof. Let $\mathcal{L}$ be an ample line bundle on X, and let $\mathcal{E}$ be a polystable vector bundle on X satisfying $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) \cdot c_1(\mathcal{L})^{n-2} = 0$, where n is the dimension of X. Let $\pi : \tilde{X} \longrightarrow X$ be the given morphism. Since the pullback $\pi^*\mathcal{E}$ is polystable (see Lemma 4.3), by Theorem 1.2, the vector bundle $\pi^*\mathcal{E}$ is given by a unitary representation ρ of $\pi_1^{\text{top}}(\tilde{X}, x)$ (for some geometric point x of $\tilde{X}$). However, since the vector bundle arising from this representation is a pullback from X, it is clear that for every point y of $\tilde{X}$, the isotropy group G_y at y lies in the kernel of this representation ρ. Thus by using p. 90, § 8.1 of [14], the representation ρ descends to give a unitary representation of $\pi_1^{\mathrm{top}}(X, x)$. This representation of $\pi_1^{\mathrm{top}}(X, x)$ gives the vector bundle $\mathcal{E}$. □

References

[1] Abramovich D, Corti A and Vistoli A, Twisted bundles and admissible covers, *Comm. Alg.* **31** (2003) 3547–3618

[2] Białynicki-Birula A, Carrell J B and McGovern W M, Algebraic quotients. Torus actions and cohomology. The adjoint representation and the adjoint action, Encyclopaedia of Mathematical Sciences (2002) (Berlin: Springer-Verlag) vol. 131

[3] Donaldson S K, Infinite determinants, stable bundles and curvature, *Duke Math. J.* **54** (1987) 231–247

[4] Huybrechts D and Lehn M, The geometry of moduli spaces of sheaves, Aspects of Mathematics (1997) (Braunschweig: Friedr. Vieweg & Sohn) vol. E31

[5] Keel S and Mori S, Quotients by groupoids, *Ann. Math.* **145** (1997) 193–213

[6] Kobayashi S, Differential Geometry of Complex Vector Bundles, Publications of the Math. Society of Japan (1987) (Princeton, NJ: Iwanami Shoten Publishers, Tokyo, and Princeton University Press) vol. 15

[7] Kollár J, Lectures on resolution of singularities, Annals of Mathematics Studies (2007) (Princeton, NJ: Princeton University Press) vol. 166

[8] Kresch A, On the geometry of Deligne–Mumford stacks, Algebraic geometry—Seattle 2005. Part 1, 259–271, Proc. Sympos. Pure Math. (2009) (Providence, RI: Amer. Math. Soc.) vol. 80

[9] Kresch A and Vistoli A, On coverings of Deligne–Mumford stacks and surjectivity of the Brauer map, *Bull. London Math. Soc.* **36** (2004) 188–192

[10] Matsuki K and Olsson M, Kawamata-Viehweg vanishing as Kodaira vanishing for stacks, *Math. Res. Lett.* **12** (2005) 207–217

[11] Mehta V and Ramanathan A, Restriction of stable sheaves and representations of the fundamental group, *Invent. Math.* **77** (1984) 163–172

[12] Narasimhan M and Seshadri C S, Stable and unitary vector bundles on a compact Riemann surface, *Ann. Math.* **82** (1965) 540–567

[13] Nevins T, Descent of coherent sheaves and complexes to geometric invariant theory quotients, *J. Alg.* **320** (2008) 2481–2495

[14] Noohi B, Fundamental groups of algebraic stacks, *J. Inst. Math. Jussieu* **3** (2004) 69–103

[15] Noohi B, Fundamental groups of topological stacks with the slice property, *Algebr. Geom. Topol.* **8** (2008) 1333–1370

[16] Włodarczyk J, Simple Hironaka resolution in characteristic zero, *J. Amer. Math. Soc.* **18** (2005) 779–822

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