Proc. Indian Acad. Sci. (Math. Sci.) Vol. 126, No. 4, November 2016, pp. 549–556.
DOI 10.1007/s12044-016-0305-4

Combinatorics of tenth-order mock theta functions

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MS received 2 May 2014; revised 16 March 2016

Abstract. In this paper, we provide the combinatorial interpretations of two tenth-order mock theta functions which appeared in some identities given in Ramanujan's lost notebook ((1988) Narosa Publishing House, New Delhi).

Keywords. Mock theta functions; colored partitions; Frobenius partitions.

2010 Mathematics Subject Classifications. 11P81, 82, 05A17, 19.

1. Introduction

In his last letter to G. H. Hardy, S. Ramanujan listed 17 functions of 'third', 'fifth' and 'seventh' order, which he called mock theta functions [9]. Mock theta function is a function $f(q)$ defined by a q-series which converges for $|q| < 1$ and which satisfies the following two conditions:

(1) For every root of unity ζ, there is a theta function $\theta_\zeta(q)$ such that the difference $f(q) - \theta_\zeta(q)$ is bounded as $q \to \zeta$ rapidly.
(2) There is no single theta function which works for all ζ: i.e., for every theta function $\theta(q)$ there is some root of unity ζ for which $f(q) - \theta(q)$ is unbounded as $q \to \zeta$ rapidly.

In the long list of 17 mock theta functions given by Ramanujan, few have been interpreted combinatorially. For example, $\Psi(q)$ defined by (1.1) below, has been interpreted by Fine [8] as a generating function for partitions into odd parts without gaps. Agarwal [2] gave the combinatorial interpretations of the following four mock theta functions using n-color partitions:

$$\Psi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}, \tag{1.1}$$

$$F_0(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n}, \tag{1.2}$$

$$\Phi_0(q) = \sum_{n=0}^{\infty} q^{n^2}(-q;q^2)_n, \tag{1.3}$$

$$\Phi_1(q) = \sum_{n=0}^{\infty} q^{(n+1)^2}(-q;q^2)_n, \tag{1.4}$$

where $\Psi(q)$ is of order 3, while the remaining three are of order 5. Agarwal [3] translated his results using lattice paths and then Agarwal and Narang [5] extended the results using Frobenius partitions. Agarwal and Rana [4] gave combinatorial interpretations of the following fifth order mock theta function using '$n+2$ copies of n' and lattice paths:

$$F_1(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}}. \tag{1.5}$$

They further extended the result using Frobenius partitions in [11]. Together with these seventeen mock theta functions Ramanujan [10] also gave a list of eight identities involving four tenth order mock theta functions given below:

$$\phi_R(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q:q^2)_{n+1}}, \tag{1.6}$$

$$\psi_R(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}}{(q:q^2)_{n+1}}, \tag{1.7}$$

$$X_R(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q:q)_{2n}}, \tag{1.8}$$

$$\chi_R(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q:q)_{2n+1}}. \tag{1.9}$$

These functions have also been studied analytically by Choi [7].

In this paper, we provide the combinatorial interpretations for the two tenth order mock theta functions given by (1.6) and (1.7).

2. Definitions and notations

DEFINITION 2.1 [1]

A partition with '$(n+t)$-copies of n', $t \geq 0$ is a partition in which a part of size n, $n \geq 0$ can come in $(n+t)$ different colors denoted by subscripts $n_1, n_2, \ldots, n_{n+t}$.

Thus, for example, the partitions of 2 with '$n+1$ copies of n' are

$$\begin{array}{llll} 2_1, & 2_1+0_1, & 1_1+1_1, & 1_1+1_1+0_1, \\ 2_2, & 2_2+0_1, & 1_2+1_1, & 1_2+1_1+0_1, \\ 2_3, & 2_3+0_1, & 1_2+1_2, & 1_2+1_2+0_1. \end{array}$$

Note that zeros are permitted if and only if $t \geq 1$ and in no partition zeros are permitted to repeat. The weighted difference of any pair of parts m_i, n_j is defined by $m-n-i-j$ and is denoted by $((m_i - n_j))$.

DEFINITION 2.2 [6]

A two rowed array of non-negative integers

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}, a_1 \geq a_2 \geq \cdots \geq a_r \geq 0, b_1 \geq b_2 \geq \cdots \geq b_r \geq 0,$$

is known as a generalized Frobenius partition of n if

$$n = r + \sum_{i=1}^{r} a_i + \sum_{i=1}^{r} b_i.$$

For example, $n = 32 = 4 + (7 + 6 + 3 + 0) + (5 + 4 + 2 + 1)$ and the corresponding Frobenius notation is $\begin{pmatrix} 7 & 6 & 3 & 0 \\ 5 & 4 & 2 & 1 \end{pmatrix}$.

3. Combinatorial interpretations of (1.6) and (1.7) using '$(n + t)$-color partitions'

Theorem 3.1. *For $\nu \geq 1$, let $A_1(\nu)$ denote the number of n-color partitions of ν such that for some k, k_k is a part, and the weighted difference of any two consecutive parts is -1. Then,*

$$\sum_{\nu=1}^{\infty} A_1(\nu) q^{\nu} = \psi_R(q). \tag{3.1}$$

For example, for $\nu = 6$, $A_1(6) = 4$ and the relevant n-color partitions are

$$6_6, 5_4 + 1_1, 4_1 + 2_2, 3_1 + 2_1 + 1_1.$$

Theorem 3.2. *For $\nu \geq 0$, let $A_2(\nu)$ denote the number of $(n + 1)$-color partitions of ν such that for some k, k_{k+1} appears as a part, and the weighted difference of any two consecutive parts is -1. Then,*

$$\sum_{\nu=0}^{\infty} A_2(\nu) q^{\nu} = \phi_R(q). \tag{3.2}$$

For example, for $\nu = 6$, $A_2(6) = 6$. The relevant n-color partitions are

$$6_7, 6_6 + 0_1, 4_1 + 2_2 + 0_1, 5_3 + 1_2, 3_1 + 2_1 + 1_1 + 0_1, 5_4 + 1_1 + 0_1.$$

Note. Theorems 3.1 and 3.2 are the combinatorial interpretations of (1.7) and (1.6), respectively.

Remark. In the sequel $A_i(m, \nu)$, $(1 \leq i \leq 2)$ will denote the number of partitions of ν enumerated by $A_i(\nu)$ into m parts, and we shall write

$$f_i(z, q) = \sum_{\nu=0}^{\infty} \sum_{m=0}^{\infty} A_i(m, \nu) z^m q^{\nu}. \tag{3.3}$$

Proof. Splitting the partitions enumerated by $A_1(m, \nu)$ into two classes, class (1) have 1_1 as a part, and class (2) have k_k, $(k > 1)$ as a part. Now transformed partitions of class (1), enumerated by $A_1(m-1, \nu-m)$, will be obtained by deleting the part 1_1 and then subtracting 1 from all the remaining parts ignoring the subscripts and the transformed partitions of class (2), enumerated by $A_1(m, \nu-2m+1)$, will be obtained by first replacing $(k)_k$ by $(k-1)_{k-1}$ and then subtracting 2 from all of remaining parts ignoring the subscripts. The transformations are clearly reversible and leads to the following identity:

$$A_1(m, \nu) = A_1(m-1, \nu-m) + A_1(m, \nu-2m+1). \tag{3.4}$$

From (3.3), we have

$$f_1(z, q) = \sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty} A_1(m, \nu) z^m q^\nu. \tag{3.5}$$

Substituting for $A_1(m, \nu)$ from (3.4) in (3.5) and then simplifying, we get

$$f_1(z, q) = zqf_1(zq, q) + q^{-1} f_1(zq^2, q) \tag{3.6}$$

which is a q-functional equation. Setting

$$f_1(z, q) = \sum_{n=0}^{\infty} \alpha_n(q) z^n \tag{3.7}$$

and then comparing the coefficients of z^n on both sides of (3.6), we see that

$$\alpha_n(q) = \frac{q^n}{1-q^{2n-1}} \alpha_{n-1}(q). \tag{3.8}$$

Iterating (3.8) n times and noting $\alpha_0(q) = 1$, we get

$$\alpha_n(q) = \frac{q^{n(n+1)/2}}{(q; q^2)_n}. \tag{3.9}$$

Therefore,

$$f_1(z, q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q; q^2)_n} z^n. \tag{3.10}$$

Now,

$$\begin{aligned}
\sum_{\nu=0}^{\infty} A_1(\nu) q^\nu &= \sum_{\nu=0}^{\infty}\left(\sum_{m=0}^{\infty} A_1(m, \nu)\right) q^\nu \\
&= f_1(1, q) \\
&= \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q; q^2)_n} \\
&= 1 + \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}}{(q; q^2)_{n+1}} \\
&= 1 + \psi_R(q)
\end{aligned}$$

which completes the proof of Theorem 3.1. □

Sketch of proof of Theorem 3.2. Proceeding as above, one can easily obtain the following recurrence relation:

$$A_2(m, \nu) = A_1(m-1, \nu) + A_2(m, \nu - 2m + 1). \tag{3.11}$$

Corresponding q-functional equation is

$$f_2(z, q) = zf_1(z, q) + q^{-1} f_2(zq^2, q). \tag{3.12}$$

Using the above q-functional equation and proceeding as in Theorem 3.1, we get

$$\sum_{\nu=0}^{\infty} A_2(\nu) q^{\nu} = \phi_R(q).$$

4. Combinatorial interpretation of (1.7) using generalized Frobenius partitions

Theorem 4.1. *For $\nu \geq 1$, let $B_1(\nu)$ denote the number of Frobenius partitions of ν such that*

(*a*) *For any two adjacent columns* $\begin{pmatrix} a_i \\ b_i \end{pmatrix}$ *and* $\begin{pmatrix} a_{i+1} \\ b_{i+1} \end{pmatrix}$, *we have*

- (*i*) *If $a_i \leq b_i$, then $a_{i+1} > b_{i+1}$ and $a_i = a_{i+1}$, for $1 \leq i \leq (r-2)$.*
- (*ii*) *If $a_i > b_i$, then $a_{i+1} \leq b_{i+1}$ and $b_i = b_{i+1}$, for $1 \leq i \leq (r-1)$.*

(*b*) $a_r = 0$.

Let $A_1(\nu)$ denote the number of n-color partitions of ν such that

(*c*) *The weighted difference of any two consecutive parts is -1.*
(*d*) *For some i, i_i is a part.*

Then $A_1(\nu) = B_1(\nu)$, for all ν.

Remark. Note that $A_1(\nu)$ is same as that defined in Theorem 3.1.

Proof. We establish a 1–1 correspondence between the generalized Frobenius partitions enumerated by $B_1(\nu)$ and the n-color partitions enumerated by $A_1(\nu)$. We do this by mapping each column $\begin{pmatrix} a \\ b \end{pmatrix}$ of the Frobenius partition to a single part m_i of an n-color partition enumerated by $A_1(\nu)$. The mapping ϕ is

$$\phi : \begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{cases} (a+b+1)_{b-a+1}, & a \leq b, \\ (a+b+1)_{a-b}, & a > b. \end{cases} \tag{4.1}$$

and the inverse mapping ϕ^{-1} is given by

$$\phi^{-1} : m_i \rightarrow \begin{cases} \begin{pmatrix} (m-i)/2 \\ (m+i-2)/2 \end{pmatrix}, & \text{if } m \equiv i \pmod 2; \\ \begin{pmatrix} (m+i-1)/2 \\ (m-i-1)/2 \end{pmatrix}, & \text{if } m \equiv i+1 \pmod 2. \end{cases} \tag{4.2}$$

Now suppose we have any two adjacent columns $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ in a generalized Frobenius partition enumerated by $B_1(\nu)$ with

$$\phi : \begin{pmatrix} a \\ b \end{pmatrix} = m_i \quad \text{and} \quad \phi : \begin{pmatrix} c \\ d \end{pmatrix} = n_j.$$

Then since $a \geq c$ and $b \geq d$, we have

$$((m_i - n_j)) = \begin{cases} 2(b-d) - 1, & b < a, d \geq c, \\ 2(a-c) - 1, & b \geq a, d < c. \end{cases} \tag{4.3}$$

Therefore equation (4.3) and Theorem 4.1(a) imply Theorem 4.1(c). Now if $a_r = 0$, then using equation (4.1) we have

$$\phi : \begin{pmatrix} a_r \\ b_r \end{pmatrix} = (b_r + 1)_{b_r+1}, \quad \text{for} \quad a \leq b \tag{4.4}$$

which is of the form i_i. Hence Theorem 4.1(b) and equation (4.1) imply Theorem 4.1(d).

To see the reverse implication, we consider the inverse images of two consecutive parts m_i, n_j of n-color partition enumerated by $A_1(\nu)$ such that

$$\phi^{-1} : m_i = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad \phi^{-1} : n_j = \begin{pmatrix} c \\ d \end{pmatrix}.$$

Since $((m_i - n_j)) = -1$, we see that if m and i have the same parity then n and j will have the opposite parity and vice-versa. Then

$$a - c = \begin{cases} \frac{((m_i - n_j))+1}{2}, & m \equiv i \pmod 2, n \not\equiv j \pmod 2, \\ \frac{((m_i - n_j))-1}{2} + i + j, & m \not\equiv i \pmod 2, n \equiv j \pmod 2. \end{cases} \tag{4.5}$$

$$b - d = \begin{cases} \frac{((m_i - n_j))-1}{2} + i + j, & m \equiv i \pmod 2, n \not\equiv j \pmod 2, \\ \frac{((m_i - n_j))+1}{2}, & m \not\equiv i \pmod 2, n \equiv j \pmod 2. \end{cases} \tag{4.6}$$

Also,

$$a - b = \begin{cases} -i + 1, & m \equiv i \pmod 2, \\ i, & m \not\equiv i \pmod 2. \end{cases} \tag{4.7}$$

Similarly,

$$c - d = \begin{cases} -j + 1, & n \equiv j \pmod 2, \\ j, & n \not\equiv j \pmod 2. \end{cases} \tag{4.8}$$

Now (4.5), (4.6), (4.7) and (4.8) imply Theorem 4.1(a).

Finally from Theorem 4.1(d), i.e. i_i must appear as a part, and it is the smallest part of the partition. Therefore using (4.2), we have

$$\phi^{-1} : i_i = \begin{pmatrix} 0 \\ i - 1 \end{pmatrix},$$

hence $a_r = 0$. This completes the proof of Theorem 4.1. □

Table 1. An example of direct bijection between F-partitions and colored partitions.

Generalized Frobenius partitions enumerated by $B_1(6)$	Image under ϕ
$\begin{pmatrix} 0 \\ 5 \end{pmatrix}$	6_6
$\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$	$5_4 + 1_1$
$\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$	$4_1 + 2_2$
$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$3_1 + 2_1 + 1_1$

To illustrate the bijection we have constructed, we give the example for $\nu = 6$ shown in Table 1.

5. Conclusion

In this paper, we have provided the combinatorial interpretation of two tenth order mock theta functions by using $(n + t)$-color partitions and we have also obtained the bijection between n-color partitions and Frobenius partitions for one of tenth order mock theta function given by (1.7). We are curious to know if it is possible to obtain a similar bijection for (1.6).

Acknowledgements

This work was supported by the University Grant Commission, No. F. 2-16/2011(SA-1).

References

[1] Agarwal A K and Andrews G E, Rogers–Ramanujan identities for partitions with N copies of N, *J. Comb. Theory Ser. A* **45(1)** (1987) 40–49

[2] Agarwal A K, n-Color partition theoretic interpretations of some mock theta functions, *Electron. J. Comb.* **11(1)** (2004) Note 14, 6pp (electronic)

[3] Agarwal A K, Lattice paths and mock theta functions, *Proceedings of the* 6^{th} *Int. Conf. SSFA* **6** (2005) 95–102

[4] Agarwal A K and Rana M, Two new combinatorial interpretations of a fifth order mock theta function, IMS Special Centenary Volume (1907–2007) pp. 11–24

[5] Agarwal A K and Narang G, Genralized Frobenius partitions and mock-theta functions, *Ars Comb.* **99** (2011) 439–444

[6] Andrews G E, Generalized Frobenius partitions, *Mem. Am. Math. Soc.* **49(301)** (1984) IV+44pp

[7] Choi Y S, Tenth order mock theta functions in Ramanujan's lost notebook, *Invent Math.* **136** (1999) 497–569

[8] Fine N J, Basic hypergeometric series and applications, in: Mathematical Surveys and Monographs, No. 27. AMS (1988)
[9] Hardy G H, Seshu Aiyer P V and Wilson B M, Collected papers of Srinivasa Ramanujan (Cambridge University Press) (1927)
[10] Ramanujan S, The lost notebook and other unpublished papers (1988) (New Delhi: Narosa Publishing House)
[11] Rana M and Agarwal A K, Frobenius partition theoretic interpretation of a fifth order mock theta function, *Canad. J. Pure Appl. Sci. British Columbia* **2(3)** (2009) 859–863

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