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Generalized r-Lah numbers

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Abstract. In this paper, we consider a two-parameter polynomial generalization, denoted by $\mathcal{G}_{a,b}(n,k;r)$, of the r-Lah numbers which reduces to these recently introduced numbers when $a = b = 1$. We present several identities for $\mathcal{G}_{a,b}(n,k;r)$ that generalize earlier identities given for the r-Lah and r-Stirling numbers. We also provide combinatorial proofs of some earlier identities involving the r-Lah numbers by defining appropriate sign-changing involutions. Generalizing these arguments yields orthogonality-type relations that are satisfied by $\mathcal{G}_{a,b}(n,k;r)$.

Keywords. r-Lah numbers; r-Stirling numbers; polynomial generalization.

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1. Introduction

Let $[m] = \{1, 2, \ldots, m\}$ if $m \geq 1$, with $[0] = \varnothing$. By a *partition* of $[m]$, we will mean a collection of non-empty, pairwise disjoint subsets of $[m]$, called *blocks*, whose union is $[m]$. A *Lah distribution* will refer to a partition of $[m]$ in which elements within each block are ordered (though there is no inherent ordering for the blocks themselves). Following the notation from [16], let $\left\lfloor {n \atop k} \right\rfloor_r$ denote the number of Lah distributions of the elements of $[n+r]$ having $k+r$ blocks such that the elements of $[r]$ belong to distinct (ordered) blocks. The $\left\lfloor {n \atop k} \right\rfloor_r$ are called *r-Lah numbers* and have only recently been studied.

The numbers $\left\lfloor {n \atop k} \right\rfloor_r$ were once mentioned in [6] (together with r-Whitney–Lah numbers) and appear in [17] under the name of restricted Lah numbers. A few properties of the $\left\lfloor {n \atop k} \right\rfloor_r$ were established in [3], and a systematic study of these numbers was undertaken in [16]. When $r = 0$, the r-Lah number reduces to the Lah number $\left\lfloor {n \atop k} \right\rfloor$ (named for the mathematician Ivo Lah [8] and often denoted by $L(n,k)$), which counts the number of partitions of $[n]$ into k ordered blocks (see, e.g. [4, 19]).

Earlier, analogous r-versions of the Stirling numbers of the first and second kind were introduced by Broder [5], and later rediscovered by Merris [11], where r distinguished elements have to be in distinct cycles or blocks. Following the parametrization and notation used in [16], let $\left[{n \atop k} \right]_r$ be the number of permutations of $[n+r]$ into $k+r$ cycles in which members of $[r]$ belong to distinct cycles and let $\left\{ {n \atop k} \right\}_r$ be the number of partitions of $[n+r]$ into $k+r$ blocks in which members of $[r]$ belong to distinct blocks. There are several algebraic properties for which $\left[{n \atop k} \right]_r$ and $\left\{ {n \atop k} \right\}_r$ satisfy analogous identities, among them various recurrences and connection constant relations (see Section 2 of [16] for a comparative study). Analogues of some of these properties involving r-Lah numbers were established in [16].

In this paper, we consider a two-parameter polynomial generalization of the number $\left\lfloor {n \atop k} \right\rfloor_r$ which reduces to it when both parameters are unity. Denoted by $\mathcal{G}_{a,b}(n,k;r)$, it will also be seen to specialize to $\left[{n \atop k} \right]_r$ when $a=1, b=0$ and to $\left\{ {n \atop k} \right\}_r$ when $a=0, b=1$. Since the $\mathcal{G}_{a,b}(n,k;r)$ reduce to the r-Lah numbers when $a=b=1$, we will refer to them as *generalized r-Lah numbers*. We note that the special case $r=0$ is equivalent to a re-parametrized version of the numbers $\mathfrak{S}_{s;h}(n,k)$ (see [9, 10]) which arise in conjunction with the normal ordering problem from mathematical physics. Furthermore, it is seen that $\mathcal{G}_{a,b}(n,k;r)$ is a variant of the generalized Stirling polynomial introduced by Hsu and Shiue in [7] and studied from an algebraic standpoint. Finally, we remark that $\mathcal{G}_{a,b}(n,k;r)$ can be reached by a special substitution into the partial r-Bell polynomials introduced in [15], but here we look at some specific properties of $\mathcal{G}_{a,b}(n,k;r)$ that were not considered more generally in [15] (several of which do not seem to hold in the more general setting).

The paper is organized as follows. In §2, we define the polynomial $\mathcal{G}_{a,b}(n,k;r)$ in terms of two statistics on Lah distributions and, among our results, show that $\mathcal{G}_{a,b}(n,k;r)$ is strictly log-concave when a and b are positive real numbers. In §3, we derive, by combinatorial arguments, various identities satisfied by $\mathcal{G}_{a,b}(n,k;r)$ that generalize earlier identities for the r-Stirling and r-Lah numbers (see [5, 12, 13, 16]). Some recurrences are also given for the row sum $\sum_{k=0}^{n} \mathcal{G}_{a,b}(n,k;r)$, among them a generalization of a Bell number formula of Spivey [18]. In §4, we provide combinatorial proofs of four identities from [16] involving $\left\lfloor {n \atop k} \right\rfloor_r$ by defining suitable sign-changing involutions on certain combinatorial configurations. Extending our arguments yields orthogonality-type relations satisfied by the generalized r-Lah numbers.

We will make use of the following notation and conventions. Empty sums will take the value zero, and empty products the value one. The binomial coefficient $\binom{n}{k}$ is defined as $\frac{n!}{k!(n-k)!}$ if $0 \le k \le n$, and will be taken to be zero otherwise. If m and n are positive integers, then $[m,n]=\{m,m+1,\ldots,n\}$ if $m \le n$, with $[m,n]=\varnothing$ if $m>n$. Finally, if n is an integer, then let $n^{\overline{m}}=\prod_{i=0}^{m-1}(n+i)$ and $n^{\underline{m}}=\prod_{i=0}^{m-1}(n-i)$ if $m \ge 1$, with $n^{\overline{0}}=n^{\underline{0}}=1$ for all n.

2. Definition and basic properties

Given $0 \le k \le n$ and $r \ge 0$, let $\mathcal{L}_r(n,k)$ denote the set of Lah distributions enumerated by $\left\lfloor {n \atop k} \right\rfloor_r$, i.e., partitions of $[n+r]$ into $k+r$ ordered blocks in which the elements of $[r]$ belong to distinct blocks. We will say that the elements of $[r]$ within a member of $\mathcal{L}_r(n,k)$ are *distinguished* and apply this term also to the blocks in which they belong (with blocks not containing an element of $[r]$ being described as *non-distinguished*). We will sometimes refer to the members of $\mathcal{L}_r(n,k)$ as *r-Lah distributions*.

Note that when $r=0$ or $r=1$, there is no restriction introduced by distinguished elements so that $\left\lfloor {n \atop k} \right\rfloor_0 = \left\lfloor {n \atop k} \right\rfloor$ and $\left\lfloor {n \atop k} \right\rfloor_1 = \left\lfloor {n+1 \atop k+1} \right\rfloor$. Accordingly, when $r=0$, we will often omit the subscript and let $\mathcal{L}(n,k)$ denote the set of all Lah distributions of size n having k blocks. Note that $\mathcal{L}_r(n,k)$ is a proper subset of $\mathcal{L}(n+r,k+r)$ when $r \ge 2$ and $n>k$.

We consider a generalization of the numbers $\left\lfloor {n \atop k} \right\rfloor_r$ obtained by introducing a pair of statistics on $\mathcal{L}_r(n,k)$ as follows: If $\lambda \in \mathcal{L}(n,k)$ and $i \in [n]$, then we will say that i is a *record low* of λ if there are no elements $j < i$ to the left of i within its block in λ. For example, if $n = 9$, $k = 3$ and $\lambda = \{1, 5, 3\}, \{8, 4, 7, 2, 9\}, \{6\} \in \mathcal{L}(9, 3)$, then the element 1 is a record low in the first block, 8, 4 and 2 are record lows in the second, and 6 is a record low in the third block for a total of five record lows altogether. Note that the first element within a block as well as the smallest are always record lows.

We now recall the following statistic from [9].

DEFINITION 2.1

Given $\lambda \in \mathcal{L}(n, k)$, let $\mathrm{rec}^*(\lambda)$ denote the total number of record lows of λ which are not themselves the smallest element of a block. Let $\mathrm{nrec}(\lambda)$ denote the number of elements of $[n]$ which are not record lows of λ.

To illustrate, if λ is as above, then $\mathrm{rec}^*(\lambda) = 2$ (for the 8 and 4) and $\mathrm{nrec}(\lambda) = 4$ (for 5, 3, 7 and 9). We now consider the restriction of the rec* and nrec statistics to $\mathcal{L}_r(n, k)$ and define the distribution polynomial $\mathcal{G}_{a,b}(n, k; r)$ by

$$\mathcal{G}_{a,b}(n, k; r) = \sum_{\lambda \in \mathcal{L}_r(n,k)} a^{\mathrm{nrec}(\lambda)} b^{\mathrm{rec}^*(\lambda)},$$

where a and b are indeterminates.

Note that $\mathcal{G}_{a,b}(n, k; r)$ reduces to $\left\lfloor {n \atop k} \right\rfloor_r$ when $a = b = 1$, by definition. Furthermore, it is seen that $\mathcal{G}_{a,b}(n, k; r)$ reduces to $\left[{n \atop k} \right]_r$ when $a = 1, b = 0$ and to $\left\{ {n \atop k} \right\}_r$ when $a = 0, b = 1$. Note that in the former case, the first element must be the smallest within each block in order for $\lambda \in \mathcal{L}_r(n, k)$ to have a non-zero contribution towards $\mathcal{G}_{a,b}(n, k; r)$, while in the latter case, the elements must be arranged in decreasing order within each block of λ.

Given $\lambda \in \mathcal{L}(n, k)$, let $w(\lambda) = w_{a,b}(\lambda) = a^{\mathrm{nrec}(\lambda)} b^{\mathrm{rec}^*(\lambda)}$ denote the *weight of* λ, and by the weight of a subset of $\mathcal{L}(n, k)$, we will mean the sum of the weights of all the members contained therein. Note that $\mathcal{G}_{a,b}(n, k; r)$ can only assume non-zero values when $0 \le k \le n$ and $r \ge 0$. We now write a recurrence for $\mathcal{G}_{a,b}(n + 1, k; r)$ where $1 \le k \le n + 1$. First note that the total w-weight of all members of $\mathcal{L}_r(n + 1, k)$ in which the element $n + r + 1$ belongs to its own block is $\mathcal{G}_{a,b}(n, k - 1; r)$ since $n + r + 1$ in this case contributes to neither the nrec nor rec* values. The weight of all members of $\mathcal{L}_r(n + 1, k)$ in which $n + r + 1$ starts a block containing at least one member of $[n + r]$ is $b(k + r)\mathcal{G}_{a,b}(n, k; r)$ since rec* is increased by one by the addition of $n + r + 1$. Finally, if $n + r + 1$ directly follows some member of $[n + r]$ within a block, then nrec is increased by one, which implies a contribution of $a(n + r)\mathcal{G}_{a,b}(n, k; r)$ in this case. Combining the three previous cases gives the recurrence

$$\mathcal{G}_{a,b}(n + 1, k; r) = \mathcal{G}_{a,b}(n, k - 1; r) + (an + bk + (a + b)r)\mathcal{G}_{a,b}(n, k; r), \quad 1 \le k \le n + 1, \quad (1)$$

with boundary values $\mathcal{G}_{a,b}(0, k; r) = \delta_{k,0}$ and $\mathcal{G}_{a,b}(n, 0; r) = \prod_{i=0}^{n-1}(a(i + r) + br)$.

Remark. By (1), one sees that the $\mathcal{G}_{a,b}(n, k; r)$ occur as a special case of the solution to a general bivariate recurrence in [1, 2], which was approached algebraically (wherein general formulas for the relevant exponential generating functions were found).

The $a = b = 1$ case of the following result occurs as Theorem 3.2 of [16].

Theorem 2.2. *If $n \geq 0$, then*

$$\prod_{i=0}^{n-1}(x + (a+b)r + ai) = \sum_{k=0}^{n} \mathcal{G}_{a,b}(n,k;r) \prod_{i=0}^{k-1}(x - bi). \tag{2}$$

Proof. Proceed by induction on n, the $n = 0$ case is clear. If $n \geq 0$, then

$$\begin{aligned}
\prod_{i=0}^{n}(x + (a+b)r + ai) &= (x + (a+b)r + an) \prod_{i=0}^{n-1}(x + (a+b)r + ai) \\
&= (x + (a+b)r + an) \sum_{k=0}^{n} \mathcal{G}_{a,b}(n,k;r) \prod_{i=0}^{k-1}(x - bi) \\
&= \sum_{k=0}^{n} \mathcal{G}_{a,b}(n,k;r) \left[\prod_{i=0}^{k}(x - bi) + (an + bk + (a+b)r) \prod_{i=0}^{k-1}(x - bi) \right] \\
&= (an + (a+b)r)\mathcal{G}_{a,b}(n,0;r) + \sum_{k=1}^{n+1} \mathcal{G}_{a,b}(n,k-1;r) \prod_{i=0}^{k-1}(x - bi) \\
&\quad + \sum_{k=1}^{n+1}(an + bk + (a+b)r)\mathcal{G}_{a,b}(n,k;r) \prod_{i=0}^{k-1}(x - bi) \\
&= \sum_{k=0}^{n+1} G_{a,b}(n+1,k;r) \prod_{i=0}^{k-1}(x - bi),
\end{aligned}$$

by (1), which completes the induction. □

We have the following explicit formula for the numbers $u(n,k) = \mathcal{G}_{a,b}(n,k;r)$.

Lemma 2.3. *If $n, k \geq 0$ and $b \neq 0$, then*

$$u(n,k) = \frac{1}{b^k k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \prod_{i=0}^{n-1}(ai + bj + (a+b)r). \tag{3}$$

Proof. We proceed by induction on n. If $n = 0$, then formula (3) holds for all k since

$$u(0,k) = \delta_{k,0} = \frac{1}{b^k k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j}, \qquad k \geq 0.$$

Note that (3) also holds for $k = 0$ since $u(n, 0) = \prod_{i=0}^{n-1}(ai + (a + b)r)$. By (1), in order to complete the induction for $n \geq 1$, we must show

$$\frac{1}{b^{k-1}(k-1)!}\sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}\prod_{i=0}^{n-2}(ai+bj+(a+b)r)$$
$$+\frac{a(n-1)+bk+(a+b)r}{b^k k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\prod_{i=0}^{n-2}(ai+bj+(a+b)r)$$
$$=\frac{1}{b^k k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\prod_{i=0}^{n-1}(ai+bj+(a+b)r), \qquad k \geq 1.$$

By the fact $\binom{k}{j} - \binom{k-1}{j} = \binom{k-1}{j-1}$, the preceding equality holds if and only if

$$bk\sum_{j=1}^{k}(-1)^{k-j}\binom{k-1}{j-1}\prod_{i=0}^{n-2}(ai+bj+(a+b)r)$$
$$+(a(n-1)+(a+b)r)\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\prod_{i=0}^{n-2}(ai+bj+(a+b)r)$$
$$=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\prod_{i=0}^{n-1}(ai+bj+(a+b)r),$$

or equivalently, by the fact $k\binom{k-1}{j-1} = j\binom{k}{j}$,

$$\sum_{j=0}^{k}(-1)^{k-j}(bj+a(n-1)+(a+b)r)\binom{k}{j}\prod_{i=0}^{n-2}(ai+bj+(a+b)r)$$
$$=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\prod_{i=0}^{n-1}(ai+bj+(a+b)r),$$

which is obviously true and completes the induction. □

Theorem 2.4. *If a and b are non-zero, then*

$$\sum_{n\geq 0}u(n,k)\frac{x^n}{n!} = \frac{1}{b^k k!}(1-ax)^{-\left(1+\frac{b}{a}\right)r}\left((1-ax)^{-\frac{b}{a}}-1\right)^k. \tag{4}$$

Proof. The result follows from (3) since

$$\sum_{n\geq 0}\frac{x^n}{n!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\prod_{i=0}^{n-1}(ai+bj+(a+b)r)$$
$$=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\sum_{n\geq 0}\frac{x^n}{n!}\prod_{i=0}^{n-1}(ai+bj+(a+b)r)$$

$$= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \sum_{n\geq 0} \frac{(ax)^n}{n!} \prod_{i=0}^{n-1} \left(i + \frac{b}{a} j + \left(1 + \frac{b}{a}\right) r \right)$$

$$= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (1-ax)^{-\left(\frac{b}{a} j + \left(1+\frac{b}{a}\right) r\right)}$$

$$= (1-ax)^{-\left(1+\frac{b}{a}\right) r} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (1-ax)^{-\frac{b}{a} j}$$

$$= (1-ax)^{-\left(1+\frac{b}{a}\right) r} ((1-ax)^{-\frac{b}{a}} - 1)^k.$$

□

Recall that a sequence $(x_n)_{n\geq 0}$ is said to be *log-concave* if $x_n^2 \geq x_{n-1}x_{n+1}$ for all $n \geq 1$ and *strictly log-concave* if the inequality is strict.

Theorem 2.5. *If $n \geq 1$, then the sequence $(u(n,k))_{k=0}^{n}$ is strictly log-concave for all real numbers $a \geq 0$ and $b > 0$.*

Proof. Fix $n \geq 1$ and consider the sequence $a_k = b^k k! u(n,k)$. Note that for $j \geq 1$, we have

$$(ai + bj + (a+b)r)^2 - b^2 = (ai + b(j-1) + (a+b)r)(ai + b(j+1) + (a+b)r), \qquad 0 \leq i \leq n-1,$$

which implies

$$\prod_{i=0}^{n-1} (ai + bj + (a+b)r)^2 > \prod_{i=0}^{n-1} (ai + b(j-1) + (a+b)r) \times \prod_{i=0}^{n-1} (ai + b(j+1) + (a+b)r).$$

Recall Corollary 3.5 of [20], which states that if x_k and y_k are log-concave sequences, then so is their binomial convolution $z_k = \sum_{j=0}^{k} \binom{k}{j} x_j y_{k-j}$. Applying this result with

$$x_j = \prod_{i=0}^{n-1} (ai + bj + (a+b)r) \qquad \text{and} \qquad y_j = (-1)^j$$

implies that a_k is log-concave for $k \geq 0$, by (3). Since the sequence a_k is positive for $0 < k \leq n$, it follows that $u(n,k) = \frac{a_k}{b^k k!}$ is strictly log-concave. □

Remark. The previous theorem implies that $u(n,k)$, $0 \leq k \leq n$, is unimodal and can assume its maximum value for at most two (consecutive) values of k.

3. Generalized r-Lah identities

In this section, we derive various combinatorial identities involving $\mathcal{G}_{a,b}(n,k;r)$ and its row sum over k. We first consider a couple of recurrence formulas for $\mathcal{G}_{a,b}(n,k;r)$.

Theorem 3.1. *We have*

$$\mathcal{G}_{a,b}(n,k;r)=\sum_{i=k}^{n}\mathcal{G}_{a,b}(i-1,k-1;r)\prod_{j=i}^{n-1}(aj+bk+(a+b)r), \qquad 1\le k\le n, \tag{5}$$

and

$$\mathcal{G}_{a,b}(n,k;r)=\sum_{i=0}^{k}(a(n+r-i-1)+b(k+r-i))\mathcal{G}_{a,b}(n-i-1,k-i;r), \qquad 0\le k<n. \tag{6}$$

Proof. To show (5), we may assume that the blocks within an r-Lah distribution are arranged from left to right in ascending order according to the size of the smallest element. Then the right-hand side of (5) gives the total weight of all members of $\mathcal{L}_r(n,k)$ by considering the smallest element, $i+r$, belonging to the rightmost block where $k\le i\le n$. Note that there are $\mathcal{G}_{a,b}(i-1,k-1;r)$ possibilities concerning placement of the members of $[i+r-1]$ and $\prod_{j=i}^{n-1}(aj+bk+(a+b)r)$ ways in which to arrange the members of $[i+r+1,n+r]$. Summing over all possible i gives (5).

To show (6), consider the largest element, $n+r-i$, not going by itself in a block where $0\le i\le k$ (note $k<n$ implies the existence of such an element). Observe that then the elements of $[n+r-i-1]$ comprise a member of $\mathcal{L}_r(n-i-1,k-i)$ and that there are $a(n+r-i-1)+b(k+r-i)$ possibilities concerning placement of $n+r-i$. Finally, the members of $[n+r-i+1,n+r]$ must all belong to singleton blocks and hence contribute to neither the nrec nor the rec* values. □

The $a=b=1$ case of (5) occurs as in Theorem 3.3 of [16]. Extending the proofs of Theorems 3.4 and 3.6 in [16] yields the following identities.

Theorem 3.2. *If* $0\le k\le n$, *then*

$$\mathcal{G}_{a,b}(n,k;r+s)=\sum_{i=k}^{n}\binom{n}{i}\mathcal{G}_{a,b}(i,k;r)\prod_{j=0}^{n-i-1}(aj+(a+b)s). \tag{7}$$

If $0\le k\le n-m$, *then*

$$\binom{k+m}{k}\mathcal{G}_{a,b}(n,k+m;r+s)=\sum_{i=k}^{n-m}\binom{n}{i}\mathcal{G}_{a,b}(i,k;r)\mathcal{G}_{a,b}(n-i,m;s). \tag{8}$$

The $a=0,b=1$ case of the following identity is a refinement of the r-Bell number relation (Theorem 2 of [13]).

Theorem 3.3. *If* $n,m,k\ge 0$, *then*

$$\mathcal{G}_{a,b}(n+m,k;r)=\sum_{i=0}^{n}\sum_{j=0}^{m}\binom{n}{i}\mathcal{G}_{a,b}(m,j;r)\mathcal{G}_{a,b}(i,k-j;0)\times\prod_{\ell=0}^{n-i-1}(a\ell+a(m+r)+b(j+r)). \tag{9}$$

Proof. Given $\lambda \in \mathcal{L}_r(n+m,k)$, consider the number, $n-i$, of elements in $I = [m+r+1, n+m+r]$ that lie in a block containing an element of $[m+r]$ and the number, $j+r$, of blocks occupied by the members of $[m+r]$. There are then $\mathcal{G}_{a,b}(m,j;r)$ possibilities regarding placement of the members of $[m+r]$. Once these positions have been determined, there are $\binom{n}{i}\prod_{\ell=0}^{n-i-1}(a\ell+a(m+r)+b(j+r))$ ways in which to choose and arrange the aforementioned elements of I. Finally, the remaining elements of I can be arranged in $\mathcal{G}_{a,b}(i,k-j;0)$ ways as none of them can belong to distinguished blocks. Summing over all possible i and j gives (9). □

If $n \geq 0$, then let $\mathcal{G}_{a,b}(n;r) = \sum_{k=0}^{n}\mathcal{G}_{a,b}(n,k;r)$. Note that $\mathcal{G}_{a,b}(n;0)$ reduces to the Bell number, A000110 of [17], when $a=0$, $b=1$ and to the sequence A000262 of [17], when $a=b=1$. Summing (7) and (9) over k gives, respectively, the formulas

$$\mathcal{G}_{a,b}(n;r+s) = \sum_{i=0}^{n}\binom{n}{i}\mathcal{G}_{a,b}(i;r)\prod_{j=0}^{n-i-1}(aj+(a+b)s) \tag{10}$$

and

$$\begin{aligned}\mathcal{G}_{a,b}(n+m;r) &= \sum_{i=0}^{n}\sum_{j=0}^{m}\binom{n}{i}\mathcal{G}_{a,b}(m,j;r)\mathcal{G}_{a,b}(i;0)\\ &\quad\times\prod_{\ell=0}^{n-i-1}(a\ell+a(m+r)+b(j+r)).\end{aligned} \tag{11}$$

Note that the $a=0, b=1, s=1$ case of (10) occurs as Theorem 7.1 of [12] (see also Theorem 1 of [14]). Moreover, the $a=0, b=1$ case of (11) occurs as Theorem 2 of [13] (see also [18]). The polynomials $\mathcal{G}_{a,b}(n;r)$ also satisfy the following recurrence formulas.

PROPOSITION 3.4

If $n \geq 0$, *then*

$$\mathcal{G}_{a,b}(n;r) = \sum_{i=0}^{n}\binom{n}{i}\mathcal{G}_{a,b}(n-i;0)\prod_{j=0}^{i-1}(aj+(a+b)r) \tag{12}$$

and

$$\begin{aligned}\mathcal{G}_{a,b}(n+1;r) &= r\sum_{i=0}^{n}\binom{n}{i}\mathcal{G}_{a,b}(n-i;r-1)\prod_{j=0}^{i}(aj+a+b)\\ &\quad+\sum_{i=0}^{n}\binom{n}{i}\mathcal{G}_{a,b}(n-i;r)\prod_{j=0}^{i-1}(aj+a+b).\end{aligned} \tag{13}$$

Proof. To show (12), consider the number, i, of elements in $[r+1, r+n]$ that belong to distinguished blocks within $\mathcal{L}_r(n) = \cup_{k=0}^{n}\mathcal{L}_r(n,k)$. Note that there are $\binom{n}{i}$ ways to select these elements and $\prod_{j=0}^{i-1}(aj+(a+b)r)$ ways in which to arrange them, once selected, within the distinguished blocks. (Note that the j-th smallest element chosen is the j-th to be arranged and thus contributes $a(j-1)+(a+b)r$ for $1 \leq j \leq i$.) The remaining $n-i$

elements of $[r+1, r+n]$ may then be partitioned in $\mathcal{G}_{a,b}(n-i;0)$ ways. Summing over all possible i gives (12).

To show (13), we consider whether or not the element $n+r+1$ belongs to a distinguished block within a member of $\mathcal{L}_r(n+1)$. If it does, then there are r choices for the block, which we will denote by B. If there are i other elements of $[r+1, r+n+1]$ in B, then there are $\binom{n}{i}$ ways in which to select these elements and $\prod_{\ell=0}^{i}(a\ell+a+b)$ ways in which to arrange all $i+2$ elements within B. The remaining $n-i$ elements of $[r+1, r+n+1]$ and the other $r-1$ elements of $[r]$ can then be arranged together according to any member of $\mathcal{L}_{r-1}(n-i)$. Thus, the first sum on the right-hand side of (13) gives the weight of all members of $\mathcal{L}_r(n+1)$ in which $n+r+1$ belongs to a distinguished block. By similar reasoning, the second sum gives the weight of all members of $\mathcal{L}_r(n+1)$ in which $n+r+1$ belongs to a non-distinguished block according to the number i of other elements in this block. □

Taking $a=0, b=1$ in (12) gives

$$B_{n,r} = \sum_{i=0}^{n} r^i \binom{n}{i} B_{n-i}, \qquad n \geq 0,$$

which is equivalent to the $x=1$ case of equation (4) of [12], where $B_{n,r} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r$ denotes the r-Bell number and B_n denotes the usual Bell number.

Taking $a=0, b=1$ in (13), and applying (10) when $s=1$ to both sums, gives

$$B_{n+1,r} = rB_{n,r} + B_{n,r+1}, \qquad n \geq 0,$$

which is as in Theorem 8.1 of [12].

Remark. Adding a variable x that marks the number of non-distinguished blocks within members of $\mathcal{L}_r(n)$, identity (13) can be generalized to

$$\begin{aligned}\mathcal{G}_{a,b,x}(n+1;r) &= r\sum_{i=0}^{n}\binom{n}{i}\mathcal{G}_{a,b,x}(n-i;r-1)\prod_{j=0}^{i}(aj+a+b)\\ &\quad +x\sum_{i=0}^{n}\binom{n}{i}\mathcal{G}_{a,b,x}(n-i;r)\prod_{j=0}^{i-1}(aj+a+b),\end{aligned}$$

which reduces to Theorem 4.2 of [12] when $a=0, b=1$. The other identities above for $\mathcal{G}_{a,b}(n;r)$ can also be similarly generalized. We have the following additional formula for $\mathcal{G}_{a,b}(n;r)$.

PROPOSITION 3.5

If $n \geq 0$, then

$$\begin{aligned}\mathcal{G}_{a,b}(n;r) &= \mathcal{G}_{a,b}(n;0) + (a+b)r\sum_{i=0}^{n-1}\sum_{j=0}^{i}\sum_{\ell=0}^{n-i-1}\binom{n-i-1}{\ell}\mathcal{G}_{a,b}(i,j;0)\\ &\quad \times\mathcal{G}_{a,b}(n-i-\ell-1;0)\prod_{t=0}^{\ell-1}v(i,j,t),\end{aligned} \tag{14}$$

where $v(i,j,t) = a(i+r+t+1) + b(j+r)$.

Proof. If $n = 0$ or $r = 0$, then the identity is immediate, so assume $n, r \geq 1$. First note that the weight of all members of $\mathcal{L}_r(n)$ in which no element of $[r + 1, r + n]$ belongs to a distinguished block is $\mathcal{G}_{a,b}(n; 0)$. So we must show that the sum on the right-hand side of (14) gives the weight of all members of $\mathcal{L}_r(n)$ in which at least one element of $[r+1, r+n]$ belongs to a distinguished block. To enumerate such distributions π, consider the smallest element, $r + i + 1$, of $[r + 1, r + n]$ lying within a distinguished block of π, where $0 \leq i \leq n - 1$. Then the elements of $[r + 1, r + i]$ constitute a distribution enumerated by $\mathcal{G}_{a,b}(i, j; 0)$ for some $0 \leq j \leq i$, and there are $(a + b)r$ possibilities regarding the position of the element $r + i + 1$.

Concerning the positions of the members of $[r+i+2, r+n]$ within π, suppose further that exactly ℓ of them belong to a block containing at least one member of $[r + i + 1]$, where $0 \leq \ell \leq n - i - 1$. Then there are $\binom{n-i-1}{\ell}$ choices regarding the selection of these elements, the subset of which we will denote by $U = \{u_0 < u_1 < \cdots < u_{\ell-1}\}$. Then the contribution of element u_t towards the weight is seen to be $v(i, j, t)$ for each t, as there are $i+r+t+1$ elements already belonging to the first $j+r$ blocks at the time that u_t is inserted (assume that members of U are inserted in increasing order, starting with the smallest). Thus, the contribution of all members of U towards the weight is $\prod_{t=0}^{\ell-1} v(i, j, t)$. Finally, there are $\mathcal{G}_{a,b}(n - i - \ell - 1; 0)$ possibilities concerning the positions of the members of $[r+i+2, r+n]-U$ since they may be arranged according to any Lah distribution with no restriction as to the number of blocks. Summing over all possible i, j, and ℓ gives (14). □

The $a = 0, b = 1$ case of (14) yields the following formula for the r-Bell numbers which we were unable to find in the literature.

COROLLARY 3.6

If $n \geq 0$, *then*

$$B_{n,r} = B_n + r \sum_{i=0}^{n-1} \sum_{j=0}^{i} \sum_{\ell=0}^{n-i-1} \binom{n-i-1}{\ell} \left\{ {i \atop j} \right\} (j+r)^{\ell} B_{n-i-\ell-1}, \tag{15}$$

where $\left\{ {i \atop j} \right\}$ *denotes the classical Stirling number of the second kind.*

4. Combinatorial proofs of *r*-Lah formulas

In this section, we provide combinatorial proofs of the following relations involving the r-Lah numbers which were given in Theorem 3.11 of [16].

Theorem 4.1. *Let* $0 \leq k \leq n$ *and* $r, s \geq 0$. *Then*

(*i*) $$\binom{n}{k}(2r-2s)^{\overline{n-k}} = \sum_{j=k}^{n} (-1)^{j-k} \left\lfloor {n \atop j} \right\rfloor_r \left\lfloor {j \atop k} \right\rfloor_s,$$

(*ii*) $$\left\lfloor {n \atop k} \right\rfloor_{2r-s} = \sum_{j=k}^{n} (-1)^{j-k} \left\lfloor {n \atop j} \right\rfloor_r \left\lfloor {j \atop k} \right\rfloor_s, \quad \text{if } 2r \geq s,$$

(*iii*) $$\left\{ {n \atop k} \right\}_{2s-r} = \sum_{j=k}^{n} (-1)^{n-j} \left\{ {n \atop j} \right\}_r \left\lfloor {j \atop k} \right\rfloor_s, \quad \text{if } 2s \geq r,$$

$$(iv) \qquad \left\lfloor {n \atop k} \right\rfloor_{\frac{r+s}{2}} = \sum_{j=k}^{n} \left[{n \atop j} \right]_r \left\{ {j \atop k} \right\}_s,$$

if r and s have the same parity.

Proof of (*i*): First suppose $r \geq s$. Consider the set of ordered pairs (α, β), where $\alpha \in \mathcal{L}_r(n, j)$ for some $k \leq j \leq n$ and β is an arrangement of the $j + s$ blocks of α not containing the elements of $[s+1, r]$ according to some member of $\mathcal{L}_s(j, k)$. Note that within β, the blocks of α are ordered according to the size of the smallest elements. Let $\mathcal{A}$ $(= \mathcal{A}_{n,k})$ denote the set of all such ordered pairs (α, β). Define the sign of $(\alpha, \beta) \in \mathcal{A}$ by $(-1)^{j-k}$, where j denotes the number of non-distinguished blocks of α. Then the right-hand side of (i) gives the sum of the signs of all members of $\mathcal{A}$.

Let $\mathcal{A}^* \subseteq \mathcal{A}$ comprise those pairs in which each block of β contains only one block of α, with this block being a singleton. Then each member of $\mathcal{A}^*$ has a positive sign and $|\mathcal{A}^*| = \binom{n}{k}(2r - 2s)^{\overline{n-k}}$. To show the latter statement, first note that the blocks of β for each $(\alpha, \beta) \in \mathcal{A}^*$ contain k elements of $[r+1, r+n]$, together with the members of $[s]$. Thus, there are $\binom{n}{k}$ choices concerning the elements of $[r+1, r+n]$ to go in these blocks. The remaining $n-k$ elements of $[r+1, r+n]$ then belong to the blocks of α containing the members of $[s+1, r]$. Note that these $n-k$ elements may be positioned in any one of $(2r-2s)^{\overline{n-k}}$ ways amongst these blocks, as there are $2r-2s+i-1$ ways to position the i-th smallest element for $1 \leq i \leq n-k$ (upon selecting the position first for the smallest element and then for the second smallest and so on). This implies the cardinality formula for $|\mathcal{A}^*|$ above.

We now define a sign-changing involution of $\mathcal{A} - \mathcal{A}^*$, which will complete the proof for the case $r \geq s$. To do so, given $(\alpha, \beta) \in \mathcal{A} - \mathcal{A}^*$, suppose that the blocks of β are arranged from left to right in increasing order according to the size of the smallest element of $[n+r]$ contained therein. Identify the leftmost block of β containing at least two elements of $[n+r]$ altogether, which we will denote by B. If the first block of α within B is a singleton, whence B contains at least two blocks of α, then erase brackets and move the element contained therein to the initial position of the block that follows. If the first block of α within B contains at least two elements of $[n+r]$, then form a singleton block using the initial element which then becomes the first block within the sequence of blocks comprising B. One may verify that this mapping provides the desired involution, which completes the proof in the case when $r \geq s$.

Note that since both sides of (i) are polynomials in r and s, establishing the $r \geq s$ case for all non-negative integers r and s completes the proof of (i). However, it is instructive to also provide a combinatorial proof in the case $r < s$. To do so, we show equivalently

$$\binom{n}{k}(2s - 2r)^{\underline{n-k}} = \sum_{j=k}^{n} (-1)^{n-j} \left\lfloor {n \atop j} \right\rfloor_r \left\lfloor {j \atop k} \right\rfloor_s. \tag{16}$$

Define ordered pairs $(\alpha, \beta) \in \mathcal{A}$, where α is as before and β is an arrangement of all of the blocks of α, together with $s - r$ singleton blocks $\{-1\}, \{-2\}, \ldots, \{-(s-r)\}$ (which we will refer to as *special*), according to some member of $\mathcal{L}_s(j, k)$. Note that the distinguished blocks of α, together with the special blocks, are to be regarded as distinguished elements within β (with similar terminology applied to the blocks of β). Furthermore, let us refer to the blocks of β containing $\{-i\}$ for some i as *special* and the other blocks of β as *non-special*. Define the sign by $(-1)^{n-j}$, where j is the number of non-distinguished blocks of α.

Let $\mathcal{A}^*$ consist of all ordered pairs (α, β) in which all blocks of α are singletons and are distributed within β such that the non-special blocks of β contain only one block of α, while the special blocks of β have (i) at most one block of α to the right of the special singleton contained therein, and (ii) at most one to the left of it. Then each member of $\mathcal{A}^*$ has a positive sign and $|\mathcal{A}^*| = \binom{n}{k}(2s-2r)^{\underline{n-k}}$. This follows by first observing that there are $\binom{n}{k}$ ways in which to choose and arrange the elements of $[r+1, r+n]$ that are not to be contained within the special blocks of β. It is then seen that there are $(2s-2r)^{\underline{n-k}}$ ways in which to arrange the remaining elements of $[r+1, r+n]$ in the special blocks of β according to the restrictions above.

To define the sign-changing involution of $\mathcal{A} - \mathcal{A}^*$, first apply the mapping used in the previous case if some non-special block of β contains two or more elements of $[n+r]$ altogether. Otherwise, identify the smallest $i \in [r-s]$, which we will denote by i_0, such that the block of β containing $\{-i\}$ violates condition (i) or (ii). Apply the involution used in the prior case to the blocks of α to the left of $\{-i_0\}$ within its block in β if (i) is violated, and if not, then apply this mapping to the blocks of α occurring to the right of $\{-i_0\}$. Combining the two mappings yields the desired involution of $\mathcal{A} - \mathcal{A}^*$ and completes the proof in the case when $r < s$. □

Proof of (*ii*): In what follows, let $\mathcal{C}_r(n,k)$ denote the subset of $\mathcal{L}_r(n,k)$ enumerated by $\left[{n \atop k}\right]_r$, i.e., those distributions in which the smallest element is first within each block. First assume $r = s$. In this case, let $\mathcal{B}$ $(= \mathcal{B}_{n,k})$ denote the set of ordered pairs (γ, δ) such that $\gamma \in \mathcal{L}_r(n, j)$ for some $k \le j \le n$ and δ is an arrangement of all the blocks of γ according to some member of $\mathcal{C}_r(j,k)$. Here, it is understood that the blocks of γ are ordered according to the size of the smallest element, with the distinguished blocks of γ considered distinguished as elements of δ. Furthermore, the first block of γ within each cycle of δ is the smallest (i.e., it contains the smallest element of $[n+r]$ contained within all of the blocks in the cycle). Define the sign of (γ, δ) as $(-1)^{j-k}$, where j denotes the number of non-distinguished blocks of γ. Then the right-hand side of (ii) when $r = s$ is the sum of the signs of all members of $\mathcal{B}$.

Let $\mathcal{B}^* \subseteq \mathcal{B}$ comprise those pairs $(\gamma, \delta) \in \mathcal{B}$ satisfying the conditions: (i) within each block of γ, the first element is smallest, and (ii) no block of δ contains two or more blocks of γ. Note that members of $\mathcal{B}^*$ must contain k non-distinguished blocks, for otherwise, (ii) would be violated, whence each member of $\mathcal{B}^*$ has a positive sign. Furthermore, members of $\mathcal{B}^*$ are seen to be synonymous with members of $\mathcal{C}_r(n,k)$.

We define a sign-changing involution of $\mathcal{B} - \mathcal{B}^*$ as follows. Suppose that the cycles of δ within $(\gamma, \delta) \in \mathcal{B} - \mathcal{B}^*$ are arranged in increasing order according to the size of the smallest element of $[n+r]$ contained therein. Let B be the left-most cycle of δ that either contains at least two blocks of δ or contains a block of δ in which the first element fails to be the smallest. Consider the first block x within B. If x is of the form $\{a, \ldots, b, \ldots\}$, where b is the smallest element of the block and $b \ne a$, then replace it within B with the two blocks $\{b, \ldots\}, \{a, \ldots\}$. On the other hand, if the first element of x is the smallest, whence B contains at least two blocks, then write all elements of x in order at the end of the second block of B. This mapping provides the desired involution and establishes the result in the case $r = s$.

Now suppose $r < s \le 2r$ and write $s = 2r - \ell$ for some $0 \le \ell \le r-1$. We modify the proof given in the prior case as follows. Let $\mathcal{B}$ consist of all ordered pairs (γ, δ), where $\gamma \in \mathcal{L}_r(n, j)$ and δ is an arrangement of all the blocks of γ, together with the special singletons $\{-i\}$ for $i \in [r-\ell]$, arranged according to some member of $\mathcal{C}_s(j,k)$. Blocks are ordered according to the size of the smallest elements contained therein and the cycles of δ are arranged as before. The distinguished elements of δ are the distinguished blocks of γ, together with the special singletons. Let $\mathcal{B}^* \subseteq \mathcal{B}$ consist of those pairs satisfying conditions (i) and (ii) above, where for (ii), we exclude from consideration cycles of δ containing $\{-i\}$ for some i. Given $(\gamma, \delta) \in \mathcal{B}-\mathcal{B}^*$, apply the prior involution to the cycles of δ not containing the special singletons.

Let $\mathcal{B}' \subseteq \mathcal{B}^*$ consist of those (γ, δ) in which the numbers $\pm i$ for $i \in [r-\ell]$ all belong to singleton blocks of γ each occupying its own cycle of δ. Note that $|\mathcal{B}'| = \begin{bmatrix} n \\ k \end{bmatrix}_{2r-s}$ since there are $\ell = 2r - s$ distinguished cycles (i.e., those containing a block with an element of $[r-\ell+1, r]$ in it), with all cycles containing a single contents-ordered block whose first element is also the smallest. To complete the proof in this case, we extend the involution to $\mathcal{B}^* - \mathcal{B}'$ as follows. Let i_0 be the smallest $i \in [r-\ell]$ such that either (a) a cycle of δ containing $\{-i\}$ also has one or more elements of $[r+1, r+n]$ in it, or (b) a cycle of δ containing $\{-i\}$ has only that block in it, with i not occurring as a singleton block of γ.

If (a) occurs and there are at least two elements of $[r+1, r+n]$ altogether in the cycle of δ containing $\{-i_0\}$, then apply the mapping used in the proof of the $r \geq s$ case of (i) above to the blocks of this cycle excluding $\{-i_0\}$. Otherwise, if (a) occurs and there is only one element of $[r+1, r+n]$ in the cycle containing $\{-i_0\}$ or if (b) occurs, then replace one option with the other by either moving the element in the other block of the cycle containing $\{-i_0\}$ to the last position of the block containing i_0 within its cycle or vice-versa. Combining the last two mappings provides the desired involution of $\mathcal{B}^* - \mathcal{B}'$ and completes the proof in the $r < s \leq 2r$ case.

Finally, if $r > s$, then consider ordered pairs (γ, δ), where δ is an arrangement in cycles of all the non-distinguished blocks of γ, together with those containing i for some $i \in [s]$. Apply the involution used in the $r = s$ case above, but this time excluding from consideration those blocks of γ containing i for some $i \in [s+1, r]$. The set of survivors (γ, δ) of this involution then consists of all (γ, δ) where any block of γ (other than the one containing some $i \in [s+1, r]$) has its smallest element first, with the cycles of δ each containing one block of γ. We add $r - s$ to all of the elements in $[r+1, r+n]$. Then to a block of γ containing $i \in [s+1, r]$, we write $i + r - s$ at the front of it. For each i, split the block now containing $i + r - s$ and i into two separate blocks starting with these elements. Designate all blocks starting with $i \in [2r-s]$ as distinguished. From this, it is seen that the set of survivors of the involution in this case are synonymous with members of $\mathcal{C}_{2r-s}(n, k)$, which completes the proof. □

Proof of (iii): One can give a similar proof to (ii) above. We describe the main steps. Let $\mathcal{S}_r(n, k)$ denote the subset of $\mathcal{L}_r(n, k)$ enumerated by $\left\{ n \atop k \right\}_r$, i.e., those distributions in which the elements occur in increasing order within each block. In the case $r = s$, consider the set $\mathcal{D}$ of ordered pairs (ρ, τ) such that $\rho \in \mathcal{S}_r(n, j)$ for some j and τ is an arrangement of the blocks of ρ according to some member of $\mathcal{L}_r(j, k)$. Define an involution of $\mathcal{D}$ by considering the first block of τ containing a non-singleton block of ρ or in which the blocks of ρ are not arranged in increasing order of smallest elements (possibly both). Within this block of τ, in a left-to-right scan of the blocks of ρ contained therein, consider the first occurrence of either (i) consecutive blocks of the form $B = \{x\}$, $C = \{y, \ldots\}$, where $x > \max(C)$, or (ii) $C = \{y, \ldots\}$, where $|C| \geq 2$ and the block directly preceding C (if it exists) contains a single element x that is strictly smaller than $\max(C)$. We replace one option with the other by either moving the element in B to the end of block C in (i) or taking the last element of block C as in (ii) and forming a singleton that directly precedes it.

If $r < s$, then add $s - r$ special singleton blocks to the arrangement τ to be regarded as distinguished. Apply the involution used in the previous case to the blocks of τ not containing a special singleton. To the blocks of τ containing a special singleton, we apply the involution separately to the sections to the left and to the right of it. Given a survivor of this involution, we break the blocks of τ into two sections with the special singleton starting the second section and then add a distinguished element to the first section. Note that this results in $r + 2(s - r) = 2s - r$ distinguished blocks in all.

If $s < r \leq 2s$, then consider ordered pairs (ρ, τ), where τ consists of contents-ordered blocks whose elements are the non-distinguished blocks of ρ, together with the first s distinguished blocks of ρ. Apply the involution used in the $r = s$ case, excluding from consideration those blocks of ρ containing a member of $[s + 1, r]$. We extend this involution by considering the smallest $i \in [r - s]$, if it exists, such that there is at least one member of $[r+1, r+n]$ in either the block of τ containing i or in the block of ρ containing $r+1-i$ (possibly both). Let M denote the largest element of $[r+1, r+n]$ contained in either of these blocks. If M belongs to the block of τ containing i, necessarily as a singleton $\{M\}$, then erase the brackets enclosing M and move it to the final position of the block of ρ containing $r + 1 - i$, and vice-versa, if M belongs to the block of ρ containing $r + 1 - i$. The set of survivors of this extended involution is seen to have cardinality $\left\{ {n \atop k} \right\}_{2s-r}$. □

Proof of (*iv*): Suppose r and s have the same parity. First assume $r \geq s$. Let $\mathcal{E}$ denote the set of ordered pairs (α, β) such that $\alpha \in \mathcal{C}_r(n, j)$ for some $k \leq j \leq n$ and β is an arrangement of the j non-distinguished cycles of α, together with s special singleton cycles $(-1), (-2), \ldots, (-s)$, into $k + s$ blocks according to some member of $\mathcal{S}_s(j, k)$. Then the right-hand side of (iv) gives the cardinality of $\mathcal{E}$. To complete the proof in this case, we define a bijection between the sets $\mathcal{E}$ and $\mathcal{L}_{\frac{r+s}{2}}(n, k)$.

To do so, given $(\alpha, \beta) \in \mathcal{E}$, let C_i, $1 \leq i \leq r$, denote the cycles of α containing the members of $[r]$. We assume that the smallest element is written first within a cycle. If $1 \leq i \leq s$, then let $C_1^{(i)}, C_2^{(i)}, \ldots, C_{t_i}^{(i)}$ for some $t_i \geq 0$ denote the other cycles (if any) within the block of β containing $(-i)$, arranged in decreasing order of smallest elements. For each i, we remove the parentheses enclosing these cycles and concatenate the resulting words into one long word, which we write to the left of the elements in cycle C_i in a single block. (If there are no other cycles in the block of β containing $(-i)$, then only the elements in cycle C_i are written.) This yields s contents-ordered blocks containing the distinguished elements $1, \ldots, s$.

For $\frac{r+s}{2} < j \leq r$, consider the word W_j obtained by reading the contents of cycle C_j from left to right, excluding the initial letter j. We then write the letters of W_j in order, followed by the contents of cycle $C_{j-\frac{r-s}{2}}$, in a single block for each j. This yields $\frac{r-s}{2}$ additional blocks containing the distinguished elements $s + 1, \ldots, \frac{r+s}{2}$. For the other k blocks of β (which contain cycles of α having only elements in $[r + 1, r + n]$), express the permutation corresponding to the sequence of cycles contained therein as a word. Putting these blocks together with the prior ones yields a Lah distribution containing all the elements of the set $[\frac{r+s}{2}] \cup [r + 1, r + n]$ in which members of $[\frac{r+s}{2}]$ all belong to distinct blocks. Subtracting $\frac{r-s}{2}$ from each letter in $[r + 1, r + n]$ yields a member of $\mathcal{L}_{\frac{r+s}{2}}(n, k)$, which we will denote by $f(\alpha, \beta)$.

To reverse the mapping f, given $L \in \mathcal{L}_{\frac{r+s}{2}}(n,k)$, we reconstruct its pre-image $(\alpha,\beta) \in \mathcal{E}$ as follows. First observe that within the block of L containing i for some $i \in [s]$, a left-to-right minima (excepting i), taken together with the sequence of letters between it and the next minima, corresponds to a cycle belonging to the block of β containing $(-i)$, with the letters to the right of and including i forming the cycle C_i of α. Within blocks of L containing i for $i \in [s+1, \frac{r+s}{2}]$, elements to the right of and including i constitute cycle C_i of α, while those to the left of i (if any) constitute the letters beyond the first letter of cycle $C_{i+\frac{r-s}{2}}$ in α. Finally, writing the permutations in the undistinguished blocks of L as cycles (and adding $\frac{r-s}{2}$ to each letter in $[\frac{r+s}{2}+1, \frac{r+s}{2}+n]$) yields the remaining cycles of α and blocks of β.

Now assume $r < s$ and let $\mathcal{E}$ consist of the ordered pairs (α,β) as before. To define the mapping f in this case, we proceed as follows. For each $i \in [\frac{s-r}{2}+1, s]$, we delete the cycle $(-i)$ from its block within β and then concatenate the contents of the remaining cycles, where cycles within a block are arranged in decreasing order of size of their first elements. We then write the resulting word in a block followed by the contents of cycle $C_{i-\frac{s-r}{2}}$ if $\frac{s-r}{2} < i \le \frac{s+r}{2}$, or followed by the contents of the cycles in the block containing the special cycle $(-(i-\frac{s+r}{2}))$ if $\frac{s+r}{2} < i \le s$. In the latter case, cycles within a block are arranged by decreasing order of first elements, except for the special cycle, which is first.

For each of the remaining blocks of β, we express the permutation corresponding to the sequence of cycles contained therein as a word. At this point, we have $k+\frac{s+r}{2}$ contents-ordered blocks of the set $S \cup [r+n]$ in which members of $S \cup [r]$ belong to distinct blocks, where $S = \{-1,-2,\ldots,-\frac{s-r}{2}\}$. To each element of $[r+1, r+n]$, we add $\frac{s-r}{2}$, and to each element of S, we add $\frac{s+r}{2}+1$. This results in a member of $\mathcal{L}_{\frac{r+s}{2}}(n,k)$ and the prior steps are seen to be reversible. This completes the proof in the case $r < s$. □

It is possible to extend the proofs given above for Theorem 4.1 and show the following generalization in terms of $\mathcal{G}_{a,b}(n,k;r)$.

Theorem 4.2. *If* $0 \le k \le n$ *and* $r, s \ge 0$, *then*

$$\binom{n}{k}\prod_{i=0}^{n-k-1}(ai+(a+b)(r-s)) = \sum_{j=k}^{n}(-1)^{j-k}\mathcal{G}_{a,b}(n,j;r)\mathcal{G}_{b,a}(j,k;s) \quad (17)$$

and

$$\mathcal{G}_{a,b}(n,k;r) = \sum_{j=k}^{n}\mathcal{G}_{a,t}(n,j;r)\mathcal{G}_{-t,b}(j,k;r). \quad (18)$$

Proof. We first show (17) in the case when $r = s$. To do so, we extend the proof given for the first part of Theorem 4.1 above. Let $\mathcal{A}$ denote the set of ordered pairs $\pi = (\gamma,\delta)$, where $\gamma \in \mathcal{L}_r(n,j)$ for some $k \le j \le n$ and δ is an r-Lah distribution having $k+r$ blocks whose elements are the blocks of γ. It is understood that the blocks of γ within δ are ordered according to the size of their smallest elements and that the blocks of γ containing members of $[r]$ are considered distinguished as elements of δ. Define the (signed) weight of π by

$$v(\pi) = (-1)^{j-k}w_{a,b}(\gamma)w_{b,a}(\delta).$$

Then the right-hand side of (17) when $r = s$ gives the sum of the weights of all members of $\mathcal{A}$, by the definition of $\mathcal{G}_{a,b}(n, k; r)$. Note that (17) is trivial if $k = n$. To complete the proof, we define a sign-changing involution of $\mathcal{A}$ when $k < n$.

Given $\pi = (\gamma, \delta) \in \mathcal{A}$, let B denote the leftmost block of δ containing at least two elements of $[n+r]$ altogether (assume that the blocks of δ are ordered from left to right in ascending order of minimal elements). Let block B contain t elements of $[n + r]$, which we will denote by $b_1 < b_2 < \cdots < b_t$, where $t \geq 2$. Now consider the block R of γ within B that contains b_1. First suppose that either (i) b_1 is not the first element of R, or (ii) b_1 is the first element of R, but R is not the rightmost block of γ within B.

We pair members of $\mathcal{A}$ for which (i) or (ii) applies by either replacing the block

$$R = \{a, p_1, p_2, \ldots, b_1, q_1, q_2, \ldots\}$$

with $R = \{b_1, q_1, q_2, \ldots\}$, $S = \{a, p_1, p_2, \ldots\}$, if (i) occurs, or replacing blocks R and S with R, if (ii) occurs. Let $\pi' = (\gamma', \delta')$ denote the resulting member of $\mathcal{A}$; observe that since no two distinguished blocks of γ belong to the same block of δ in π, the same is true of γ' and δ' in π'. Note further that π and π' are of opposite sign since the number of blocks of γ and γ' differ by one, but that

$$w_{a,b}(\gamma)w_{b,a}(\delta) = w_{a,b}(\gamma')w_{b,a}(\delta').$$

To realize the last statement, note that the S block in case (ii) contributes a factor of b towards the v-weight not witnessed in (i) since it is a non-record low within B, whereas the smallest element amongst those to the left of b_1 within R is a record low that is not minimal and hence contributes a factor of b in case (i) that is not witnessed in (ii).

Suppose now that R starts with b_1 and is the last block of B. Now apply the same involution as in the preceding paragraph, using b_2, to the blocks of γ within B excluding R. This extended involution is not defined if it is the case that (I) b_2 belongs to R, or (II) b_2 is the first element of the penultimate block of γ within B. We map members of $\mathcal{A}$ for which (I) holds to those for which (II) holds by removing b_2 and all elements to its right from R and forming a separate block which we place directly to the left of R, and vice-versa, if (II) holds. Note that once again only the sign of the weight is changed. Combining the previous mappings yields the desired involution of $\mathcal{A}$ and completes the proof of (17) in the case when $r = s$.

Now assume $r > s$. In this case, we consider the set $\mathcal{A}$ of ordered pairs (γ, δ), where $\gamma \in \mathcal{L}_r(n, j)$ for some $k \leq j \leq n$ and δ is an s-Lah distribution having $k + s$ blocks whose elements are the non-distinguished blocks and the first s distinguished blocks of γ. Applying the same involution as in the $r = s$ case implies that the set of survivors consists of those ordered pairs (γ, δ) in which $\gamma \in \mathcal{L}_r(n, k)$ is such that its first s distinguished blocks and all of its non-distinguished blocks are singletons, which determines δ. Note that the weight of such ordered pairs is $\binom{n}{k}\prod_{i=0}^{n-k-1}(ai + (a + b)(r - s))$, as there are $\binom{n}{k}$ choices for the elements that occupy the non-distinguished blocks of γ and, once they have been selected, $\prod_{i=0}^{n-k-1}(ai + (a + b)(r - s))$ possibilities concerning the positions of the remaining $n - k$ elements of $[r + 1, r + n]$ within the final $r - s$ distinguished blocks of γ. This establishes (17) when $r \geq s$, which implies (17) in general, since both sides are polynomials in r and s.

To show (18), we again consider the set $\mathcal{A}$ from the proof of the $r = s$ case of (17), but this time define the weight of $\pi = (\gamma, \delta) \in \mathcal{A}$ to be $u(\pi) = w_{a,t}(\gamma)w_{-t,b}(\delta)$, where t is an indeterminate. Then the right-hand side of (18) gives the sum of the u-weights of all members of $\mathcal{A}$. We define an involution on $\mathcal{A}$ as follows. Apply the first involution used in the proof of the $r = s$ case of (17) considering the block R in B. On the set of survivors, repeat this involution by considering all blocks within B except for R, which is last. In general, repeat this involution an ℓ-th time, if necessary, on the set of configurations in $\mathcal{A}$ for which the following is true: if $R = R_1, R_2, \ldots, R_{\ell-1}$ are the final $\ell - 1$ blocks within B, then each block R_i has its smallest element first, with $\min(R_1) < \min(R_2) < \cdots < \min(R_{\ell-1})$.

The procedure above ends when it is no longer possible to apply the aforementioned involution, which is seen to always change the sign of the u-weight since either a factor of t is replaced by $-t$, or conversely. The set of survivors of the involution obtained by applying this procedure are precisely those members of $\mathcal{A}$ in which each block of γ has its smallest element first and the blocks of γ within each block of δ are arranged from left to right by decreasing order of smallest elements. Upon erasing the parentheses enclosing the blocks of γ within each block of δ and concatenating the resulting words, the survivors of the involution may be identified with members of $\mathcal{L}_r(n, k)$ and have weight $\mathcal{G}_{a,b}(n, k; r)$.

To see this, note that within a survivor (γ, δ) of the involution, every block of γ within a block of δ is a record low, while within each block of γ, every non-minimal element is a non-record low since the minimal element starts the block. Thus, once the parentheses are removed and words are concatenated, each element that did not start some block of γ becomes a non-record low in the longer word, while each block starter becomes a record low. Therefore, the w-weight of the distribution in $\mathcal{L}_r(n, k)$ that results from the concatenation of the blocks of γ within (γ, δ) is $w_{a,t}(\gamma)w_{-t,b}(\delta)$ for all possible (γ, δ). This implies that the set of survivors has weight $\mathcal{G}_{a,b}(n, k; r)$, as claimed, which completes the proof of (18). □

Note that (17) reduces to part (i) of Theorem 4.1 when $a = b = 1$, while taking $a = t = 1, b = 0$ or $b = -t = 1, a = 0$ or $a = b = 1, t = 0$ in formula (18) gives the $r = s$ cases of parts (ii), (iii), and (iv) of Theorem 4.1, respectively. One also has the following generalizations of parts (ii) and (iii):

$$\mathcal{G}_{a,0}(n, k; 2r - s) = \sum_{j=k}^{n} (-1)^{j-k} \mathcal{G}_{a,a}(n, j; r) \mathcal{G}_{a,0}(j, k; s), \qquad 2r \geq s, \quad (19)$$

and

$$\mathcal{G}_{0,b}(n, k; 2s - r) = \sum_{j=k}^{n} (-1)^{n-j} \mathcal{G}_{0,b}(n, j; r) \mathcal{G}_{b,b}(j, k; s), \qquad 2s \geq r, \quad (20)$$

though we were unable to find a bivariate generalization involving both a and b of either identity.

The following orthogonality relation is a consequence of the $r = s$ case of (17).

COROLLARY 4.3

Let $(a_n)_{n=0}^{\infty}$ *and* $(b_n)_{n=0}^{\infty}$ *be sequences of complex numbers. Then we have* $b_n = \sum_{k=0}^{n} \mathcal{G}_{a,b}(n, k; r) a_k$, $n \geq 0$, *if and only if* $a_n = \sum_{k=0}^{n} (-1)^{n-k} \mathcal{G}_{b,a}(n, k; r) b_k$, $n \geq 0$.

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