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Certain variants of multipermutohedron ideals

AJAY KUMAR[1,2] and CHANCHAL KUMAR[1,*]

[1]Indian Institute of Science Education and Research, Mohali, Knowledge City, Sector 81, SAS Nagar, Punjab 140 306, India
[2]Present Address: DAV University, Jalandhar, Punjab 144 012, India
[*]Corresponding author.
E-mail: ajaykumar@iisermohali.ac.in; chanchal@iisermohali.ac.in

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Abstract. Multipermutohedron ideals have rich combinatorial properties. An explicit combinatorial formula for the multigraded Betti numbers of a multipermutohedron ideal and their Alexander duals are known. Also, the dimension of the Artinian quotient of an Alexander dual of a multipermutohedron ideal is the number of generalized parking functions. In this paper, monomial ideals which are certain variants of multipermutohedron ideals are studied. Multigraded Betti numbers of these variant monomial ideals and their Alexander duals are obtained. Further, many interesting combinatorial properties of multipermutohedron ideals are extended to these variant monomial ideals.

Keywords. Betti numbers; multipermutohedron; Alexander dual; parking functions.

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1. Introduction

Postnikov and Shapiro [10] studied two Artinian k-algebras $\mathcal{A}_G$ and $\mathcal{B}_G$ associated to a finite graph G such that $\dim_k(\mathcal{A}_G) = \dim_k(\mathcal{B}_G)$ and a k-basis of $\mathcal{A}_G$ or $\mathcal{B}_G$ consists of monomials of the form $\mathbf{x}^{\mathbf{b}-\mathbf{1}}$, where $\mathbf{b} = (b_1, b_2, \ldots, b_n)$; $b_i \geq 1$ is a G-parking function (a slightly different notion of a G-parking function $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ is given in [10], where it is assumed that $b_i \geq 0$ for all i). It is shown that the number of G-parking functions equals the number of spanning trees of the graph G. If $G = K_{n+1}$, the complete graph on $n+1$ vertices, then K_{n+1}-parking functions are the ordinary parking function of length n. More generally, a notion of λ-parking or generalized parking functions is studied in [9, 11]. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, a sequence $(a_1, a_2, \ldots, a_n)$ of positive integers is said to be a *λ-parking function of length n*, if its non-decreasing rearrangement $(b_1, b_2, \ldots, b_n)$ satisfies $b_i \leq \lambda_{n-i+1}$ for all i. The ordinary parking functions correspond to $\lambda = (n, n-1, \ldots, 1)$. The number of ordinary parking functions of length n equals $(n+1)^{n-1}$. If $\lambda_i = a + (n-i)b$, where a, b are positive integers, then Pitman and Stanley [9], and independently Yan [11, 12] showed that the number of λ-parking functions of length n is $a(a+nb)^{n-1}$. Pitman and Stanley [9] also showed that the number of λ-parking functions of length n is given by a combinatorial formula $\sum_{\mathbf{a}} \prod_{i=1}^{n} (\lambda_{n-a_i+1} - \lambda_{n-a_i+2})$, where summation is carried over all ordinary parking functions $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ of length n. Postnikov and Shapiro [10] considered a k-algebra $A_\lambda = R/I_\lambda$, where $R = k[x_1, x_2, \ldots, x_n]$ is the standard polynomial ring over a field k and I_λ is a monomial ideal given by

$$I_\lambda = \left\langle \left(\prod_{i\in A} x_i\right)^{\lambda_{n-|A|+1}} : \emptyset \neq A \subset [n] \right\rangle.$$

They showed that the Hilbert series of A_λ is given by

$$H(A_\lambda, t) = \sum_{\mathbf{a}} \prod_{i=1}^{n} \left(\frac{t^{\lambda_{n-a_i+2}} - t^{\lambda_{n-a_i+1}}}{1-t} \right).$$

Since standard monomials of A_λ are precisely of the form $\mathbf{x}^{\mathbf{b-1}}$, where $\mathbf{b}$ is a λ-parking function, a combinatorial formula for counting λ-parking functions is obtained at once, by computing $H(A_\lambda, 1)$.

Further if $\lambda_1 > \lambda_2 > \cdots > \lambda_n$, then it is shown in [10] that the monomial ideal I_λ is a strictly monotone monomial ideal. Also, the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ of an $n-1$-simplex Δ_{n-1} labeled with monomial $\left(\prod_{i\in A} x_i\right)^{\lambda_{n-|A|+1}}$ on the vertex corresponding to $\emptyset \neq A \subseteq [n]$, supports the minimal free resolution of I_λ. Thus i-th Betti number $\beta_i(R/I_\lambda) = (i!)S_{n+1,i+1}; i \geq 0$, where $S_{n+1,i+1}$ is the Stirling number of the second kind. In [4], the monomial ideal I_λ is expressed as an Alexander dual of a multipermutohedron ideal and a combinatorial description of all the multigraded Betti numbers of I_λ is obtained in the general case $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Further, using a fine Hilbert series of R/I_λ, a simple proof of Steck determinant formula for counting λ-parking functions is obtained.

For a finite graph G, a binomial ideal I_G called the *toppling ideal* of the graph G was considered by Manjunath and Sturmfels [5]. For a fixed vertex q of G, the toppling ideal I_G has a distinguished initial (monomial) ideal M_G^q such that the quotient R/M_G^q is the Artinian k-algebra $\mathcal{A}_G$. For complete graphs, the minimal polyhedral cellular free resolution of M_G^q and I_G was given in [10] and [5], respectively. Recently, Mohammadi and Shokrieh [6] generalized these constructions to arbitrary finite graphs and resolved several questions and conjectures from [10] and [5]. In particular, minimal free resolutions of I_G and M_G^q were obtained for an arbitrary finite graph G in [6]. The Riemann–Roch theorem on graph G was shown to be linked to Alexander duality for the monomial ideal M_G^q in [5] and a polyhedral minimal co-cellular resolution of an Alexander dual $(M_G^q)^{[\mathbf{a}]}$ of M_G^q with respect to a suitable multidegree $\mathbf{a}$ was given in [6].

It is clearly indicated in [10] that combinatorially defined monomial ideals have homological invariants described in terms of combinatorially interesting objects. Multipermutohedron ideals have nice combinatorial properties. In this paper, we have studied monomial ideals which are certain variants of multipermutohedron ideals and described their homological invariants in terms of combinatorial objects. We now give a brief overview of this paper.

Let $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{N}^n$ such that $u_1 \leq u_2 \leq \cdots \leq u_n$ and let the first m_1 coordinates be equal to u_1 and next m_2 coordinates be equal to u_{m_1+1} and so on. In other words,

$$u_1 = \cdots = u_{s_1} < u_{s_1+1} = \cdots = u_{s_2} < u_{s_2+1} = \cdots < u_{s_{l-1}+1} = \cdots = u_n,$$

where $s_i = \sum_{\alpha=1}^{i} m_\alpha$ for $0 \leq i \leq l$ and $s_0 = 0$. Note that $m_i \geq 1$ and $s_l = \sum_{i=1}^{l} m_i = n$. Now set $\mathbf{m}(\text{or } \mathbf{m_u}) = (m_1, m_2, \ldots, m_l) \in \mathbb{N}^l$. For a permutation π of $\mathbf{u}$, let $\pi\mathbf{u} = (\pi u_1, \ldots, \pi u_n) \in \mathbb{R}^n$. The convex hull of all points $\pi\mathbf{u}$; π a permutation of $\mathbf{u}$, is an $(n-1)$-dimensional (except when $u_1 = u_2 = \ldots = u_n$) polytope $P(\mathbf{u})$ called *a multipermutohedron*. When all u_i's are distinct, we call $P(\mathbf{u})$ a *permutohedron*. Each vertex $\pi\mathbf{u}$ of the multipermutohedron $P(\mathbf{u})$ is naturally labeled with the monomial $\mathbf{x}^{\pi\mathbf{u}}$ making it a labeled polyhedral cell complex. The associated monomial ideal $I(\mathbf{u}) = \langle \mathbf{x}^{\pi\mathbf{u}} : \pi \text{ a permutation of } \mathbf{u} \rangle$ is called a *multipermutohedron ideal*. The cellular free complex $\mathbb{F}_*(P(\mathbf{u}))$ associated to the multipermutohedron $P(\mathbf{u})$ gives a (finite) free resolution of the multipermutohedron ideal $I(\mathbf{u})$ (see (3.1) for a definition of $\mathbb{F}_*(\mathbf{X})$). In the second section, we define a notion of *split-multipermutohedron ideal*. Let $n = r + s$ with $r, s \geq 1$, be a positive integer and $\mathfrak{S}_n$ be the set of all permutations of $\{1, 2, \ldots, n\}$. Consider the set H of all permutations of $\{1, 2, \ldots, n\}$ of type (σ_1, σ_2), where σ_1 is a permutation of $\{1, 2, \ldots, r\}$ and σ_2 is a permutation of $\{r+1, \ldots, r+s = n\}$. For $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{N}^n$, the convex hull of points $\sigma\mathbf{u}$ for $\sigma \in$H is also a polytope, which is the product of multipermutohedrons $P(\mathbf{v}) \times P(\mathbf{w})$, where $\mathbf{v} = (u_1, \ldots, u_r)$ and $\mathbf{w} = (u_{r+1}, \ldots, u_n)$. The associated monomial ideal $\mathcal{I} = \langle \mathbf{x}^{\sigma_1\mathbf{v}}\mathbf{y}^{\sigma_2\mathbf{w}} : (\sigma_1, \sigma_2) \in H \rangle \subseteq k[x_1, x_2, \ldots, x_r, y_1, \ldots, y_s]$, where $y_j = x_{r+j}$ is a called a *split-multipermutohedron ideal*. Clearly $\mathcal{I} = I(\mathbf{v}) \otimes_k I(\mathbf{w})$ is a tensor product of multipermutohedron ideals $I(\mathbf{v})$ and $I(\mathbf{w})$, so it is straight forward to compute the Betti numbers of $\mathcal{I}$ in terms of Betti numbers of $I(\mathbf{v})$ and $I(\mathbf{w})$. However, using combinatorial descriptions of multigraded Betti numbers of multipermutohedron ideals and their Alexander duals as given in [3] and [4] respectively, we have given an explicit description of the multigraded Betti numbers of $\mathcal{I}$ and its Alexander duals. The multigraded Betti numbers of multipermutohedron ideal and its duals are also needed for describing the Betti numbers of the sum of two multipermutohedron ideals and its duals in the last section.

In the third section, for $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ with $1 \leq u_1 < u_2 < \ldots < u_n$, we considered a monomial ideal $I_W = \langle \mathbf{x}^{\sigma\mathbf{u}} : \sigma \in W \rangle$, where W is the set of all permutations $\sigma \in \mathfrak{S}_n$ such that $\sigma(1)$ is arbitrary and $\sigma(j) = k$ for $j > 1$ if either $\sigma(i) = k+1$ or $\sigma(i) = k-1$ for some $i < j$. It is easy to see that $|W| = 2^{n-1}$ and the convex hull of the 2^{n-1} points $\sigma\mathbf{u}$; $\sigma \in W$ is an $(n-1)$-dimensional hypercube $\mathcal{H}(\mathbf{u})$ in $\mathbb{R}^n$. Therefore, the monomial ideal $I_W = \langle \mathbf{x}^{\sigma\mathbf{u}} : \sigma \in W \rangle$ is called a *hypercubic ideal* and henceforth denoted by $J(\mathbf{u})$. It is easy to show that the cellular free complex $\mathbb{F}_*(\mathcal{H}(\mathbf{u}))$ associated to the labeled hypercube $\mathcal{H}(\mathbf{u})$ is the minimal free resolution of $R/J(\mathbf{u})$. An Alexander dual of a (general) monomial ideal was introduced by Miller [8]. We have obtained the following results on the Alexander dual $J(\mathbf{u})^{[\mathbf{u_n}]}$ of the hypercubic ideal $J(\mathbf{u})$ with respect to $\mathbf{u_n} = (u_n, u_n, \ldots, u_n)$.

Theorem 1.1. *The minimal generators of the Alexander dual $J(\mathbf{u})^{[\mathbf{u_n}]}$ of the hypercubic ideal are given by*

$$J(\mathbf{u})^{[\mathbf{u_n}]} = \left\langle \prod_{j \in T} x_j^{\mu_{j,T}} : \emptyset \neq T = \{j_1, j_2, \ldots, j_t\} \subseteq [n];\ j_1 < j_2 < \cdots < j_t \right\rangle,$$

where $\mu_{j_1,T} = u_n - u_t + 1$ and $\mu_{j_i,T} = u_n - u_{t+j_i-i} + 1$ for $i \in \{2, 3, \ldots, t\}$.

For each non-empty set $B \subseteq \mathfrak{S}_n$, let $I_B = \langle \mathbf{x}^{\sigma(\mathbf{u})}; \sigma \in B \rangle$ be a monomial ideal in R. Then the hypercubic ideal $I_W = J(\mathbf{u})$ has the following minimal property.

Theorem 1.2. *If $B \supseteq W$, then a cellular resolution $\mathbb{F}_*(\mathbf{Bd}(\Delta_{n-1}))$ of the quotient $R/I_B^{[\mathbf{u_n}]}$ supported on $\mathbf{Bd}(\Delta_{n-1})$ is the minimal free resolution. On the other hand, if*

$B \subsetneq W$, then a cellular resolution of the quotient $R/I_B^{[\mathbf{u_n}]}$ supported on $\mathbf{Bd}(\Delta_{n-1})$ is always non-minimal.

We defined a notion of *restricted λ-parking function* (see Definition 4.1) so that the dimension $\dim_k\big(R/J(\mathbf{u})^{[\mathbf{u_n}]}\big)$ equals the number of restricted λ-parking functions for $\lambda = (u_n - u_1 + 1, u_n - u_2 + 1, \ldots, u_n - u_n + 1)$. More precisely, the standard monomials of the Artinian quotient $R/J(\mathbf{u})^{[\mathbf{u_n}]}$ are of the form $\mathbf{x}^{\mathbf{p}-\mathbf{1}}$, where $\mathbf{p}$ is a restricted λ-parking function. Using a fine-graded Hilbert series of $R/J(\mathbf{u})^{[\mathbf{u_n}]}$, we obtained a formula for counting restricted λ-parking functions.

Theorem 1.3.

$$\dim_k\big(R/J(\mathbf{u})^{[\mathbf{u_n}]}\big) = \sum_{i=1}^{n}(-1)^{n-i} \sum_{\emptyset \subsetneq A_1 \subsetneq A_2 \subsetneq \ldots \subsetneq A_i = [n]} \prod_{q=1}^{i} \prod_{j \in A_q - A_{q-1}} \mu_{j,\, A_q},$$

where $\mu_{j,\, A_q}$ is as in Theorem 1.1.

The combinatorial formula for dimension $\dim_k\big(R/J(\mathbf{u})^{[\mathbf{u_n}]}\big)$ in Theorem 1.3 is different from a similar formula in Proposition 8.4 of [10].

In the fifth and final section, we have computed the Betti numbers of sum of two multipermutohedron ideals and their Alexander duals. It may be an interesting problem to compute the Betti numbers of powers of a multipermutohedron ideals and determine the dimension $\dim_k\big(\frac{R}{(I(\mathbf{u})^l)^{[l\mathbf{u_n}]}}\big)$ of the Artinian quotient of the Alexander dual of the l-th power $I(\mathbf{u})^l$ of the multipermutohedron ideal $I(\mathbf{u})$ with respect to $l\mathbf{u_n} = (lu_n, lu_n, \ldots, lu_n)$. Since powers of a multipermutohedron ideal can be expressed as a sum of finitely many multipermutohedron ideals, we have studied one simple case of a sum of two multipermutohedron ideals.

2. Split-multipermutohedron ideals and their Alexander duals

For a monomial ideal I in the polynomial ring $k[x_1, x_2, \ldots, x_n]$ and $\mathbf{b} \in \mathbb{N}^n$, we consider the *upper Koszul simplicial complex* $K^{\mathbf{b}}(I) = \{\text{squarefree vectors } \tau : \mathbf{x}^{\mathbf{b}-\tau} \in I\}$ and the *lower Koszul simplicial complex* $K_{\mathbf{b}}(I) = \{\text{squarefree vectors } \tau \preceq \mathbf{b} : \mathbf{x}^{\mathbf{b}'+\tau} \notin I\}$ of I in degree $\mathbf{b}$, where $\mathbf{b}' = \max\{\mathbf{b}-\mathbf{1}, \mathbf{0}\} = (b_1', b_2', \ldots, b_n')$ with $b_i' = \max\{b_i - 1, 0\}$ for all i. The multigraded Betti numbers of I in degree $\mathbf{b}$ are given by

$$\beta_{i,\mathbf{b}}(I) = \dim_k \tilde{H}_{i-1}(K^{\mathbf{b}}(I); k)$$

and

$$\beta_{i-1,\mathbf{b}}(I) = \dim_k \tilde{H}^{|\mathrm{Supp}(\mathbf{b})|-i-1}(K_{\mathbf{b}}(I); k); \quad i \geq 1,$$

where the support $\mathrm{Supp}(\mathbf{b}) = \{i : b_i > 0\}$ (see Theorem 5.11 of [7]).

Now consider a split-multipermutohedron ideal $\mathcal{I} = \langle \mathbf{x}^{\sigma_1 \mathbf{v}} \mathbf{y}^{\sigma_2 \mathbf{w}} : (\sigma_1, \sigma_2) \in \mathrm{H} \rangle$, as defined in the Introduction. Clearly, $\mathcal{I} = I(\mathbf{v}) \otimes_k I(\mathbf{w})$, where $I(\mathbf{v})$ and $I(\mathbf{w})$ are the multipermutohedron ideals. Thus the split-multipermutohedron ideal $\mathcal{I}$ depends on the splitting of the vector $\mathbf{u} = (\mathbf{v}, \mathbf{w})$, but instead of denoting it by a cumbersome notation $\mathcal{I}_{(\mathbf{v},\mathbf{w})}$, we simply write $\mathcal{I}$ without causing any confusion. Let $R_1 = k[x_1, x_2, \ldots, x_r]$ and $R_2 = k[y_1, y_2, \ldots, y_s]$ be polynomial rings over a field k and $R = R_1 \otimes_k R_2 = k[\mathbf{x}] \otimes_k k[\mathbf{y}] \cong k[\mathbf{x}, \mathbf{y}]$. Let $\mathbf{b} \in \mathbb{N}^n$. Then we can write $\mathbf{b} = (\mathbf{b_v}, \mathbf{b_w})$, where $\mathbf{b_v} = (b_1, b_2, \ldots, b_r)$ and $\mathbf{b_w} = (b_{r+1}, b_{r+2}, \ldots, b_n)$.

Lemma 2.1. *The upper Koszul simplicial complex $K^{\mathbf{b}}(\mathcal{I})$ of a split-multipermuto-hedron ideal $\mathcal{I}$ in degree $\mathbf{b} = (\mathbf{b_v}, \mathbf{b_w})$ is given by*

$$K^{\mathbf{b}}(\mathcal{I}) = K^{\mathbf{b_v}}(I(\mathbf{v})) * K^{\mathbf{b_w}}(I(\mathbf{w})),$$

where $*$ denotes the simplicial join.

Proof. For a square-free vector $\tau = (\tau_{\mathbf{v}}, \tau_{\mathbf{w}})$, we have $\tau \in K^{\mathbf{b}}(\mathcal{I})$ if and only if $\tau_{\mathbf{v}} \in K^{\mathbf{b_v}}(I(\mathbf{v}))$ and $\tau_{\mathbf{w}} \in K^{\mathbf{b_w}}(I(\mathbf{w}))$. □

The multigraded Betti numbers of a split-multipermutoheron ideal $\mathcal{I} = I(\mathbf{v}) \otimes_k I(\mathbf{w})$ can be expressed in terms of multigraded Betti numbers of multipermutohedron ideals $I(\mathbf{v})$ and $I(\mathbf{w})$. The multigraded Betti numbers of multipermutohedron ideals and their Alexander duals are described in [3] and [4], respectively. For the sake of completeness, we shall recall relevant results on multipermutohedron ideals and their Alexander duals. Let $e_1, e_2, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. For $0 \leq j \leq i$, set $E(j,i) = \sum_{\alpha=j+1}^{i} e_\alpha \in \mathbb{R}^n$. Consider a multipermutohedron ideal $I(\mathbf{h})$ with $\mathbf{h} = (h_1, h_2, \ldots, h_n) \in \mathbb{N}^n$ as given in the Introduction ($\mathbf{u}$ replaced with $\mathbf{h}$). The multigraded Betti numbers of $I(\mathbf{h})$ are described in terms of the so-called $\mathbf{m_h}$-isolated sets.

DEFINITION 2.2

A subset $J = \{j_1, j_2, \ldots, j_t\} \subseteq [n]$ is said to be $\mathbf{m_h}$-*isolated* if $j_t = n$ and $J \cap (s_{j-1}, s_j]$ is either empty or singleton for $1 \leq j \leq l$, where $(s_{j-1}, s_j] = \{a \in \mathbb{N} : s_{j-1} + 1 \leq a \leq s_j\}$. In other words, for each α there is a unique i_α with $s_{i_\alpha - 1} + 1 \leq j_\alpha \leq s_{i_\alpha}$. For $\mathbf{h} = (h_1, h_2, \ldots, h_n)$, set $\mathbf{b}(J) = \sum_{\alpha=1}^{t} u_{j_\alpha} E(j_{\alpha-1}, j_\alpha)$ and set $\mathbf{m_h}$-weight $wt_{\mathbf{m_h}}(J) = wt_{\mathbf{m_h}}(\mathbf{b}(J)) = \sum_{\alpha=1}^{t} (s_{i_\alpha - 1} - j_{\alpha-1})$, where $j_0 = 0$. The set of all $\mathbf{m_h}$-isolated subsets of $[n]$ is denoted by $\mathcal{I}_{\mathbf{m_h}}$.

For multipermutohedron ideal $I(\mathbf{h})$ the upper Koszul complex $K^{\mathbf{b}}(I(\mathbf{h}))$ is a join of skeletons of simplices and the multigraded Betti numbers $\beta_{i,\mathbf{b}}(I(\mathbf{h}))$ are given as follows. For $J = \{j_1, j_2, \ldots, j_t\} \in \mathcal{I}_{\mathbf{m_h}}$,

$$\beta_{i,\mathbf{b}(J)}(I(\mathbf{h})) = \left[\prod_{\alpha=1}^{t} \binom{j_\alpha - j_{\alpha-1} - 1}{s_{i_\alpha - 1} - j_{\alpha-1}}\right] \delta_{i, wt_{\mathbf{m_h}}(J)},$$

where $J \cap (s_{i_\alpha - 1}, s_{i_\alpha}] = \{j_\alpha\}$ and $\delta_{i,q}$ is the Kronecker delta. If π is a permutation of $\mathbf{b}(J)$, then $\beta_{i,\pi\mathbf{b}(J)}(I(\mathbf{h})) = \beta_{i,\mathbf{b}(J)}(I(\mathbf{h}))$. Also, $\beta_{i,\mathbf{b}}(I(\mathbf{h})) = 0$ if $\mathbf{b} \neq \pi\mathbf{b}(J)$ for all $J \in \mathcal{I}_{\mathbf{m_h}}$ and any permutation π of $\mathbf{b}(J)$. From the above formula, it is clear that the multigraded Betti numbers $\beta_{i,\mathbf{b}(J)}(I(\mathbf{h}))$ could be non-zero if and only if $i = wt_{\mathbf{m_h}}(\mathbf{b}(J))$, where $wt_{\mathbf{m_h}}(J) - 1 = \dim K^{\mathbf{b}(J)}(I(\mathbf{h}))$.

On similar lines, multigraded Betti numbers of an Alexander dual $I(\mathbf{h})^{[\mathbf{h_n}+\mathbf{c}-\mathbf{1}]}$ of multipermutohedron ideal $I(\mathbf{h})$ are described in [4].

DEFINITION 2.3

Let $J = \{j_1, j_2, \ldots, j_t\} \subseteq [n]$ with $0 = j_0 < j_1 < j_2 < \cdots < j_t \leq n$. Then J is said to be *dual* $\mathbf{m_h}$-*isolated* if $J \cap (s_{j-1}, s_j]$ is either empty or singleton for $1 \leq j \leq l$. Thus for each α, there is a unique i_α with $s_{i_\alpha - 1} + 1 \leq j_\alpha \leq s_{i_\alpha}$. For $\mathbf{h} = (h_1, h_2, \ldots, h_n)$, set $\widetilde{\mathbf{b}}(J) = \sum_{\alpha=1}^{t} \lambda_{j_\alpha} E(j_{\alpha-1}, j_\alpha)$, $\lambda_i = h_n - h_i + c$ and set *dual* $\mathbf{m_h}$-weight $dwt_{\mathbf{m_h}}(J) = dwt_{\mathbf{m_h}}$

$(\tilde{\mathbf{b}}(J)) = \left[\sum_{\alpha=1}^{t}(j_\alpha - s_{i_\alpha - 1})\right] - 1$. Also, the size of the support $|\mathrm{Supp}(\tilde{\mathbf{b}}(J))| = j_t$. The set of all dual $\mathbf{m_h}$-isolated subsets of $[n]$ is denoted by $\mathcal{I}^*_{\mathbf{m_h}}$. If $J \subseteq [n]$ is a dual $\mathbf{m_h}$-isolated subset with $dwt_{\mathbf{m_h}}(J) = i$, we write $J \in \mathcal{I}^*_{\mathbf{m_h}}(\langle i \rangle)$.

The lower Koszul simplicial complex $K_{\tilde{\mathbf{b}}(J)}(I(\mathbf{h})^{[\mathbf{h_n}+\mathbf{c}-\mathbf{1}]})$ of an Alexander dual $I(\mathbf{h})^{[\mathbf{h_n}+\mathbf{c}-\mathbf{1}]}$ of a multipermutohedron ideal $I(\mathbf{h})$ is a join of skeletons of simplices and $\dim K_{\tilde{\mathbf{b}}(J)}(I(\mathbf{h})^{[\mathbf{h_n}+\mathbf{c}-\mathbf{1}]}) = j_t - dwt_{\mathbf{m_h}}(J) - 2$. The multigraded Betti numbers of $I(\mathbf{h})^{[\mathbf{h_n}+\mathbf{c}-\mathbf{1}]}$ are given as follows. For $J = \{j_1, j_2, \ldots, j_t\} \in \mathcal{I}^*_{\mathbf{m_h}}$,

$$\beta_{i-1,\tilde{\mathbf{b}}(J)}(I(\mathbf{h})^{[\mathbf{h_n}+\mathbf{c}-\mathbf{1}]}) = \left[\prod_{\alpha=1}^{t}\binom{j_\alpha - j_{\alpha-1} - 1}{s_{i_\alpha - 1} - j_{\alpha-1}}\right]\delta_{i-1,dwt_{\mathbf{m_h}}(J)},$$

where $J \cap (s_{i_\alpha - 1}, s_{i_\alpha}] = \{j_\alpha\}$. If π is a permutation of $\tilde{\mathbf{b}}(J)$, then we have $\beta_{i-1,\pi\tilde{\mathbf{b}}(J)}(I(\mathbf{h})^{[\mathbf{h_n}+\mathbf{c}-\mathbf{1}]}) = \beta_{i-1,\tilde{\mathbf{b}}(J)}(I(\mathbf{h})^{[\mathbf{h_n}+\mathbf{c}-\mathbf{1}]})$. If $\mathbf{b} \neq \pi\tilde{\mathbf{b}}(J)$, for all $J \in \mathcal{I}^*_{\mathbf{m_h}}(\langle i-1 \rangle)$ and any permutation π of $\tilde{\mathbf{b}}(J)$, then $\beta_{i-1,\mathbf{b}}(I(\mathbf{h})^{[\mathbf{h_n}+\mathbf{c}-\mathbf{1}]}) = 0$.

The Künneth formula for the homology vector space of the join $\Sigma_1 * \Sigma_2$ of two simplicial complexes Σ_1 and Σ_2 is given by

$$\tilde{H}_{i-1}(\Sigma_1 * \Sigma_2; k) = \bigoplus_{p+q=i-2} \tilde{H}_p(\Sigma_1; k) \otimes_k \tilde{H}_q(\Sigma_2; k).$$

We now describe the multigraded Betti numbers of split-multipermutohedron ideals and their Alexander duals.

PROPOSITION 2.4

The multigraded Betti numbers of a split-multipermutohedron ideal $\mathcal{I}$ exist only in the multidegree $\mathbf{b}$ of the form $\mathbf{b} = (\mathbf{b_v}, \mathbf{b_w})$ and

$$\beta_{i,\mathbf{b}}(\mathcal{I}) = \beta_{p,\mathbf{b_v}}(I(\mathbf{v}))\beta_{q,\mathbf{b_w}}(I(\mathbf{w})),$$

where $\mathbf{b_v} = \mathbf{b}(J), \mathbf{b_w} = \mathbf{b}(J')$ for $J \in \mathcal{I}_{\mathbf{m_v}}$, $J' \in \mathcal{I}_{\mathbf{m_w}}$ and $wt_{\mathbf{m_v}}(\mathbf{b_v}) = p$, $wt_{\mathbf{m_w}}(\mathbf{b_w}) = q$ with $p + q = i$.

Proof. In view of the above discussion, it is sufficient to take $\mathbf{b} = (\mathbf{b_v}, \mathbf{b_w})$, where $\mathbf{b_v} = \mathbf{b}(J)$, $\mathbf{b_w} = \mathbf{b}(J')$ for $J \in \mathcal{I}_{\mathbf{m_v}}$, $J' \in \mathcal{I}_{\mathbf{m_w}}$. From Lemma 2.1, we have $K^{\mathbf{b}}(\mathcal{I}) = K^{\mathbf{b_v}}(I(\mathbf{v})) * K^{\mathbf{b_w}}(I(\mathbf{w}))$. Thus using Künneth formula for the join of simplicial complexes, we have $\beta_{i,\mathbf{b}}(\mathcal{I}) = \sum_{p+q=i}\beta_{p,\mathbf{b_v}}(I(\mathbf{v}))\beta_{q,\mathbf{b_w}}(I(\mathbf{w}))$. Also $\beta_{p,\mathbf{b_v}}(I(\mathbf{v})) = 0$ if $p \neq wt_{\mathbf{m_v}}(\mathbf{b_v})$. □

Remark 2.5. From the last proposition, i-th Betti number $\beta_i(\mathcal{I})$ can easily be calculated. We have

$$\beta_i(\mathcal{I}) = \sum_{\mathbf{b}}\beta_{i,\mathbf{b}}(\mathcal{I}) = \sum_{\mathbf{b}=(\mathbf{b_v},\mathbf{b_w})}\sum_{p+q=i}(\beta_{p,\mathbf{b_v}}(I(\mathbf{v}))\beta_{q,\mathbf{b_w}}(I(\mathbf{w})))$$

$$= \sum_{p+q=i}\left(\sum_{\mathbf{b_v}}\beta_{p,\mathbf{b_v}}(I(\mathbf{v}))\right)\left(\sum_{\mathbf{b_w}}\beta_{q,\mathbf{b_w}}(I(\mathbf{w}))\right) = \sum_{p+q=i}\beta_p(I(\mathbf{v}))\beta_q(I(\mathbf{w})).$$

However, the above result on Betti numbers $\beta_i(\mathcal{I})$ of split-multipermutohedron ideals is valid for a large class of ideals. Let I_1 and I_2 be ideals in polynomial rings $k[x_1, x_2, \dots, x_r]$ and $k[y_1, y_2, \dots, y_s]$, respectively. Suppose $\mathbb{F}_* \to I_1$ and $\mathbb{G}_* \to I_2$ are minimal free resolutions. Then it is known that the tensor complex $\mathbb{F}_* \otimes_k \mathbb{G}_*$ is the minimal free resolution of $I_1 \otimes_k I_2$. Hence formula for the i-th Betti number of $\mathcal{I}$ holds in general for ideals of the form $I_1 \otimes_k I_2$.

For $\mathbf{u} = (\mathbf{v}, \mathbf{w})$, set $\mathbf{v_r} = (u_r, u_r, \dots, u_r) \in \mathbb{N}^r$, $\mathbf{w_s} = (u_n, u_n, \dots, u_n) \in \mathbb{N}^s$ and $\mathbf{a} = (\mathbf{v_r}, \mathbf{w_s}) = (u_r, u_r, \dots, u_r, u_n, u_n, \dots, u_n)$. Now consider the Alexander dual $\mathcal{I}^{[\mathbf{a}]}$ of split-multipermutohedron ideal $\mathcal{I}$ with respect to $\mathbf{a}$.

Lemma 2.6. *The Alexander dual of $\mathcal{I}$ with respect to* $\mathbf{a}$ *is given by*

$$\mathcal{I}^{[\mathbf{a}]} = I(\mathbf{v})^{[\mathbf{v_r}]} \otimes_k R_2 + R_1 \otimes_k I(\mathbf{w})^{[\mathbf{w_s}]}.$$

Proof. Our aim is to find a maximal vector $\mathbf{b}$ with $\mathbf{b} \preceq \mathbf{a}$ such that $\mathbf{x}^{\mathbf{b}} \notin \mathcal{I}$ (see Proposition 5.23 of [7]). Thus either $\mathbf{b} = (\mathbf{b_1}, \mathbf{w_s})$, where $\mathbf{b_1}$ is maximal with $\mathbf{x}^{\mathbf{b_1}} \notin I(\mathbf{v})$ or $\mathbf{b} = (\mathbf{v_r}, \mathbf{b_2})$, where $\mathbf{b_2}$ is maximal with $\mathbf{x}^{\mathbf{b_2}} \notin I(\mathbf{w})$. □

As in Lemma 2.1, it can be easily verified that the lower Koszul simplicial complex $K_{\mathbf{b}}(\mathcal{I}^{[\mathbf{a}]})$ of $\mathcal{I}^{[\mathbf{a}]}$ in degree $\mathbf{b} = (\mathbf{b_v}, \mathbf{b_w})$ is given by $K_{\mathbf{b}}(\mathcal{I}^{[\mathbf{a}]}) = K_{\mathbf{b_v}}(I(\mathbf{v})^{[\mathbf{v_r}]}) * K_{\mathbf{b_w}}(I(\mathbf{w})^{[\mathbf{w_s}]})$, where $*$ denotes the simplicial join.

Theorem 2.7. *The multigraded Betti numbers of $R/\mathcal{I}^{[\mathbf{a}]}$ exist only in the multidegree* $\mathbf{b}$ *of the form* $\mathbf{b} = (\mathbf{b_v}, \mathbf{b_w})$ *and*

$$\beta_{i,\mathbf{b}}(R/\mathcal{I}^{[\mathbf{a}]}) = \beta_{p,\mathbf{b_v}}(R_1/I(\mathbf{v})^{[\mathbf{v_r}]})\beta_{q,\mathbf{b_w}}(R_2/I(\mathbf{w})^{[\mathbf{w_s}]}),$$

where $\mathbf{b_v} = \tilde{\mathbf{b}}(J)$, $\mathbf{b_w} = \tilde{\mathbf{b}}(J')$ *for* $J \in \mathcal{I}^*_{\mathbf{m_v}}$, $J' \in \mathcal{I}^*_{\mathbf{m_w}}$ *and* $p = dwt_{\mathbf{m_v}}(\mathbf{b_v})$, $q = dwt_{\mathbf{m_w}}(\mathbf{b_w})$ *with* $p + q = i$.

Proof. In view of the discussion of Lemma 2.1, it is sufficient to take $\mathbf{b} = (\mathbf{b_v}, \mathbf{b_w})$, where $\mathbf{b_v} = \widetilde{\mathbf{b}}(J)$, $\mathbf{b_w} = \widetilde{\mathbf{b}}(J')$ for $J \in \mathcal{I}^*_{\mathbf{m_v}}$, $J' \in \mathcal{I}^*_{\mathbf{m_w}}$. We have $K_{\mathbf{b}}(\mathcal{I}^{[\mathbf{a}]}) = K_{\mathbf{b_v}}(I(\mathbf{v})^{[\mathbf{v_r}]}) * K_{\mathbf{b_w}}(I(\mathbf{w})^{[\mathbf{w_s}]})$. Now using Künneth formula, we have $\beta_{i,\mathbf{b}}(R/\mathcal{I}^{[\mathbf{a}]}) = \sum_{p+q=i} \beta_{p,\mathbf{b_v}}(R_1/I(\mathbf{v})^{[\mathbf{v_r}]})\beta_{q,\mathbf{b_w}}(R_2/I(\mathbf{w})^{[\mathbf{w_s}]})$. Also $\beta_{p,\mathbf{b_v}}(\mathcal{I}^{[\mathbf{a}]}) = 0$ if $p \neq dwt_{\mathbf{m_v}}(\mathbf{b_v})$. □

Remark 2.8.

(1) As in the case of split-multipermutohedron ideal $\mathcal{I}$, we can prove that

$$\beta_i(R/\mathcal{I}^{[\mathbf{a}]}) = \sum_{p+q=i} \beta_p(R_1/I(\mathbf{v})^{[\mathbf{v_r}]})\beta_q(R_2/I(\mathbf{w})^{[\mathbf{w_s}]}).$$

It follows from the above formula that if the cellular free resolution of $R_1/I(\mathbf{v})^{[\mathbf{v_r}]}$ supported on a (labeled) polyhedral cell complex $\triangle_1$ and the cellular free resolution of $R_2/I(\mathbf{w})^{[\mathbf{w_s}]}$ supported on a (labeled) polyhedral cell complex $\triangle_2$ are both minimal, then the cellular free complex supported on the join $\triangle_1 * \triangle_2$ gives the minimal free resolution of $R/\mathcal{I}$.

(2) The dimension $\dim_k(R/\mathcal{I}^{[\mathbf{a}]})$ of an Artinian quotient $R/\mathcal{I}^{[\mathbf{a}]}$ equals the product $\big(\dim_k(R_1/I(\mathbf{v})^{[\mathbf{v_r}]})\big)\big(\dim_k(R_2/I(\mathbf{w})^{[\mathbf{w_s}]})\big)$, where each of the two factors is given by the Steck determinant formula (see Theorem 2.8 of [4]).

3. Hypercubic ideals and their Alexander duals

Let $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{N}^n$, $1 \leq u_1 < u_2 < \ldots < u_n$. For each non empty set $B \subseteq \mathfrak{S}_n$, we associate a monomial ideal $I_B = \langle \mathbf{x}^{\sigma\mathbf{u}} : \sigma \in B \rangle$. Let $I_W = J(\mathbf{u})$ be the *hypercubic ideal* corresponding to the subset $W \subset \mathfrak{S}_n$ given by

$$W = \{\sigma \in \mathfrak{S}_n : \sigma(1) \text{ is arbitrary and } \sigma(j) = k \text{ for } j > 1 \text{ if either } \sigma(i) = k+1 \text{ or } \sigma(i) = k-1 \text{ for some } i < j\},$$

as in the Introduction. In other words, W consists of all permutations $\sigma \in \mathfrak{S}_n$ such that apart from the first term, a number k appears in the word representation $\sigma(1)\sigma(2)\ldots\sigma(n)$ of σ if either $k+1$ or $k-1$ has already appeared before k. It can be shown that $|W| = 2^{n-1}$. The convex polytope spanned by 2^{n-1} points $\sigma\mathbf{u}, \sigma \in W$ is a $(n-1)$-dimensional hypercube and we denote it by $\mathcal{H}(\mathbf{u})$. A vertex $\sigma\mathbf{u}$ of $\mathcal{H}(\mathbf{u})$ is naturally labeled with monomial $\mathbf{x}^{\sigma\mathbf{u}}$. Thus the monomial ideal $J(\mathbf{u})$ is generated by vertex labels of the $(n-1)$-dimensional hypercube, which is a reason for calling it a *hypercubic ideal*. We shall associate a cellular complex $\mathbb{F}_*(\mathcal{H}(\mathbf{u}))$ of free R-modules to $\mathcal{H}(\mathbf{u})$. More generally, to each labeled simplicial complex or a labeled polyhedral cell complex $\mathbf{X}$, one can associate a free complex of R-modules [1, 2]. Let $\mathcal{F}_{i-1}(\mathbf{X})$ (or simply $\mathcal{F}_{i-1}$) be the set of $i-1$-faces of $\mathbf{X}$. Then the associated free complex $\mathbb{F}_*(\mathbf{X})$ is given by

$$\mathbb{F}_*(\mathbf{X}) : \cdots \longrightarrow F_i \xrightarrow{\delta_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0, \tag{3.1}$$

where $F_i = \bigoplus_{\sigma \in \mathcal{F}_{i-1}(\mathbf{X})} R[-\nu(\sigma)]$, monomial $\mathbf{x}^{\nu(\sigma)}$ is the label on the face σ and δ_i is a differential.

Now we proceed to systematically study hypercubic ideals. The following result is known, although its simple proof is included for lack of proper reference.

Lemma 3.1. *There is a one-to-one correspondence between the set W and the power set* $\mathbf{P}[n-1]$ *of* $[n-1] = \{1, 2, \ldots, n-1\}$.

Proof. Let B = set of $(n-1)$-tuples consisting of '0' and '1'. Clearly $|B| = 2^{n-1}$. Define $f : W \to B$ as follows: For $\sigma \in W$, $f(\sigma) \in B$ is given by

$$(f(\sigma))(i) = i\text{-th coordinate of } f(\sigma) = \begin{cases} 1, & \text{if } \sigma(i) > \sigma(i+1), \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathbf{b} = (b_1, b_2, \ldots, b_{n-1}) \in B$ and $\sum_{i=0}^{n-1} b_i = l$. Then define $g : B \to W$ as follows:

$$(g(\mathbf{b}))(1) = l+1, \text{ and for } j \geq 2,$$

$$(g(\mathbf{b}))(j) = \begin{cases} \max\limits_{1\leq k\leq j-1} (g(\mathbf{b}))(k) + 1 & \text{if } b_{j-1} = 0, \\ \min\limits_{1\leq k\leq j-1} (g(\mathbf{b}))(k) - 1 & \text{if } b_{j-1} = 1. \end{cases}$$

Clearly $g \circ f = 1_W$ and $f \circ g = 1_B$. □

We now give facial description of a hypercube. The Hasse diagram of the Boolean poset $(\mathbf{P}[n-1], \subseteq)$ constitutes of vertices and edges of an $(n-1)$-dimensional hypercube. The faces of the hypercube can be described by chains of subsets of $[n-1]$. A 0-dimensional face (vertex) of a hypercube corresponds to a subset of $[n-1]$. An 1-dimensional face (edge) corresponds to a chain $A_0 \subsetneq A_1 \subseteq [n-1]$ such that $|A_1| - |A_0| = 1$. In general, for $1 \leq d \leq n-1$, a d-dimensional face corresponds to a chain $A_0 \subsetneq A_d \subseteq [n-1]$ such that $|A_d| - |A_0| = d$. As the number of ways of choosing d elements out of $n-1$ elements is $\binom{n-1}{d}$ and the total number of subsets of $[n-1-d]$ is 2^{n-1-d}, the number of d-dimensional faces of a hypercube is $2^{n-1-d}\binom{n-1}{d}$.

Theorem 3.2. *The cellular resolution $\mathbb{F}_*(\mathcal{H}(\mathbf{u}))$ supported on the hypercube $\mathcal{H}(\mathbf{u})$ is the minimal free resolution of the hypercubic ideal $J(\mathbf{u})$.*

Proof. A vertex $\sigma\mathbf{u}$ in the hypercube $\mathcal{H}(\mathbf{u})$ is naturally labeled with monomial $\mathbf{x}^{\sigma\mathbf{u}}$ and monomial label on each face F is the LCM of monomial labels on vertices of F. Thus $\mathcal{H}(\mathbf{u})$ is a labeled polyhedral cell complex. It is clear that for any vector $\mathbf{b} \in \mathbb{N}^n$, either $\mathcal{H}(\mathbf{u})_{\preceq\mathbf{b}}$ is contractible or void. Also, the label on a face of $\mathcal{H}(\mathbf{u})$ is different from labels on its proper subfaces. Thus the cellular free complex $\mathbb{F}_*(\mathcal{H}(\mathbf{u}))$ is the minimal free resolution of $J(\mathbf{u})$ (see Proposition 1.2 of [2]). □

Theorem 3.3. *The minimal generators of the Alexander dual $J(\mathbf{u})^{[\mathbf{u_n}]}$ of the hypercubic ideal $J(\mathbf{u})$ are given by*

$$J(\mathbf{u})^{[\mathbf{u_n}]} = \left\langle \prod_{j\in T} x_j^{\mu_{j,T}} : \emptyset \neq T = \{j_1, j_2, \ldots, j_t\} \subseteq [n];\ \ j_1 < j_2 < \cdots < j_t \right\rangle,$$

where $\mu_{j_1,T} = u_n - u_t + 1$ and $\mu_{j_i,T} = u_n - u_{t+j_i-i} + 1$ for $i \in \{2, 3, \ldots, t\}$.

Proof. Let $T \subseteq [n]$ be a non-empty subset such that $T = \{j_1, j_2, \ldots, j_t\}$, where $j_1 < j_2 < \cdots < j_t$. Consider the vector

$$\mathbf{b}_T = \sum_{j\notin T} u_n e_j + (u_t - 1)e_{j_1} + \sum_{\alpha=2}^{t} (u_{t+j_\alpha-\alpha} - 1)e_{j_\alpha}. \tag{3.2}$$

Since $1 \leq j_1 < j_2 < \cdots < j_t \leq n$, we see that $j_2 - 2 \leq j_3 - 3 \leq \cdots \leq j_t - t$ and hence $t + j_\alpha - \alpha \leq j_t \leq n$ for $2 \leq \alpha \leq t$. Thus $\mathbf{b}_T \preceq \mathbf{u_n}$.

Claim. $\mathbf{x}^{\mathbf{b}_T} \notin J(\mathbf{u})$.

In order to prove this claim, we express T as a disjoint union of integer intervals. For $a, b \in \mathbb{Z}$, the set $[a, b] = \{x \in \mathbb{Z} : a \leq x \leq b\}$ is an integer interval. Let

$$T = [j_1, j_{n_1}] \amalg [j_{n_1+1}, j_{n_2}] \amalg \ldots \amalg [j_{n_{r-1}+1}, j_{n_r}] = T_1 \amalg T_2 \amalg \ldots \amalg T_r,$$

where $T_i = [j_{n_{i-1}+1}, j_{n_i}] = \{j_{n_{i-1}+1}, j_{n_{i-1}+2}, \ldots, j_{n_i}\}$ for $i = 1, 2, \ldots, r$; $n_0 = 0$ and $n_r = t$. Set $S = [1, n] - T$ and write S also as a disjoint union of integer intervals, i.e., $S = S_0 \amalg S_1 \amalg \ldots \amalg S_r$, where $S_\alpha = [j_{n_\alpha} + 1, j_{n_{\alpha+1}} - 1];\ 0 \leq \alpha \leq r$. Clearly, $S_0 = \emptyset$ if and only if $j_1 = 1$ and $S_r = \emptyset$ if and only if $j_t = n$. Further, $S_\alpha \neq \emptyset$ for $1 \leq \alpha \leq r-1$. Since $[1, n] - T = S$, we have

$$n = \sum_{i=0}^{r} |S_i| + t. \tag{3.3}$$

Also, $[1, j_{n_\alpha+1} - 1] = \left(\amalg_{i=1}^{\alpha} T_i\right) \amalg \left(\amalg_{i=0}^{\alpha} S_i\right)$ and $\amalg_{i=1}^{\alpha} T_i = \{j_1, j_2, \ldots, j_{n_\alpha}\}$ implies that

$$j_{n_\alpha+1} - 1 = n_\alpha + \sum_{i=0}^{\alpha} |S_i|. \tag{3.4}$$

We note that the vector $\mathbf{b}_T$ can also be expressed as

$$\begin{aligned} \mathbf{b}_T &= \sum_{j \notin T} u_n e_j + (u_t - 1)e_{j_1} + \sum_{j_1 < j \leq j_{n_1}} (u_{t+j_1-1} - 1)e_j \\ &\quad + \sum_{\alpha=2}^{r} \sum_{j \in T_\alpha} (u_{t+j_{n_\alpha}-n_\alpha} - 1)e_j. \end{aligned}$$

Suppose, if possible, $\mathbf{x}^{\mathbf{b}_T} \in J(\mathbf{u})$. Then there exists a vector $\mathbf{c} = \sum_{i=1}^{n} c_i e_i = \sigma \mathbf{u}$, $\sigma \in W$ such that $\mathbf{b}_T \succeq \mathbf{c}$. Since $\mathbf{c} = \sigma \mathbf{u}$ for some $\sigma \in W$, there exists some $i_0 \in [1, n]$ such that $c_{i_0} = u_n$. Also j-th coordinate $(\mathbf{b}_T)_j$ of the vector $\mathbf{b}_T$ is u_n if and only if $j \in S = [1, n] - T$. Thus $i_0 \in S_\alpha$ for some $0 \leq \alpha \leq r$. We now proceed to show that

$$\max_{j \in S_0} c_j \;\geq\; u_{n-(|S_1|+|S_2|+\cdots+|S_\alpha|)} \quad \text{if } S_0 \neq \emptyset \tag{3.5}$$

or, in the other case,

$$\max_{j \in S_1} c_j \;\geq\; u_{n-(|S_2|+\cdots+|S_\alpha|)} \qquad \text{if } S_0 = \emptyset. \tag{3.6}$$

Assuming the contrary, suppose $\max_{j \in S_0} c_j < u_{n-(|S_1|+|S_2|+\cdots+|S_\alpha|)}$ if $S_0 \neq \emptyset$ or $\max_{j \in S_1} c_j < u_{n-(|S_2|+\cdots+|S_\alpha|)}$ if $S_0 = \emptyset$. For any $\beta \leq \alpha$, from (3.3) and (3.4), we have

$$\begin{aligned} n - (|S_\beta| + |S_{\beta+1}| + \cdots + |S_\alpha|) \;&\geq\; n - \sum_{i=\beta}^{r} |S_i| \\ &= t + \sum_{i=0}^{\beta-1} |S_i| = t + j_{n_{\beta-1}+1} - 1 - n_{\beta-1}. \end{aligned}$$

Thus $u_{n-(|S_\beta|+\cdots+|S_\alpha|)} \geq u_{t+j_{n_{\beta-1}+1}-(n_{\beta-1}+1)} > u_{t+j_{n_{\beta-1}+1}-(n_{\beta-1}+1)} - 1$. Since the j-th coordinate of $\mathbf{b}_T$ is $(\mathbf{b}_T)_j = u_{t+j_{n_{\beta-1}+1}-(n_{\beta-1}+1)} - 1$ for $j \in T_\beta$, $\beta \geq 1$ (except $j = j_1$) and $\mathbf{c} \preceq \mathbf{b}_T$, we have $c_j \leq u_{t+j_{n_{\beta-1}+1}-(n_{\beta-1}+1)} - 1 < u_{n-(|S_\beta|+\cdots+|S_\alpha|)}$ for $j \in T_\beta$ (except $j = j_1$). Further $c_{j_1} < u_t < u_{t+|S_0|} \leq u_{n-(|S_1|+\cdots+|S_\alpha|)}$. Thus

$$c_j < u_{n-(|S_\beta|+\cdots+|S_\alpha|)} \text{ for } j \in T_\beta,\, 1 \leq \beta \leq r. \tag{3.7}$$

Suppose $S_0 \neq \emptyset$ and $\max_{j \in S_0} c_j = u_{m_0}$. Then $u_{m_0} < u_{n-(|S_1|+\cdots+|S_\alpha|)}$. Also, using (3.7), we get $c_j < u_{n-(|S_1|+\cdots+|S_\alpha|)}$ for $j \in T_1$. Thus the maximum value of c_j for $j \in [1, j_{n_1}] = S_0 \cup T_1$ will remain less than $u_{n-(|S_1|+\cdots+|S_\alpha|)}$, i.e.,

$$\max_{j \in [1, j_{n_1}]} c_j = u_{m_1} < u_{n-(|S_1|+\cdots+|S_\alpha|)}.$$

As $\mathbf{c} = \sigma\mathbf{u}$, for some $\sigma \in W$, we have $c_j = u_\alpha$ for $j > 1$ if and only if c_k is either $u_{\alpha-1}$ or $u_{\alpha+1}$ for some $k < j$. If $c_{j_{n_1}+1} = u_{m_1+1}, c_{j_{n_1}+2} = u_{m_1+2}, \ldots, c_{j_{n_1+1}-1} = u_{m_1+(j_{n_1+1}-j_{n_1}-1)} = u_{m_1+|S_1|}$, then $\max_{j\in[1,j_{n_1+1}-1]} c_j = u_{m_1+|S_1|}$. Since the values of c_j as j varies over S_1 can not increase faster than the sequence $(u_{m_1+1}, \ldots, u_{m_1+|S_1|})$, we must have $\max_{j\in[1,j_{n_1+1}-1]} c_j = u_{m_1'}$, where $m_1' \leq m_1 + |S_1|$. Thus, we have

$$u_{m_1'} \leq u_{m_1+|S_1|} < u_{n-(|S_2|+\cdots+|S_\alpha|)}.$$

Since $c_j < u_{n-(|S_2|+\cdots+|S_\alpha|)}$ for $j \in T_2$, $\max_{j\in[1,j_{n_2}]} c_j = u_{m_2} < u_{n-(|S_2|+\cdots+|S_\alpha|)}$.

Again, if $c_{j_{n_2}+1} = u_{m_2+1}, c_{j_{n_2}+2} = u_{m_2+2}, \ldots, c_{j_{n_2+1}} - 1 = u_{m_2+|S_2|}$, then we see that $\max_{j\in[1,j_{n_2+1}-1]} c_j = u_{m_2+|S_2|}$. Since the values of c_j as j varies over S_2 can not increase faster than the sequence $(u_{m_2+1}, \ldots, u_{m_2+|S_2|})$, we must have $\max_{j\in[1,j_{n_2+1}-1]} c_j = u_{m_2'}$, for some $m_2' \leq m_2 + |S_2|$. Thus

$$u_{m_2'} \leq u_{m_2+|S_2|} < u_{n-(|S_3|+\cdots+|S_\alpha|)}.$$

Repeating this argument, we conclude that $\max_{j\in[1,j_{n_\alpha+1}-1]} c_j < u_n$, which contradicts the fact that $c_{i_0} = u_n$ for $i_0 \in S_\alpha \subseteq [1, j_{n_\alpha+1} - 1]$. Hence we must have

$$\max_{j\in S_0} c_j \;\geq\; u_{n-(|S_1|+\cdots+|S_\alpha|)} \quad \text{if } S_0 \neq \emptyset.$$

Similarly, the inequality $\max_{j\in S_1} c_j \geq u_{n-(|S_2|+\cdots+|S_\alpha|)}$ if $S_0 = \emptyset$ can also be proved. For $S_0 \neq \emptyset$, we have $\max_{j\in S_0} c_j \geq u_{n-(|S_1|+\cdots+|S_\alpha|)} \geq u_{n-(\sum_{i=1}^r |S_i|)} = u_{t+|S_0|}$. This is possible only if $c_j \geq u_{t+1}$ for all $j \in S_0 = [1, j_1 - 1]$. As $c_{j_1} = u_t - 1$ and $c_j > u_t$ for $j < j_1$, we arrive at a contradiction. On the other hand, for $S_0 = \emptyset$, we have $\max_{j\in S_1} c_j \geq u_{n-(|S_2|+\cdots+|S_\alpha|)} \geq u_{n-(\sum_{i=2}^r |S_i|)} = u_{t+|S_1|}$. This shows that $c_j \geq u_{t+1}$ for all $j \in S_1 = [j_1 + 1, j_{n_1} - 1]$. Since $c_j \leq u_t - 1 < u_t$ for all $j \in T_1 = [1, j_1]$, we get a contradiction again. Therefore, $\mathbf{x}^{\mathbf{b}_T} \notin J(\mathbf{u})$. This proves the claim.

We now proceed to show that $\mathbf{b}_T \preceq \mathbf{u_n}$ is a maximal vector such that $\mathbf{x}^{\mathbf{b}_T} \notin J(\mathbf{u})$. In other words, we shall prove that $\mathbf{x}^{\mathbf{b}_T+e_j} \in J(\mathbf{u})$ for any $j \in T$.

Let $j \in T$. Then we construct a vector $\mathbf{c} = (c_1, c_2, \ldots, c_n) \in \mathbb{N}^n$ as follows. For $i \in T - \{j\}$, $c_i = u_{\mathrm{rank}(i)}$, where $\mathrm{rank}(i)$ is the rank of an element 'i' in the set $T - \{j\}$. If $A = \{a_1, a_2, \ldots, a_l\}$ with $a_1 < a_1 < \cdots < a_l$, then the rank of an element is given by $\mathrm{rank}(a_k) = l - k + 1;\ 1 \leq k \leq l$. Also, $c_j = (\mathbf{b}_T)_j + 1$ and the value of c_i for $i \notin T$ are obtained by arranging remaining $u_\alpha{}'s$ in an increasing order. We illustrate the choice of vector $\mathbf{c}$ by an example. Let $n = 7$ and $T = \{2, 3, 5, 6\}$. Then $\mathbf{b}_T = (u_7, u_4 - 1, u_5 - 1, u_7, u_6 - 1, u_6 - 1, u_7)$. Let $j = 5$. Then $\mathbf{b}_T + e_5 = (u_7, u_4 - 1, u_5 - 1, u_7, u_6, u_6 - 1, u_7)$ and $c_6 = u_1, c_3 = u_2, c_2 = u_3, c_5 = u_6, c_1 = u_4, c_4 = u_5, c_7 = u_7$. Let $T = \{j_1, j_2, \ldots, j_t\}$ and $j = j_k \in T$. Then

$$c_{j_k} = \begin{cases} u_t; & k = 1, \\ u_{t+j_k-k}; & k > 1, \end{cases}$$

and for $i \neq k$,

$$c_{j_i} = \begin{cases} u_{t-i+1}; & i > k, \\ u_{t-i}; & i < k. \end{cases}$$

As $(\mathbf{b}_T)_{j_i} = u_{t+j_i-i} - 1 \geq u_{t+j_i-i-1} \geq c_{j_i}$ and

$$(\mathbf{b}_T + e_{j_k})_{j_k} = \begin{cases} u_t; & k = 1, \\ u_{t+j_k-k}; & k > 1, \end{cases}$$

we deduce that $\mathbf{b}_T + e_j \succeq \mathbf{c}$. It is clear that $u_1, u_2, \ldots, u_{t-1}$ appear in the vector $\mathbf{c}$ in a decreasing order at '$t-1$' places $j_1, j_2, \ldots, j_{k-1}, j_{k+1}, \ldots, j_t$. Thus if u_α for $1 \leq \alpha \leq t-1$ appears at the ith position in the vector $\mathbf{c}$ for $i > 1$, then certainly $u_{\alpha-1}$ appears in $\mathbf{c}$ before the ith position. Also the values of c_j for $j \notin T$ are obtained by putting the remaining u_i's in an increasing order. This shows that $\mathbf{c} = \sigma\mathbf{u}$ for some $\sigma \in W$ and $\mathbf{x}^{\mathbf{c}} \in J(\mathbf{u})$. Thus $\mathbf{x}^{\mathbf{b}_T + e_j} \in J(\mathbf{u})$. Hence $\mathbf{x}^{\mathbf{u_n} - \mathbf{b}_T}$ is a minimal generator of $J(\mathbf{u})^{[\mathbf{u_n}]}$ (see Proposition 5.23 of [7]). □

It is easy to see that $J(\mathbf{u})^{[\mathbf{u_n}]}$ is a strictly monotone monomial ideal and thus a cellular free complex $\mathbb{F}_*(\mathbf{Bd}(\Delta_{n-1}))$ associated to the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ of an $n-1$-simplex Δ_{n-1} is the minimal free resolution of the quotient $R' = R/J(\mathbf{u})^{[\mathbf{u_n}]}$ (see Corollary 6.4 of [10]).

Let $I(\mathbf{u})$ be a permutohedron ideal. Then $J(\mathbf{u}) \subsetneqq I(\mathbf{u})$ and the minimal generators of the Alexander duals $I(\mathbf{u})^{[\mathbf{u_n}]}$ and $J(\mathbf{u})^{[\mathbf{u_n}]}$ are parametrized by non-empty subsets of $[n]$. In fact, we have the following result.

PROPOSITION 3.4

Let $K(\mathbf{u})$ be a monomial ideal such that $J(\mathbf{u}) \subseteqq K(\mathbf{u}) \subseteqq I(\mathbf{u})$. Then minimal generators of the Alexander dual $K(\mathbf{u})^{[\mathbf{u_n}]}$ are parametrized by non-empty subsets of $[n]$.

Proof. Let $\emptyset \neq T \subseteq [n]$. Then by Lemma 2.3 and Theorem 3.3 of [4], there are unique vectors $\mathbf{b}_T \preceq \mathbf{u_n}$ and $\mathbf{b}'_{\mathbf{T}} \preceq \mathbf{u_n}$ maximal with property that $\mathbf{x}^{\mathbf{b}_T} \notin I(\mathbf{u})$ and $\mathbf{x}^{\mathbf{b}'_{\mathbf{T}}} \notin J(\mathbf{u})$. Now $\mathbf{x}^{\mathbf{b}_T} \notin I(\mathbf{u})$ implies that $\mathbf{x}^{\mathbf{b}_T} \notin K(\mathbf{u})$. If $\mathbf{b}_T \preceq \mathbf{u_n}$ is maximal such that $\mathbf{x}^{\mathbf{b}_T} \notin K(\mathbf{u})$, then $\mathbf{x}^{\mathbf{u_n} - \mathbf{b}_T} \in K(\mathbf{u})^{[\mathbf{u_n}]}$ (see Proposition 5.23 of [7]). Otherwise, there exists a vector $\mathbf{w}_T$ with $\mathbf{u_n} \succcurlyeq \mathbf{w}_T \succ \mathbf{b}_T$, so that $\mathbf{w}_T$ is maximal with the property that $\mathbf{x}^{\mathbf{w}_T} \notin K(\mathbf{u})$. Further, we notice that $\mathbf{w}_T \preceq \mathbf{b}'_{\mathbf{T}}$. Otherwise, $\mathbf{x}^{\mathbf{w}_T} \in J(\mathbf{u}) \subseteqq K(\mathbf{u})$, a contradiction. Hence we conclude that for all non-empty subsets $T \subseteq [n]$ there is a unique maximal vector $\mathbf{w}_T$ with $\mathbf{b}_T \preceq \mathbf{w}_T \preceq \mathbf{b}'_{\mathbf{T}}$, such that $\mathbf{x}^{\mathbf{w}_T} \notin K(\mathbf{u})$. Thus $\mathbf{x}^{\mathbf{u_n} - \mathbf{w}_T}$ is a minimal generator of $K(\mathbf{u})^{[\mathbf{u_n}]}$. □

For each non-empty set $B \subseteq \mathfrak{S}_n$, let $I_B = \langle \mathbf{x}^{\sigma(\mathbf{u})} : \sigma \in B \rangle$ be a monomial ideal in R. Then the hypercubic ideal $I_W = J(\mathbf{u})$ has the following minimal property.

Theorem 3.5. *If $B \supseteq W$, then a cellular resolution $\mathbb{F}_*(\mathbf{Bd}(\Delta_{n-1}))$ of the quotient $R/I_B^{[\mathbf{u_n}]}$ supported on $\mathbf{Bd}(\Delta_{n-1})$ is the minimal free resolution. On the other hand, if $B \subsetneqq W$, then a cellular resolution of the quotient $R/I_B^{[\mathbf{u_n}]}$ supported on $\mathbf{Bd}(\Delta_{n-1})$ is always non-minimal.*

Proof. A proof of this theorem depends heavily on the proof of Theorem 3.3. Therefore, we continue to use notations and terminologies used in the proof of Theorem 3.3. Let $\mathcal{T} = \{T \subseteq [n]$: either $1 \in T$ or $2 \in T$ but$\{1, 2\} \nsubseteq T\}$. Then $|\mathcal{T}| = 2^{n-1}$. For $T \in \mathcal{T}$, we define a vector $\mathbf{v}_T$ as

$$\mathbf{v}_T = \sum_{j \notin T} u_n e_j + (u_{t+1} - 1)e_{j_1} + \sum_{\alpha=2}^{t} (u_{t+j_\alpha-\alpha} - 1)e_{j_\alpha},$$

where $T = \{j_1, j_2, \ldots, j_t\} \subseteq [n]$ with $j_1 < j_2 < \cdots < j_t$ and either $j_1 = 1$ but $j_2 > 2$ or $j_1 = 2$. We notice that there are exactly 2^{n-1} associated vectors $\mathbf{v}_T$.

Claim. There exists a unique vector $\mathbf{c} = \rho\mathbf{u}$ for $\rho \in W$ such that $\mathbf{v}_T \succeq \mathbf{c}$.

We now proceed to prove this claim.

Case 1. Take $T = \{j_1, j_2, \ldots, j_t\}$, where $2 = j_1 < j_2 < \cdots < j_t$. Write

$$T = [j_1, j_{n_1}] \amalg [j_{n_1+1}, j_{n_2}] \amalg \ldots \amalg [j_{n_{r-1}+1}, j_{n_r}] = T_1 \amalg T_2 \amalg \ldots \amalg T_r$$

and set $S = [1, n] - T = S_0 \amalg S_1 \amalg \ldots \amalg S_r$. Then $S_0 = \{1\}$. Set $T' = \{1\} \cup T$ and consider $\mathbf{b}_{T'} = \sum_{j \notin T'} u_n e_j + (u_{t+1} - 1)e_1 + \sum_{\alpha=1}^{t} (u_{t+1+j_\alpha-(\alpha+1)} - 1)e_{j_\alpha}$ as in (3.2). Clearly $\mathbf{v}_T \succ \mathbf{b}_{T'}$. Thus $\mathbf{x}^{\mathbf{v}_T} \in J(\mathbf{u})$. Since the ideal $J(\mathbf{u})$ is minimally generated by $\mathbf{x}^{\sigma\mathbf{u}}$; $\sigma \in W$, there exists $\mathbf{c} = \rho\mathbf{u}$ with $\rho \in W$ such that $\mathbf{v}_T \succeq \mathbf{c}$. Consider a vector $\mathbf{b}_T = \sum_{j \notin T} u_n e_j + (u_t - 1)e_{j_1} + \sum_{\alpha=2}^{t} (u_{t+j_\alpha-\alpha} - 1)e_{j_\alpha}$ as in (3.2). Note that $\mathbf{v}_T$ and $\mathbf{b}_T$ differ only in one coordinate ($j = j_1$). Proceeding as in Theorem 3.3, we can show that $\max_{j \in S_0} c_j \geq u_{n-(|S_1|+|S_2|+\cdots+|S_\alpha|)}$ for some $1 \leq \alpha \leq r$. In fact,

$$\max_{j \in S_0} c_j \geq u_{n-(|S_1|+|S_2|+\cdots+|S_\alpha|)} \geq u_{n-(|S_1|+|S_2|+\cdots+|S_r|)} = u_{t+|S_0|}.$$

Thus $c_1 \geq u_{t+1}$. Also $c_2 \leq (\mathbf{v}_T)_2 = u_{t+1} - 1$. This shows that $c_1 = u_{t+1}$ and $c_2 = u_t$. From (3.4), we have $t + j_{n_{\beta-1}+1} - (n_{\beta-1}+1) = t + \sum_{i=0}^{\beta-1} |S_i|$ for $1 \leq \beta \leq r$. Thus $(\mathbf{v}_T)_j = u_{t+j_{n_{\beta-1}+1}-(n_{\beta-1}+1)} - 1 = u_{t+\sum_{i=0}^{\beta-1}|S_i|} - 1$; $j \in T_\beta$ (except $j = j_1$). Since $\mathbf{c} \preceq \mathbf{v}_T$, we have $c_j < u_{t+\sum_{i=0}^{\beta-1}|S_i|}$; $j \in T_\beta$. As $c_2 = c_{j_1} < u_{t+1}$, we have $c_j < u_{t+1}$; $j \in T_1$. This shows that $c_j = u_{t-j+2}$ for $j \in T_1$. Again proceeding as in Theorem 3.3, we can show that $\max_{j \in [1, j_{n_1+1}-1]} c_j \geq u_{n-(|S_2|+\cdots+|S_\alpha|)} = u_{t+|S_0|+|S_1|}$. Hence $c_j = u_{t+s+|S_0|}$, where $j = j_{n_1} + s \in S_1$ and $s = 1, 2, \ldots, |S_1|$. Continuing in this manner, it can be shown that

$$\mathbf{c} = u_{t+1}e_1 + \sum_{h=0}^{r-1} \sum_{l=n_h+1}^{n_{h+1}} u_{t-l+1}e_{j_l} + \sum_{i=1}^{r} \sum_{s=1}^{|S_i|} (u_{t+|S_0|+|S_1|+\cdots+|S_{i-1}|+s})e_{j_{n_i}+s}.$$

Case 2. Take $T = \{j_1, j_2, \ldots, j_t\}$, where $j_1 = 1$ and $2 < j_2 < \cdots < j_t$. Set $T'' = \{2\} \cup T$ and consider a vector $\mathbf{b}_{T''}$ as in (3.2). Now proceeding as in Case 1, we can show that

$$\mathbf{c} = \sum_{h=0}^{r-1} \sum_{l=n_h+1}^{n_{h+1}} u_{t-l+1}e_{j_l} + \sum_{i=1}^{r} \sum_{s=1}^{|S_i|} (u_{t+|S_0|+|S_1|+\cdots+|S_{i-1}|+s})e_{j_{n_i}+s},$$

where $n_1 = 1$, $n_0 = 0$ and $|S_0| = 0$.

From the above discussion, we conclude that for each $T \in \mathcal{T}$ there exists a unique vector $\mathbf{c} = \rho\mathbf{u}$, $\rho \in W$ such that $\mathbf{v}_T \succeq \mathbf{c}$. Thus the minimal generators of the ideal $J(\mathbf{u})$ are in one-to-one correspondence with the vectors $\mathbf{v}_T$, $T \in \mathcal{T}$. Suppose we form an ideal $J'(\mathbf{u}) = I_{W'} = \langle \mathbf{x}^{\sigma\mathbf{u}} | \sigma \in W' \rangle$, by deleting a generator of $J(\mathbf{u})$, i.e., $W' = W - \{\rho\}$ for some $\rho \in W$. Then there will be exactly one $\mathbf{v}_T$ such that $\mathbf{x}^{\mathbf{v}_T} \notin J'(\mathbf{u})$. Suppose $T \in \mathcal{T}$ is of the form $T = \{j_1 = 2, j_2, \ldots, j_t\}$. Now in view of Theorem 3.3, $\mathbf{x}^{\mathbf{b}_{T'}+e_j} \in J(\mathbf{u})$ for $j \in T$ where $T' = \{1\} \cup T$. In other words, $\mathbf{b}_{T'} + e_j \succeq \mathbf{d}$, for some $\mathbf{d} = \sigma\mathbf{u}$, $\sigma \in W$ with

$j \in T$. Since $(\mathbf{b}_{T'} + e_j)_1 = u_{t+1} - 1$, for $j \in T$ and $c_1 = u_{t+1}$ (Case 1), we have $\mathbf{d} \neq \mathbf{c}$. Thus $\mathbf{d} = \sigma\mathbf{u}, \sigma \in W'$. Hence $\mathbf{x}^{\mathbf{b}_{T'}+e_j} \in J'(\mathbf{u}) = I_{W'}$, $j \in T$. Since $(\mathbf{v}_T)_j = (\mathbf{b}_{T'})_j$ for all $j \neq 1$, $u_n = (\mathbf{v}_T)_1 > (\mathbf{b}_{T'})_1$. We conclude that $\mathbf{v}_T \preceq \mathbf{u_n}$ is maximal such that $\mathbf{x}^{\mathbf{v}_T} \notin J'(\mathbf{u}) = I_{W'}$. Thus $\mathbf{x}^{\mathbf{u_n}-\mathbf{v}_T}$ is a minimal generator of $J'(\mathbf{u})^{[\mathbf{u_n}]}$. Also $\mathbf{x}^{\mathbf{u_n}-\mathbf{b}_T} \in J'(\mathbf{u})^{[\mathbf{u_n}]}$ and $\mathbf{x}^{\mathbf{u_n}-\mathbf{b}_{T'}} \in J'(\mathbf{u})^{[\mathbf{u_n}]}$. Since $\mathbf{v}_T \succ \mathbf{b}_{T'}$ and $\mathbf{v}_T \succ \mathbf{b}_T$, the monomial $\mathbf{x}^{\mathbf{u_n}-\mathbf{v}_T}$ strictly divides $\mathbf{x}^{\mathbf{u_n}-\mathbf{b}_{T'}}$ and $\mathbf{x}^{\mathbf{u_n}-\mathbf{b}_T}$. Thus there will be no minimal generator of $J'(\mathbf{u})^{[\mathbf{u_n}]}$ corresponding to the nonempty subset T'. Similarly for $T = \{j_1 = 1, j_2, \ldots, j_t\};\ j_2 > 2$, we have $(\mathbf{b}_{T''} + e_j)_2 = u_{t+1} - 1$ for $j \in T$ and $c_2 = u_{t+1}$ (Case 2). Proceeding as above, it is easy to see that $\mathbf{v}_T \preceq \mathbf{u_n}$ is maximal such that $\mathbf{x}^{\mathbf{v}_T} \notin J'(\mathbf{u}) = I_{W'}$. Thus $\mathbf{x}^{\mathbf{u_n}-\mathbf{v}_T}$ is a minimal generator of $J'(\mathbf{u})^{[\mathbf{u_n}]}$. Also $\mathbf{x}^{\mathbf{u_n}-\mathbf{b}_T} \in J'(\mathbf{u})^{[\mathbf{u_n}]}$ and $\mathbf{x}^{\mathbf{u_n}-\mathbf{b}_{T''}} \in J'(\mathbf{u})^{[\mathbf{u_n}]}$. Since $\mathbf{v}_T \succ \mathbf{b}_{T''}$ and $\mathbf{v}_T \succ \mathbf{b}_T$, the monomial $\mathbf{x}^{\mathbf{u_n}-\mathbf{v}_T}$ strictly divides $\mathbf{x}^{\mathbf{u_n}-\mathbf{b}_{T''}}$ and $\mathbf{x}^{\mathbf{u_n}-\mathbf{b}_T}$. Thus there will be no minimal generator of $J'(\mathbf{u})^{[\mathbf{u_n}]}$ corresponding to the nonempty subset T''.

For $B \subsetneq W$ with $|B| < |W| - 1$, we can choose a subset $W' = W - \{\rho\}$ such that $B \subsetneq W' \subsetneq W$. Now $I_B \subsetneq I_{W'} \subsetneq I_W$. From Theorem 3.3, the minimal generators of $J(\mathbf{u}) = I_W$ are parametrized by non-empty subsets of $[n]$. If minimal generators of $I_B^{[\mathbf{u_n}]}$ are also parametrized by non-empty subsets of $[n]$, then in view of Proposition 3.4, the same hold for $I_{W'}^{[\mathbf{u_n}]}$, a contradiction. Hence the cellular resolution of the quotient $R/I_B^{[\mathbf{u_n}]}$ can never be minimally supported on $\mathbf{Bd}(\Delta_{n-1})$.

Now if $B \supseteq W$, then in view of Proposition 3.4, the Alexander dual $I_B^{[\mathbf{u_n}]}$ is a strictly monotone monomial ideal. Thus, we conclude that a cellular free complex $\mathbb{F}_*(\mathbf{Bd}(\Delta_{n-1}))$ supported on $\mathbf{Bd}(\Delta_{n-1})$ is the minimal free resolution of the quotient $R/I_B^{[\mathbf{u_n}]}$. □

Remark 3.6. The minimality property of Theorem 3.5 does not characterize the hypercubic ideal $J(\mathbf{u})$. There are subsets $B \subsetneq \mathfrak{S}_\mathfrak{n}$ such that ideal I_B $(\neq J(\mathbf{u}))$ also have the same minimality property. We illustrate it with the following example.

Example 3.7. Let $B = \{(1,3,2), (2,1,3), (2,3,1), (3,1,2)\}$ be a subset of $\mathfrak{S}_3$. Then $I_B = \langle xy^3z^2, x^2yz^3, x^2y^3z, x^3yz^2\rangle$ and the Alexander dual of I_B with respect to $\mathbf{u_3} = (3,3,3)$ is $I_B^{[\mathbf{u_3}]} = \langle x^3, y^3, z^3, x^2y, yz^2, x^2z^2, xyz\rangle$. It is straight forward to verify that the ideal I_B also has desired minimality property.

4. Restricted λ-parking functions

The number of standard monomials in an Artinian quotient R/I of the polynomial ring R by a monomial ideal I is an interesting numerical invariant of I. We recall that the standard monomials of an Artinian quotient $R/I(\mathbf{u})^{[\mathbf{u_n}-\mathbf{c}+\mathbf{1}]}$ of the Alexander duals of multipermutohedron ideals correspond bijectively to generalized parking functions [4]. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. A sequence $(p_1, p_2, \ldots, p_n)$ of positive integers is said to be a λ-parking function of length n if its non-decreasing rearrangement $(q_1 \leq q_2 \leq \cdots \leq q_n)$ satisfies $q_i \leq \lambda_{n-i+1}$ for all i. We now introduce a notion of restricted λ-parking functions.

DEFINITION 4.1

A sequence $(p_1, p_2, \ldots, p_n)$ of positive integers is called a *restricted λ-parking function* if there exists some permutation $\alpha \in \mathfrak{S}_\mathfrak{n}$ such that

$$p_{\alpha_i} - 1 < \mu_{\alpha_i, T_i},$$

where $\alpha(i) = \alpha_i$ and $T_i = [n] - \{\alpha_1, \alpha_2, \ldots, \alpha_{i-1}\}$, for all $1 \le i \le n$. The expression μ_{α_i, T_i} is as in Theorem 3.3.

In Definition 4.1, $\lambda = (u_n - u_1 + 1, u_n - u_2 + 1, \ldots, u_n - u_n + 1)$. Although, λ is not explicit in the definition of restricted λ-parking functions, we shall see later that every restricted λ-parking function is indeed a λ-parking function.

The following simple lemma will be used in characterizing the standard monomials in the Artinian quotient $R/J(\mathbf{u})^{[\mathbf{u_n}]}$.

Lemma 4.2. *Let $S = \{a_1, \ldots, a_{i_1}, a_{i_1+1}, \ldots, a_{i_2}, \ldots, a_{i_m}, \ldots, a_n\}$ be a subset of positive integers (arranged in an increasing order) and $T = \{a_{i_1}, a_{i_2}, \ldots, a_{i_m}\} \subset S$. Then $m + a_{i_r} - r \le n + a_{i_r} - i_r$, for $1 \le r \le m$.*

Proof. We need only to show that $i_r - r \le n - m$. Suppose, if possible, $i_r - r > n - m$. Then $i_r \ge n - m + (r+1)$. Also $i_m \ge i_r + (m-r)$ implies that $i_m \ge n - m + (r+1) + m - r = n+1$, which is a contradiction as $i_m \le n$. □

Theorem 4.3. *A monomial $\mathbf{x}^{\mathbf{p}} = x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}$ is a standard monomial in the Artinian k-algebra $R' = R/J(\mathbf{u})^{[\mathbf{u_n}]}$ if and only if $\mathbf{p} + \mathbf{1} = (p_1 + 1, p_2 + 1, \ldots, p_n + 1)$ is a restricted λ-parking function.*

Proof. Let $\mathbf{x}^{\mathbf{p}-\mathbf{1}}$ be a standard monomial in Artinian k-algebra $R/J(\mathbf{u})^{[\mathbf{u_n}]}$. Thus $\mathbf{x}^{\mathbf{p}-\mathbf{1}} \notin J(\mathbf{u})^{[\mathbf{u_n}]}$. Therefore for every non-empty subset $T \subseteq [n]$ there exists some $\alpha \in T$ such that

$$p_\alpha - 1 < \mu_{\alpha, T}. \tag{4.1}$$

If $T_1 = \{1, 2, \ldots, n\}$, then there exists some $\alpha_1 \in T_1$ such that $p_{\alpha_1} - 1 < \mu_{\alpha_1, T_1}$. Now take $T_2 = T_1 - \{\alpha_1\}$, then there exists some $\alpha_2 \in T_2$ such that $p_{\alpha_2} - 1 < \mu_{\alpha_2, T_2}$. Continuing in this manner, by choosing $T_i = T_1 - \{\alpha_1, \alpha_2, \ldots, \alpha_{i-1}\}$ for $i = 1, 2, \ldots, n$, we have the desired result. Conversely, let $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ be a restricted λ-parking function. Let $T = \{j_1, j_2, \ldots, j_t\}$ be a non-empty subset of $[n]$, where $j_1 < j_2 < \cdots < j_t$.

Claim. $p_j - 1 < \mu_{j,T}$ for some $j \in T$.

Let $t = n - q;\ q \ge 0$. If $T = [n] - \{\alpha_1, \alpha_2, \ldots, \alpha_q\} = T_{q+1}$, then by definition there exists some $\alpha_{q+1} = j_s$ (say) $\in T_{q+1}$ such that $p_{j_s} - 1 < \mu_{j_s, T}$. Otherwise suppose $\{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_r}\} \subseteq T;\ i_1 < i_2 < \cdots < i_r \le q$. Let $s = \max\{i : T \subseteq T_i\}$. Then $T \subsetneq T_s$ but $T \nsubseteq T_{s+1}$. Thus $T_{s+1} = T_s - \{\alpha\}$ for some $\alpha \in T$. Clearly $\alpha = \alpha_{i_l}$ for some $l \in \{1, 2, \ldots, r\}$. Now from the definition of restricted λ-parking function, it follows that $p_{\alpha_{i_l}} - 1 < \mu_{\alpha_{i_l}, T_s}$. Further, by Lemma 4.2, we have $\mu_{\alpha_{i_l}, T_s} \le \mu_{\alpha_{i_l}, T}$. □

Remark 4.4.

(1) We notice that $J(\mathbf{u}) \subsetneq I(\mathbf{u}) \Rightarrow I(\mathbf{u})^{[\mathbf{u_n}]} \subsetneq J(\mathbf{u})^{[\mathbf{u_n}]}$. Thus the restricted λ-parking functions are indeed λ-parking functions.

(2) It is easy to see that the notion of restricted λ-parking functions is not a particular case of G-parking functions defined in [10].

We now derive a combinatorial formula for counting restricted λ-parking functions. Let ${\Lambda_n}'$ be the set of restricted λ-parking functions of length n.

We know that the free complex $\mathbb{F}_*(\mathbf{Bd}(\Delta_{n-1}))$ associated to the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ is the minimal free resolution of the quotient $R' = R/J(\mathbf{u})^{[\mathbf{u_n}]}$. The multigraded Hilbert series $H(R', \mathbf{x})$ of the quotient R' can be calculated using the free resolution $\mathbb{F}_*(\mathbf{Bd}(\Delta_{n-1}))$ and we obtain

$$H(R', \mathbf{x}) = \frac{1}{\prod_{l=1}^{n}(1-x_l)} \sum_{i=0}^{n} (-1)^i \sum_{(A_1, A_2, \ldots, A_i) \in \mathcal{F}_{i-1}} \prod_{q=1}^{i} \Bigg(\prod_{j \in A_q - A_{q-1}} x_j^{\mu_{j,A_q}} \Bigg). \tag{4.2}$$

PROPOSITION 4.5

Let $J(\mathbf{u})$ be a hypercubic ideal and $J(\mathbf{u})^{[\mathbf{u_n}]}$ be its Alexander dual with respect to $\mathbf{u_n}$ and $R' = R/J(\mathbf{u})^{[\mathbf{u_n}]}$. Then the number of restricted λ-parking functions of length n is given by

$$|\Lambda'_n| = \sum_{i=1}^{n} (-1)^{n-i} \sum_{\emptyset \subsetneq A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_i = [n]} \prod_{q=1}^{i} \prod_{j \in A_q - A_{q-1}} \mu_{j,\, A_q},$$

where $\mu_{j,\, A_q}$ is defined above.

Proof. In view of Theorem 4.3, we see that $H(R', \mathbf{x}) = \sum_{\mathbf{p} \in \Lambda'_n} \mathbf{x}^{\mathbf{p}-\mathbf{1}}$. Thus $H(R', \mathbf{1}) = |\Lambda'_n|$. Now passing to the limit $x_i \to 1$ simultaneously for each i in the rational function expression for the Hilbert series $H(R', \mathbf{x})$, we obtain

$$H(R', \mathbf{1}) = \lim_{\substack{x_1 \to 1,\\ \ldots,\\ x_n \to 1}} H(R', \mathbf{x}) = \lim_{\substack{x_1 \to 1,\\ \ldots,\\ x_n \to 1}} \frac{Q(\mathbf{x})}{\prod_{l=1}^{n}(1-x_l)},$$

where

$$Q(\mathbf{x}) = \sum_{i=0}^{n} (-1)^i \sum_{(A_1, A_2, \ldots, A_i) \in \mathcal{F}_{i-1}(\mathbf{Bd}(\Delta_{n-1}))} \prod_{q=1}^{i} \prod_{j \in A_q - A_{q-1}} x_j^{\mu_{j,\, A_q}}.$$

Now applying L'Hospital rule, we see that

$$|\Lambda'_n| = \frac{1}{(-1)^n} \left. \frac{\partial^n Q(\mathbf{x})}{\partial x_1 \partial x_2 \cdots \partial x_n} \right|_{\mathbf{x}=\mathbf{1}}.$$

As in the partial derivative $\frac{\partial^n Q(\mathbf{x})}{\partial x_1 \partial x_2 \cdots \partial x_n}$, the term corresponding to tuple $(A_1, A_2, \ldots, A_i)$ survives if and only if $|A_i| = n$. We get the desired result. □

Remark 4.6.

(1) The combinatorial formula for counting the restricted λ-parking functions obtained above is different from a similar formula in Proposition 8.4 of [10].

(2) It is easy to see that for $n = 3$,

$$|\Lambda_3'| = \text{Per}\begin{bmatrix} 1 & u_3 - u_2 & u_2 - u_1 \\ 1 & u_3 - u_2 + 1 & u_3 - u_2 \\ 1 & 1 & u_3 - u_1 + 1 \end{bmatrix},$$

provided $u_2 - u_1 = 1$, where 'Per' denotes the permanent of a matrix. This indicates that for some special values of u_i's, the notion of restricted λ-parking functions could be combinatorially interesting.

5. Betti numbers of the sum of two multipermutohedron ideals

Let $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{N}^n$ with $1 \le u_1 \le u_2 \le \cdots \le u_n$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{N}^n$ with $1 \le v_1 \le v_2 \le \cdots \le v_n$. Consider the multipermutohedron ideals $I(\mathbf{u})$ and I($\mathbf{v}$). Then their intersection $I(\mathbf{u}) \cap I(\mathbf{v})$ is again a multipermutohedron ideal $I(\mathbf{w})$, where $\mathbf{w} = \mathbf{u} \vee \mathbf{v}$ with $w_i = \max(u_i, v_i)$. Let

$$B_{\mathbf{u}} = \{\mathbf{b}(J) \in \mathbb{N}^n : J \in \mathcal{I}_{\mathbf{m_u}}\}.$$

Similarly, we define subsets $B_{\mathbf{v}}$ and $B_{\mathbf{w}}$ of $\mathbb{N}^n$ and let $B = B_{\mathbf{u}} \cup B_{\mathbf{v}} \cup B_{\mathbf{w}}$.

We see that the upper Koszul simplicial complexes of the sum $I(\mathbf{u}) + I(\mathbf{v})$ and the intersection $I(\mathbf{u}) \cap I(\mathbf{v}) = I(\mathbf{w})$ are given by

$$K^{\mathbf{b}}(I(\mathbf{u})+I(\mathbf{v})) = K^{\mathbf{b}}(I(\mathbf{u})) \cup K^{\mathbf{b}}(I(\mathbf{v})) \text{ and } K^{\mathbf{b}}(I(\mathbf{w})) = K^{\mathbf{b}}(I(\mathbf{u})) \cap K^{\mathbf{b}}(I(\mathbf{v})).$$

Suppose $\triangle_1$ and $\triangle_2$ be two simplicial complexes and let $\triangle = \triangle_1 \cup \triangle_2$ and $\Gamma = \triangle_1 \cap \triangle_2$. Then there is a long exact sequence of the form

$$\begin{aligned} \cdots \to \tilde{H}_i(\Gamma; k) &\xrightarrow{\partial} \tilde{H}_i(\triangle_1; k) \oplus \tilde{H}_i(\triangle_2, k) \\ &\xrightarrow{\partial} \tilde{H}_i(\triangle, k) \xrightarrow{\delta} \tilde{H}_{i-1}(\Gamma; k) \to \cdots, \end{aligned}$$

where δ is the connecting homomorphism. This sequence is called the *reduced Mayer–Vietoris sequence*. Since the multigraded Betti numbers of a monomial ideal are given in terms of the dimension of reduced homology groups of the upper Koszul simplicial complex with coefficients in the field k, the Mayer–Vietoris sequence can be used to compute $\tilde{H}_i(K^{\mathbf{b}}(I(\mathbf{u})+I(\mathbf{v})); k)$. In fact, we can take $\triangle_1 = K^{\mathbf{b}}(I(\mathbf{u}))$, $\triangle_2 = K^{\mathbf{b}}(I(\mathbf{v}))$, then $\triangle = K^{\mathbf{b}}(I(\mathbf{u})) \cup K^{\mathbf{b}}(I(\mathbf{v})) = K^{\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v}))$ and $\Gamma = K^{\mathbf{b}}(I(\mathbf{u})) \cap K^{\mathbf{b}}(I(\mathbf{v})) = K^{\mathbf{b}}(I(\mathbf{w}))$.

In view of discussion after Lemma 2.1, we observe that for any degree $\mathbf{b} \in \mathbf{N}^n$, there is at most one non-zero multigraded Betti number $\beta_{i,\mathbf{b}}(I(\mathbf{u}))$. Therefore many terms in the Mayer–Vietoris sequence are zero and it is possible to obtain the dimension $\dim_k \tilde{H}_i(\triangle; k)$. On considering various possibilities for $\mathbf{b} \in \mathbb{N}^n$, we have the following theorem.

Theorem 5.1. *For $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{N}^n$ and $i \ge 0$, let $\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v}))$ be the i-th multigraded Betti number of the sum of two multipermutohedron ideals $I(\mathbf{u})$ and $I(\mathbf{v})$ in degree $\mathbf{b}$. As $\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = \beta_{i,\pi\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v}))$ for any permutation π of $\mathbf{b}$ and*

$\beta_{i,\mathbf{b}}(I(\mathbf{u})+I(\mathbf{v}))=0$ *if* $\mathbf{b}\neq\pi\mathbf{b}'$ *for any* $\mathbf{b}'\in B$ *and* π *any permutation of* $\mathbf{b}'$. *Thus it is enough to take* $\mathbf{b}\in B$ *and we have the following cases*:

Case 1. $\mathbf{b}\in B_{\mathbf{u}},\mathbf{b}\notin B_{\mathbf{v}},\mathbf{b}\notin B_{\mathbf{w}}$ *and* $wt_{\mathbf{m_u}}(\mathbf{b})=q$. *Then*

$$\beta_{i,\mathbf{b}}(I(\mathbf{u})+I(\mathbf{v}))=\begin{cases}\beta_{i,\mathbf{b}}(I(\mathbf{u})) & \text{if } i=q,\\ 0 & \text{otherwise}.\end{cases}$$

Case 2. $\mathbf{b}\notin B_{\mathbf{u}},\mathbf{b}\in B_{\mathbf{v}},\mathbf{b}\notin B_{\mathbf{w}}$ *and* $wt_{\mathbf{m_v}}(\mathbf{b})=q$. *Then*

$$\beta_{i,\mathbf{b}}(I(\mathbf{u})+I(\mathbf{v}))=\begin{cases}\beta_{i,\mathbf{b}}(I(\mathbf{v})) & \text{if } i=q,\\ 0 & \text{otherwise}.\end{cases}$$

Case 3. $\mathbf{b}\notin B_{\mathbf{u}},\mathbf{b}\notin B_{\mathbf{v}},\mathbf{b}\in B_{\mathbf{w}}$ *and* $wt_{\mathbf{m_w}}(\mathbf{b})=q$. *Then*

$$\beta_{i,\mathbf{b}}(I(\mathbf{u})+I(\mathbf{v}))=\begin{cases}\beta_{i-1,\mathbf{b}}(I(\mathbf{w})) & \text{if } i=q+1,\\ 0 & \text{otherwise}.\end{cases}$$

Case 4. $\mathbf{b}\in B_{\mathbf{u}}\cap B_{\mathbf{v}}\cap B_{\mathbf{w}}$. *Then we have the following sub-cases*:

(*i*) $wt_{\mathbf{m_u}}(\mathbf{b})=q$, $wt_{\mathbf{m_v}}(\mathbf{b})=q-p$, $wt_{\mathbf{m_w}}(\mathbf{b})=q-p-1$ *for* $p\geq 1$. *Then*

$$\beta_{i,\mathbf{b}}(I(\mathbf{u})+I(\mathbf{v}))=\begin{cases}\beta_{i,\mathbf{b}}(I(\mathbf{u})) & \text{if } i=q,\\ \beta_{i,\mathbf{b}}(I(\mathbf{v}))+\beta_{i-1,\mathbf{b}}(I(\mathbf{w})) & \text{if } i=q-p,\\ 0 & \text{otherwise}.\end{cases}$$

(*ii*) $wt_{\mathbf{m_u}}(\mathbf{b})=q$, $wt_{\mathbf{m_v}}(\mathbf{b})=q$, $wt_{\mathbf{m_w}}(\mathbf{b})=q$. *Then*

$$\beta_{i,\mathbf{b}}(I(\mathbf{u})+I(\mathbf{v}))=\begin{cases}\beta_{i,\mathbf{b}}(I(\mathbf{u}))+\beta_{i,\mathbf{b}}(I(\mathbf{v}))-\beta_{i,\mathbf{b}}(I(\mathbf{w})) & \text{if } i=q,\\ 0 & \text{otherwise}.\end{cases}$$

(*iii*) $wt_{\mathbf{m_u}}(\mathbf{b})=q$, $wt_{\mathbf{m_v}}(\mathbf{b})=q-p$, $wt_{\mathbf{m_w}}(\mathbf{b})=q-(p+g)$, *where* $p\geq 1$, $g\geq 2$. *Then*

$$\beta_{i,\mathbf{b}}(I(\mathbf{u})+I(\mathbf{v}))=\begin{cases}\beta_{i,\mathbf{b}}(I(\mathbf{u})) & \text{if } i=q,\\ \beta_{i,\mathbf{b}}(I(\mathbf{v})) & \text{if } i=q-p,\\ \beta_{i-1,\mathbf{b}}(I(\mathbf{w})) & \text{if } i=q-(p+g)+1,\\ 0 & \text{otherwise}.\end{cases}$$

(*iv*) $wt_{\mathbf{m_u}}(\mathbf{b})=q=wt_{\mathbf{m_v}}(\mathbf{b})$, $wt_{\mathbf{m_w}}(\mathbf{b})=q-1$. *Then*

$$\beta_{i,\mathbf{b}}(I(\mathbf{u})+I(\mathbf{v}))=\begin{cases}\beta_{i,\mathbf{b}}(I(\mathbf{u}))+\beta_{i,\mathbf{b}}(I(\mathbf{v}))+\beta_{i-1,\mathbf{b}}(I(\mathbf{w})) & \text{if } i=q,\\ 0 & \text{otherwise}.\end{cases}$$

(*v*) $wt_{\mathbf{m_u}}(\mathbf{b})=q=wt_{\mathbf{m_v}}(\mathbf{b})$, $wt_{\mathbf{m_w}}(\mathbf{b})=q-p$, *where* $p\geq 2$. *Then*

$$\beta_{i,\mathbf{b}}(I(\mathbf{u})+I(\mathbf{v}))=\begin{cases}\beta_{i,\mathbf{b}}(I(\mathbf{u}))+\beta_{i,\mathbf{b}}(I(\mathbf{v})) & \text{if } i=q,\\ \beta_{i-1,\mathbf{b}}(I(\mathbf{w})) & \text{if } i=q-p+1,\\ 0 & \text{otherwise}.\end{cases}$$

(*vi*) $wt_{\mathbf{m_u}}(\mathbf{b}) = q$, $wt_{\mathbf{m_v}}(\mathbf{b}) = wt_{\mathbf{m_w}}(\mathbf{b}) = q - p$, *where* $p \geq 1$. *Then*

$$\beta_{i,b}(I(\mathbf{u}) + I(\mathbf{v})) = \begin{cases} \beta_{i,b}(I(\mathbf{u})) & \text{if } i = q, \\ \beta_{i,b}(I(\mathbf{v})) - \beta_{i,b}(I(\mathbf{w})) & \text{if } i = q - p, \\ 0 & \text{otherwise.} \end{cases}$$

Case 5. $\mathbf{b} \in B_{\mathbf{u}} \cap B_{\mathbf{w}}, \mathbf{b} \notin B_{\mathbf{v}}$. *Then we have the following sub-cases*:

(*i*) $wt_{\mathbf{m_u}}(\mathbf{b}) = q$, $wt_{\mathbf{m_w}}(\mathbf{b}) = q$. *Then*

$$\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{u})) - \beta_{i,\mathbf{b}}(I(\mathbf{w})) & \text{if } i = q, \\ 0 & \text{otherwise.} \end{cases}$$

(*ii*) $wt_{\mathbf{m_u}}(\mathbf{b}) = q$, $wt_{\mathbf{m_w}}(\mathbf{b}) = q - 1$. *Then*

$$\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{u})) + \beta_{i-1,\mathbf{b}}(I(\mathbf{w})) & \text{if } i = q, \\ 0 & \text{otherwise.} \end{cases}$$

(*iii*) $wt_{\mathbf{m_u}}(\mathbf{b}) = q$, $wt_{\mathbf{m_w}}(\mathbf{b}) = q - p$ *for* $p \geq 2$. *Then*

$$\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{u})) & \text{if } i = q, \\ \beta_{i-1,\mathbf{b}}(I(\mathbf{w})) & \text{if } i = q - p + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Case 6. $\mathbf{b} \in B_{\mathbf{v}} \cap B_{\mathbf{w}}, \mathbf{b} \notin B_{\mathbf{u}}$. *Then we have the following sub-cases*:

(*i*) $wt_{\mathbf{m_v}}(\mathbf{b}) = q$, $wt_{\mathbf{m_w}}(\mathbf{b}) = q$. *Then*

$$\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{v})) - \beta_{i,\mathbf{b}}(I(\mathbf{w})) & \text{if } i = q, \\ 0 & \text{otherwise.} \end{cases}$$

(*ii*) $wt_{\mathbf{m_v}}(\mathbf{b}) = q$, $wt_{\mathbf{m_w}}(\mathbf{b}) = q - 1$. *Then*

$$\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{v})) + \beta_{i-1,\mathbf{b}}(I(\mathbf{w})) & \text{if } i = q, \\ 0 & \text{otherwise.} \end{cases}$$

(*iii*) $wt_{\mathbf{m_v}}(\mathbf{b}) = q$, $wt_{\mathbf{m_w}}(\mathbf{b}) = q - p$ *for* $p \geq 2$. *Then*

$$\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{v})) & \text{if } i = q, \\ \beta_{i-1,\mathbf{b}}(I(\mathbf{w})) & \text{if } i = q - p + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It is enough to take $\mathbf{b} \in B = B_{\mathbf{u}} \cup B_{\mathbf{v}} \cup B_{\mathbf{w}}$. Then we have the following cases:

Case 1. $\mathbf{b} \in B_{\mathbf{u}}, \mathbf{b} \notin B_{\mathbf{v}}, \mathbf{b} \notin B_{\mathbf{w}}$ and $wt_{\mathbf{m_u}}(\mathbf{b}) = q$. Then only relevant portion of Mayer–Vietoris sequence is $0 \to \tilde{H}_{q-1}(\triangle_1; k) \to \tilde{H}_{q-1}(\triangle; k) \to 0$. From this exact sequence, we get the desired Betti numbers.

Case 2. This is similar to Case 1.

Case 3. This is similar to Case 1.

Case 4. $\mathbf{b} \in B_{\mathbf{u}} \cap B_{\mathbf{v}} \cap B_{\mathbf{w}}$. Then we have the following sub-cases:

(i) $wt_{\mathbf{m_u}}(\mathbf{b}) = q$, $wt_{\mathbf{m_v}}(\mathbf{b}) = q - p$, $wt_{\mathbf{m_w}}(\mathbf{b}) = q - p - 1$ for $p \geq 1$. Then Mayer–Vietoris sequence gives us two exact sequences: $0 \to \tilde{H}_{q-1}(\triangle_1; k) \to \tilde{H}_{q-1}(\triangle; k) \to 0$ and $0 \to \tilde{H}_{q-p-1}(\triangle_2; k) \to \tilde{H}_{q-p-1}(\triangle; k) \to \tilde{H}_{q-p-2}(\Gamma; k) \to 0$. From the above exact sequences, we obtain the desired Betti numbers.
(ii) $wt_{\mathbf{m_u}}(\mathbf{b}) = q, wt_{\mathbf{m_v}}(\mathbf{b}) = q, wt_{\mathbf{m_w}}(\mathbf{b}) = q$. Then only relevant portion of Mayer–Vietoris sequence is $0 \to \tilde{H}_{q-1}(\Gamma; k) \to \tilde{H}_{q-1}(\triangle_1; k) \oplus \tilde{H}_{q-1}(\triangle_2; k) \to \tilde{H}_{q-1}(\triangle; k) \to 0$. From this exact sequence, we obtain the desired Betti numbers. Note that $\dim \triangle_1 = \dim \triangle_2 = q - 1$ and because $\dim \triangle = \max\{\dim \triangle_1, \dim \triangle_2\} = q - 1$, we have $\tilde{H}_q(\triangle; k) = 0$.
(iii) $wt_{\mathbf{m_u}}(\mathbf{b}) = q, wt_{\mathbf{m_v}}(\mathbf{b}) = q - p, wt_{\mathbf{m_w}}(\mathbf{b}) = q - (p + g)$, where $p \geq 1, g \geq 2$. Then only relevant portions of Mayer–Vietoris sequence are $0 \to \tilde{H}_{q-1}(\triangle_1; k) \to \tilde{H}_{q-1}(\triangle; k) \to 0$, $0 \longrightarrow \tilde{H}_{q-p-1}(\triangle_2; k) \longrightarrow \tilde{H}_{q-p-1}(\triangle; k) \longrightarrow 0$ and $0 \longrightarrow \tilde{H}_{q-(p+g)}(\triangle; k) \longrightarrow \tilde{H}_{q-(p+g+1)}(\Gamma; k) \longrightarrow 0$. These exact sequences give us the desired Betti numbers.
(iv) $wt_{\mathbf{m_u}}(\mathbf{b}) = q = wt_{\mathbf{m_v}}(\mathbf{b}), wt_{\mathbf{m_w}}(\mathbf{b}) = q - 1$. Then only relevant portion of Mayer–Vietoris sequence is $0 \to \tilde{H}_{q-1}(\triangle_1; k) \oplus \tilde{H}_{q-1}(\triangle_2; k) \to \tilde{H}_{q-1}(\triangle; k) \to \tilde{H}_{q-2}(\Gamma; k) \to 0$. From this exact sequence, we obtain the desired Betti numbers.
(v) $wt_{\mathbf{m_u}}(\mathbf{b}) = q = wt_{\mathbf{m_v}}(\mathbf{b}), wt_{\mathbf{m_w}}(\mathbf{b}) = q - p$, where $p \geq 2$. Then only relevant portions of Mayer–Vietoris sequence are $0 \to \tilde{H}_{q-1}(\triangle_1; k) \oplus \tilde{H}_{q-1}(\triangle_2; k) \to \tilde{H}_{q-1}(\triangle; k) \to 0$ and $0 \to \tilde{H}_{q-p}(\triangle; k) \to \tilde{H}_{q-p-1}(\Gamma; k) \to 0$. These exact sequences give us the desired Betti numbers.
(vi) $wt_{\mathbf{m_u}}(\mathbf{b}) = q, wt_{\mathbf{m_v}}(\mathbf{b}) = wt_{\mathbf{m_w}}(\mathbf{b}) = q - p$, where $p \geq 1$. Then Mayer–Vietoris sequence reduces to $0 \to \tilde{H}_{q-1}(\triangle_1; k) \to \tilde{H}_{q-1}(\triangle; k) \to 0 \to \tilde{H}_{q-p}(\triangle; k) \xrightarrow{\delta} \tilde{H}_{q-p-1}(\Gamma; k) \to \tilde{H}_{q-p-1}(\triangle_2; k) \to \tilde{H}_{q-p-1}(\triangle; k) \to 0$, where δ is a connecting homomorphism.

Claim. Connecting homomorphism $\delta : \tilde{H}_{q-p}(\triangle; k) \to \tilde{H}_{q-p-1}(\Gamma; k)$ is zero.

Let $z \in \tilde{H}_{q-p}(\triangle; k)$. Then $z = c_1 + c_2$ with $\partial z = 0$ and c_i is a k-valued $(q - p)$-chain of $\triangle_i$; $i = 1, 2$. As $\partial(z) = 0$, we have $\partial(c_1) = -\partial(c_2)$. Therefore the connecting homomorphism is given by $\delta(z) = \partial(c_1) = -\partial(c_2)$. Since $\dim \triangle_2 = \dim K^{\mathbf{b}}(I(\mathbf{v})) = wt_{\mathbf{m_v}}(\mathbf{b}) - 1 = q - p - 1$, we have $c_2 = 0$. This proves the claim. Thus we have the exact sequences $0 \to \tilde{H}_{q-1}(\triangle_1; k) \to \tilde{H}_{q-1}(\triangle; k) \to 0$ and $0 \to \tilde{H}_{q-p-1}(\Gamma; k) \to \tilde{H}_{q-p-1}(\triangle_2; k) \to \tilde{H}_{q-p-1}(\triangle; k) \to 0$. These exact sequences give us the desired Betti numbers.

Proofs of the remaining cases are as in the Case 4. □

We remark that $B_{\mathbf{u}} \cap B_{\mathbf{v}} \subseteq B_{\mathbf{w}}$ and all the above cases are relevant. Using Theorem 5.1, we shall illustrate the computation of Betti numbers of $I(\mathbf{u}) + I(\mathbf{v})$ in the following example.

Example 5.2. Let $\mathbf{u} = (1, 2, 3, 4, 6, 6)$ and $\mathbf{v} = (1, 4, 4, 4, 5, 6)$. Then $\mathbf{w} = (1, 4, 4, 4, 6, 6)$. Writing (a, b, b, c, c, c) compactly as (a, b^2, c^3), it can be checked that

$$\begin{aligned} B_{\mathbf{u}} = \{&(1, 6^5), (2^2, 6^4), (3^3, 6^3), (4^4, 6^2), (1, 2, 3, 4, 6^2), (1, 2, 3, 6^3), \\ &(1, 2, 4^2, 6^2), (2^2, 3, 4, 6^2), (1, 3^2, 4, 6^2), (1, 2, 6^4), (1, 3^2, 6^3), \\ &(1, 4^3, 6^2), (2^2, 3, 6^3), (2^2, 4^2, 6^2), (3^3, 4, 6^2), (6^6)\},\end{aligned}$$

$$
\begin{aligned}
B_{\mathbf{v}} &= \{(1,6^5),(4^2,6^4),(4^3,6^3),(4^4,6^2),(5^5,6),(1,4,6^4),(1,4^2,6^3),\\
&\quad (1,4^3,6^2),(1,5^4,6),(4^2,5^3,6),(4^3,5^2,6),(4^4,5,6),(1,4,5^3,6),\\
&\quad (1,4^2,5^2,6),(1,4^3,5,6),(6^6)\}, \text{ and}\\
B_{\mathbf{w}} &= \{(1,6^5),(4^2,6^4),(4^3,6^3),(4^4,6^2),(1,4,6^4),(1,4^2,6^3),(1,4^3,6^2),\\
&\quad (6^6)\}.
\end{aligned}
$$

If $\mathbf{b} \in B_{\mathbf{u}}$ only (i.e. $\mathbf{b} \notin B_{\mathbf{v}} \cup B_{\mathbf{w}}$), then $\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = \beta_{i,\mathbf{b}}(I(\mathbf{u}))$. For instance $\mathbf{b} = (2^2, 3, 4, 6^2)$, then $\beta_{1,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = \beta_{1,\mathbf{b}}(I(\mathbf{u})) = 1$, where $wt_{m_{\mathbf{u}}}(\mathbf{b}) = 1$ and $\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = 0$ for all $i \neq 1$. Similarly, we can calculate the Betti numbers of $I(\mathbf{u}) + I(\mathbf{v})$ for $\mathbf{b} \in B_{\mathbf{v}}$ only (or $\mathbf{b} \in B_{\mathbf{w}}$ only). Also

$$
\begin{aligned}
B_{\mathbf{u}} \cap B_{\mathbf{w}} &= \{(1,6^5),(4^4,6^2),(1,4^3,6^2),(6^6)\} = B_{\mathbf{u}} \cap B_{\mathbf{v}} \cap B_{\mathbf{w}}\text{ , and}\\
B_{\mathbf{v}} \cap B_{\mathbf{w}} &= \{(1,6^5),(4^2,6^2),(1,4^3,6^2),(6^6),(4^2,6^4),(4^3,6^3),(1,4,6^4),\\
&\quad (1,4^2,6^3)\}.
\end{aligned}
$$

If $\mathbf{b} \in B_{\mathbf{u}} \cap B_{\mathbf{v}} \cap B_{\mathbf{w}}$, for instance $\mathbf{b} = (4^4, 6^2)$, then $wt_{m_{\mathbf{u}}}(\mathbf{b}) = 3, wt_{m_{\mathbf{v}}}(\mathbf{b}) = 2, wt_{m_{\mathbf{w}}}(\mathbf{b}) = 1$. In view of Theorem 5.1, $\beta_{3,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = \beta_{3,\mathbf{b}}(I(\mathbf{u})) = 1$, $\beta_{2,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = \beta_{2,\mathbf{b}}(I(\mathbf{v})) + \beta_{1,\mathbf{b}}(I(\mathbf{w})) = 3 + 3 = 6$ and $\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = 0$ for all $i \neq 2, 3$. On the other hand, if $\mathbf{b} \in B_{\mathbf{v}} \cap B_{\mathbf{w}}$ but $\mathbf{b} \notin B_{\mathbf{u}}$, for instance, $\mathbf{b} = (4^3, 6^3)$, then $wt_{m_{\mathbf{v}}}(\mathbf{b}) = 3, wt_{m_{\mathbf{w}}}(\mathbf{b}) = 2$. Again by Theorem 5.1, we have $\beta_{3,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = \beta_{3,\mathbf{b}}(I(\mathbf{v})) + \beta_{2,\mathbf{b}}(I(\mathbf{w})) = 2 + 4 = 6$ and $\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = 0$ for all $i \neq 3$. Similarly, other Betti numbers can be calculated.

We recall that for any degree $\mathbf{b} \in \mathbb{N}^n$, there is at most one non-zero multigraded Betti number $\beta_{i,\mathbf{b}}(I(\mathbf{u})^{[\mathbf{a}]})$. If we take $\triangle_1 = K_{\mathbf{b}}(I(\mathbf{u})^{[\mathbf{a}]})$, $\triangle_2 = K_{\mathbf{b}}(I(\mathbf{v})^{[\mathbf{a}]})$, then $\triangle = K_{\mathbf{b}}((I(\mathbf{u}) + I(\mathbf{v}))^{[\mathbf{a}]})$ and $\Gamma = K_{\mathbf{b}}(I(\mathbf{w})^{[\mathbf{a}]})$. This again gives rise to a reduced Mayer–Vietoris sequence of homology groups, in which many terms are zero. Thus proceeding on similar lines, multigraded Betti numbers of $(I(\mathbf{u}) + I(\mathbf{v}))^{[\mathbf{a}]}$ can also be obtained.

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