Extremely strict ideals in Banach spaces

T S S R K RAO

Theoretical Statistics and Mathematics Division, Indian Statistical Institute,
R. V. College P.O., Bangalore 560 059, India
E-mail: srin@fulbrightmail.org; tss@isibang.ac.in

MS received 13 February 2015; revised 14 May 2015

Abstract. Motivated by the notion of an ideal introduced by Godefroy et al. (Studia Math. 104 (1993) 13–59), in this article, we introduce and study the notion of an extremely strict ideal. For a Poulsen simplex $K$, we show that the space of affine continuous functions on $K$ is an extremely strict ideal in the space of continuous functions on $K$. For injective tensor product spaces, we prove a cancelation theorem for extremely strict ideals. We also exhibit non-reflexive Banach spaces which are not strict ideals in their fourth dual.

Keywords. Extremely strict ideals; spaces of operators; injective tensor products.

2000 Mathematics Subject Classification. Primary: 46B20; Secondary: 47L05, 46E40.

1. Introduction

Let $X$ be a Banach space. We recall from [5] that a closed subspace $Y \subset X$ is said to be an ideal, if there is a linear projection $P : X^* \to X^*$ of norm one such that $\ker(P) = Y^\perp$. This notion is motivated by the well known fact, if $X$ is canonically embedded in $X^{**}$, then the natural projection $Q : X^{***} \to X^{**}$ defined by $Q(\Lambda) = \Lambda|X$, for $\Lambda \in X^{***}$, is a linear projection of norm one with $\ker(Q) = X^\perp$. Clearly the range of a projection of norm one is an ideal. It is also known (see [5]) that $Y \subset X$ is an ideal if and only if for any finite dimensional subspace $F \subset X$ and $\epsilon > 0$, there is a linear map $T : F \to Y$ such that $T = I$ on $F \cap Y$ and $\|T\| \leq (1 + \epsilon)$. From this, it follows easily that being an ideal is a transitive property. It is easy to see that if $Y$ is an ideal in $X$, then for any $x^* \in X^*$, $P(x^*)$ is a norm-preserving extension of $x^*|Y$ and thus $Y^*$ is isometric to a subspace of $X^*$. Henceforth we treat $Y^*$ as a subspace of $X^*$ keeping in mind the particular embedding of $Y^*$ is dependent on $P$. We note in particular that if $Y \subset X$ is an ideal and if functionals in $Y^*$ have unique norm-preserving extension in $X^*$, the projection $P$ with the above properties is unique, so that the embedding of $Y^*$ in $X^*$ is independent of the projection. Let $X_1$ denote the closed unit ball of $X$. It is well known that $X_1$ is weak*-dense in $X^{**}$. Analogous to this, the author studied in [10], ideals $Y \subset X$ for which $Y_1^*$ is weak*-dense in $X_1^*$ (equivalently, $Y_1^*$ is a norming set for $X$, i.e., $\|x\| = \sup\{|y^*(x)| : \|y^*\| \leq 1\}$, $x \in X$). These are called strict ideals and it was shown in Lemma 1 of [10], that for these ideals, under the canonical embedding, $Y \subset X \subset Y^{**}$. Let $\partial Y_1^*$ denote the set of extreme points of $Y_1^*$. We say that a strict ideal $Y \subset X$ is an extremely strict ideal in $X$, c Indian Academy of Sciences 381
if \( \partial_e Y_1^+ \) determines the norm of \( X \). Several examples of strict ideals from among function spaces and spaces of operators were exhibited in [10]. In this paper, we exhibit several examples of extremely strict ideals. Under some mild approximation theoretic conditions, we show that the space of compact operators forms an extremely strict ideal in the space of bounded operators.

For a non-reflexive Banach space \( X \) we denote its dual fourth, by \( X^{(IV)} \). For the inclusions, \( X \subset X^{**} \subset X^{(IV)} \), we have that \( X \) is an ideal in \( X^{(IV)} \). Using this set up, we give an example to show that being a strict ideal is not transitive. We show that if \( X \), under the canonical embedding, is an \( M \)-ideal in \( X^{**} \), then it is not a strict ideal in \( X^{(IV)} \).

We recall from [6] that a closed subspace \( Y \subset Z \) is said to be an \( M \)-ideal, if there is a projection \( P : Z^* \rightarrow Z^* \) such that \( \ker(P) = Y^\perp \) and \( \|P(z^*)\| = \|z^* - P(z^*)\| = \|z^*\| \) for all \( z^* \in Z^* \). Clearly \( Y \) is an ideal in \( Z \) and it is easy to see that functionals in \( Y^* \) have unique norm-preserving extension to \( Z \), this \( P \) is uniquely determined with the ideal property.

Turning to injective tensor products, we prove a cancellation theorem, that if \( Y \subset Z \) is an \( M \)-ideal and \( Y \hat{\otimes}_e X \) is an extremely strict ideal in \( Z \hat{\otimes}_e X \), then \( Y \) is an extremely strict ideal in \( Z \).

Let \( X \) be a Banach space such that \( \partial_e X_1^+ \) is weak*-dense in \( X_1^+ \). In these spaces we show that any \( M \)-ideal of finite co-dimension such that \( \partial_e (M^\perp)_1 \) is a finite set (modulo multiplication by scalars of absolute value one) is an extremely strict ideal.

### 2. Main results

Throughout this paper, we equip the dual unit ball of a Banach space with the weak*-topology. In what follows, closure operation in the weak*-topology is denoted by an over line with *.

#### DEFINITION 1

An ideal \( Y \subset X \) is said to be extremely strict, if there is a linear projection \( P : X^* \rightarrow X^* \) of norm one such that \( \ker(P) = Y^\perp \) and \( X_1^+ = \overline{CO(\partial_e P(X^*)_1)}^\ast \).

#### Example 2.

Let \( X \) be a Banach space such that \( X_1^+ = \overline{CO(\partial_e X_1^+)} \), where the closure is taken in the norm topology. Let \( Q : X^{***} \rightarrow X^{***} \) be the canonical projection \( Q(\Lambda) = \Lambda \vert X \). It is now easy to see that \( X \) is an extremely strict ideal in \( X^{**} \).

We also note that if for this projection \( Q, \partial_e (Q(X^{***}))_1 \) determines the norm of \( X^{**} \), then in the norm topology, \( X_1^+ = \overline{CO(\partial_e X_1^+)}^\ast \).

#### Remark 3.

If \( Z \) is a strict ideal in a Banach space \( X \) and \( Z_1^* = \overline{CO(\partial_e Z_1^+)} \) (closure in the norm topology) then since there is a projection \( P : X^* \rightarrow X^* \) such that \( P(X^*)_1 \) is weak*-dense in \( X^* \), it is easy to see that \( Z \) is an extremely strict ideal in \( X \). If \( X \) has no isomorphic copy of the sequence space \( \ell^1 \), for any \( Z \subset X \) we also have \( Z_1^* = \overline{CO(\partial_e Z_1^+)} \) (see page 215 of [3]) and hence if \( Z \) is a strict ideal then it is an extremely strict ideal.

#### Example 4.

Let \( M = \{ f \in C([0,1]) : f(0) = f(1) = 0 \} \). Identifying \( C([0,1])^\ast \) as the space of regular Borel measures, it is easy to see that with respect to the projection \( \mu \rightarrow \mu|\{0,1\}, M \) is an extremely strict ideal in \( C([0,1]) \) but as the Lebesgue measure is non-atomic, \( M_1^\ast \) is not the norm closed convex hull of its extreme points.
Let $K$ be a compact convex set. We next use results from Choquet theory (see [1]) to give new examples of extremely strict ideals. Let $A(K)$ denote the space of real-valued affine continuous functions on $K$, equipped with the supremum norm. We recall that $Z(K) \subset C(K)^*$ denotes the subspace of boundary measures on $K$. A metrizable Choquet simplex such that $\partial eK$ is dense in $K$ is called a Poulsen simplex (see [1]). Also for a metrizable compact convex set $K$, $\partial eK$ is a $G_\delta$-set and for any boundary measure $\nu$, $\|\nu\| = |\nu|(\partial eK)$.

**Proposition 5**

Let $K$ be an infinite dimensional Choquet simplex, such that the set of extreme points $\partial eK$ is dense in $K$. $A(K)$ is an extremely strict ideal in $C(K)$.

*Proof.* We will exhibit a projection $P : C(K)^* \to C(K)^*$ of norm one such that $\ker(P) = A(K)^\perp$ and such that the extreme points of the unit ball $P(C(K)^*)_1$ determine the norm of $C(K)$.

Let $\mu \in C(K)^*$. Let $\nu \in Z(K)$ be the boundary measure representing $\mu|A(K)$. Define $P : C(K)^* \to C(K)^*$ by $P(\mu) = \nu$. Since $K$ is a simplex, it follows from Lemma II.3.5 of [1] that this map is well-defined and is a linear projection of norm one and $A(K)^\perp \subset \ker(P)$. If $P(\mu) = 0$, then $\mu^+ = \mu^-$ on $A(K)$ and by evaluating at 1, we see that $\mu^+$ and $\mu^-$ are maximal measures with the same norm, representing the same point of $K$, as $K$ is a simplex, we get that $\mu = 0$. Thus $\ker(P) = A(K)^\perp$.

We next use the density of $\partial eK$ in $K$ to show that $P(C(K)^*)_1$ determines the norm of $C(K)$. Let $\|P(\mu)\| = \|\nu\| \leq 1$. For any $f \in C(K)$, $\int f \, d\nu \leq \int |f| \, d\nu \leq \|f\|$. On the other hand, since $K$ is a simplex, for any $k \in \partial eK$, the Dirac measure $\delta(k)$ is the unique boundary measure representing, $\delta(k)|A(K)$, i.e., $P(\delta(k)) = \delta(k)$. Thus $\|f\| = \|f\|\|\partial eK\| \geq |P(\delta(k)(f))|$. Hence $\partial eP(C(K)^*)_1$ determines the norm of $C(K)$.

□

**Remark 6.** We note that for a metrizable simplex $K$, the projection $P$ defined above has the property, for $f \in C(K)$, $\sup\{|f(x)| : x \in \partial eK\} = \sup\{|v(f)| : v \in P(C(K)^*)_1\}$. If $R : C(K)^* \to C(K)^*$ is any projection of norm one such that $\ker(R) = A(K)^\perp$, for any $k \in \partial eK$, one can show that $R(\delta(k)) = \delta(k)$.

We now give an example to show that being a strict ideal is not a transitive property. We recall from Chapter III of [6] that a Banach space $X$ is said to be $M$-embedded, if for the canonical projection $Q : X^{***} \to X^{***}$, $\|\tau\| = \|Q(\tau)\| + \|\tau - Q(\tau)\|$ for all $\tau \in X^{***}$. See Chapter III of [6] for several examples and geometric properties of these spaces. Classical $M$-embedded spaces include $c_0 \subset \ell^\infty$ and $K(\ell^p) \subset L(\ell^p)$ for $1 < p < \infty$.

**Example 7.** Let $X$ be a non-reflexive $M$-embedded space. Let $X^{(V)}$ denote the fifth dual of $X$. For the canonical embedding $X \subset X^{**} \subset X^{(IV)}$, we have that $X$ is a strict ideal in $X^{**}$ and $X^{**}$ is a strict ideal in $X^{(IV)}$. For this embedding, it was shown in [11] that $X$ is an $M$-ideal in $X^{(IV)}$. Hence if $R : X^{(V)} \to X^{(V)}$ is the canonical projection with $\ker(R) = (X^{**})^\perp$, we have that $Q \circ R : X^{(V)} \to X^{(V)}$ is the only projection of norm one in $X^{(V)}$ (which is also an $L$-projection) whose kernel is $X^{**}$. Thus $X^{(V)} = X^* \oplus_1 X^{**}$. Here $X^{**}$ is the annihilator of $X$ in the fifth dual of $X$. Now suppose $X$ is a strict ideal, since the embedding of $X^*$ in $X^{***}$ and $X^{(V)}$ is the canonical embedding, weak*-denseness of $X^*$ in $X^{(V)}$ will imply that $X^*$ is weakly dense in $X^{***}$. Hence we get that the canonical
embedding of $X^*$ in $X^{***}$ is onto and hence $X^*$, consequently $X$ is reflexive. Therefore $X$ is not a strict ideal in $X^{(V)}$.

Our next result exhibits another class of spaces where every strict ideal is an extremely strict ideal. These spaces are also extremely strict ideals in their biduals. In what follows we will be making use of results from the structure theory of von Neumann algebras.

**PROPOSITION 8**

Let $X$ be the predual of a von Neumann algebra. Every strict ideal in $X$ is an extremely strict ideal.

**Proof.** Let $Y \subset X$ be a strict ideal. Since $X^*$ is a von Neumann algebra, it follows from Proposition 5 in [10] that $Y$ is the range of a projection of norm one in $X$. Thus by the structure theorem we have that $Y$ is isometric to the predual of a JBW*-triple. Now by the Choda–Russo–Dye type theorem for JBW*-triples [12], we have that $Y^*$ is the convex hull of its extreme points. Therefore from the remarks made earlier, we get that $Y$ is an extremely strict ideal in $X$. \qed

We next consider the space of compact operators $K(X, Y)$.

**Example 9.** For $x^{**} \in X^{**}$ and $y^* \in Y^*$, let $x^{**} \otimes y^*$ be the functional defined on $L(X, Y)$ by $(x^{**} \otimes y^*)(T) = x^{**}(T^*(y^*))$. We recall that $\partial_e(K(X, Y))^* = \{x^{**} \otimes y^* : x^{**} \in \partial_e X^{**}, \quad y^* \in \partial_e Y^*\}$. See page 266 of [6]. Now suppose there is a net $(T_\alpha) \subset K(Y)^*$ such that $T_\alpha \to I$ pointwise as well as $T_\alpha \to I$ pointwise. It follows from Lemma 1 of [7] that $K(X, Y)$ is an ideal in $L(X, Y)$, by the projection $\Lambda R$ defined in that paper. Here $R : L(X, Y)^* \to K(X, Y)^*$ is the restriction map and $\Lambda : K(X, Y)^* \to L(X, Y)^*$ is an into isometry. We next note that because of the assumptions on the net $(T_\alpha)$ (see also Theorem 1 of [7]), this projection is identity on functionals of the form $x^{**} \otimes y^*$. Let $T \in L(X, Y)$ and $\epsilon > 0$. Let $x \in X_1$ be such that $\|T(x)\| > \|T\| - \epsilon$. Let $y^* \in \partial_e Y^*$ be such that $T^*(y^*)(x) = y^*(T(x)) > \|T\| - \epsilon$. Now let $x^{**} \in \partial_e X^{**}$ be such that $x^{**}(T^*(y^*)) = \|T^*(y^*)\|$. Thus for $x^{**} \otimes y^* \in \partial_e K(X, Y)^*$, we have $(x^{**} \otimes y^*)(T) = x^{**}(T^*(y^*)) = \|T^*(y^*)\| \geq y^*(T(x)) > \|T\| - \epsilon$. Thus $\partial_e \Lambda R(L(X, Y))$ is a norming set for $L(X, Y)$ and hence $K(X, Y)$ is an extremely strict ideal in $L(X, Y)$.

Our next theorem deals with the space of injective tensor products. For Banach spaces $X$ and $Y$, let $X \hat{\otimes}_e Y$ denote the injective tensor product. We refer to [2] and [6] for general theory of injective products and for some of the results that we will be using here. We recall that under assumptions of approximation property $K(X, Y)$ can be identified with $X^* \hat{\otimes}_e Y$.

For an ideal $Y \subset Z$, let $P : Z^* \to Z^*$ be a projection of norm one such that $\ker(P) = Y^\perp$. We recall from Lemma 2 of [10] that the canonical projection $Q : (Z \hat{\otimes}_e X)^* \to (Z \hat{\otimes}_e X)^*$ is defined by its value on simple tensors, by $Q(\Phi)(\sum_{i=1}^n z_i \otimes x_i) = \sum_{i=1}^n P(\Phi(x_i))(z_i)$, where $\Phi \in (Z \hat{\otimes}_e X)^*$ and for a fixed $x \in X$, $\Phi_x : Z \to \mathbb{R}$ is the functional defined by $\Phi_x(z) = \Phi(z \otimes x)$. $Q$ is a linear projection of norm one, ker $Q = (Y \hat{\otimes}_e X)^\perp$. For $z^* \in Z^*$ and $x^* \in X^*$, we recall that the functional $z^* \otimes x^* \in (Z \hat{\otimes}_e X)^*$ is defined by $(z^* \otimes x^*)(z \otimes x) = z^*(z)x^*(x)$ for $z \in Z$ and $x \in X$, $\|z^* \otimes x^*\| = \|z^*\|\|x^*\|$ for $x \in X$ and $z \in Z$. We next check that for the projection $Q$ defined above, $Q(P(z^*) \otimes x^*) = z^* \otimes x^*$, by recalling that under assumptions of approximation property $K(X, Y)$ can be identified with $X^* \hat{\otimes}_e Y$.\end{document}
Proof. Let \( P(z^*) \otimes x^* \), for \( z^* \in Z^* \) and \( x^* \in X^* \). If \( \Phi = P(z^*) \otimes x^* \), then for any \( x \in X \), \( \Phi_x(z) = P(z^*)(x) \). Hence the claim.

We next recall from Theorem VI.1.3 of [6] that for any \( z^* \in \partial_e Z_1^* \) and \( x^* \in \partial_e X_1^* \), \( z^* \otimes x^* \in \partial_e (Z \otimes_e X)_1^* \) and any extreme point is of this form.

Lemma 10. Suppose \( Y \) is an extremely strict ideal in \( Z \). Then \( Y \otimes_e X \) is an extremely strict ideal in \( Z \otimes_e X \).

Proof. As \( Y \) is an ideal in \( Z \), let \( P : Z^* \to Z^* \) be a projection of norm one such that \( \ker(P) = Y^\perp \). As defined above, \( Q : (Z \otimes_e X)^* \to (Z \otimes_e X)^* \) is a projection of norm one with \( \ker Q = (Y \otimes_e X)^\perp \). We need to show that \( \partial_e Q((Z \otimes_e X)^*)_1 \) determines the norm of \( Z \otimes_e X \). For \( z \in Z \) and \( x \in X \), let \( x^* \in \partial_e X_1^* \) be such that \( x^*(x) = \|x\| \). Since \( \partial_e P(Z^*)_1 \) determines the norm of \( Z \), for \( \epsilon > 0 \), there exists \( P(z^*) \in \partial_e P(Z^*)_1 \) such that \( P(z^*)(z) > \|z\| - \epsilon \). Now \( P(z^*) \otimes x^* \in \partial_e (Y \otimes_e X)_1^* \), and is such that \( (P(z^*) \otimes x^*)(z \otimes x) = P(z^*)(x) > \|z \otimes x\| - \epsilon \). As noted before \( Q(P(z^*) \otimes x^*) = P(z^*) \otimes x^* \).

We now consider \( u = \sum z_i \otimes x_i \) for \( z_1, \ldots, z_n \in Z \) and \( x_1, \ldots, x_n \in X \). Considering \( u : X^* \to Z \) as a finite rank operator, it is easy to see that it is weak*-norm continuous on the unit ball. Thus by weak*-compactness, \( u \) attains its norm. Assume w.l.o.g. that \( \|u\| = 1 \). It is easy to see that \( F = \{ x^* \in X_1^* : \|u(x^*)\| = 1 \} \) is a norming set. Also \( F = \cap_{n \geq 1} \{ x^* \in X_1^* : \|u(x^*)\| \geq 1 - \frac{1}{n} \} \). It now follows from a result of T. Johannesen (Theorem 5.8 of [9]), that \( F \) has an extreme point of \( X_1^* \). Now the proof can be completed, using arguments, similar to the ones given during the claims in Example 9. Therefore \( \partial_e Q((Z \otimes_e X)^*)_1 \) determines the norm of \( Z \otimes_e X \).

Remark 11. Arguments in the above proof can be considerably shortened, if one assumes that \( X_1^* \) is the norm closed convex hull of its extreme points (which is the case for several spaces considered in this paper).

Remark 12. Let \( x_0 \in X \) and \( x_0^* \in X^* \) be such that \( \|x_0\| = 1 = x_0^*(x_0) = \|x_0\| \). The canonical isometry \( z \to z \otimes x_0 \) is an embedding of \( Z \) into \( Z \otimes_e X \) and there is a canonical projection of norm one from \( Z \otimes_e X \) onto the range of the above map which takes a simple tensor \( z \otimes x_0 \) to \( z \otimes x_0^*(x)x_0 \). These maps are compatible with the corresponding embedding of \( Y \) in \( Y \otimes_e X \). Now if \( Y \otimes_e X \) is an ideal in \( Z \otimes_e X \), then using the characterization of ideals in terms of finite dimensional subspaces mentioned in the Introduction, it is easy to see \( Y \subset Z \) is an ideal.

We next use the fact that an \( M \)-ideal \( Y \subset Z \) is an ideal and the associated projection \( P : Z^* \to Z^* \) is unique with respect to the property that \( \|P\| = 1 \) and \( \ker(P) = Y^\perp \), to obtain a partial converse of the above lemma.

Theorem 13. Let \( Y \subset Z \) be an \( M \)-ideal and suppose \( Y \otimes_e X \) is an extremely strict ideal in \( Z \otimes_e X \). Then \( Y \) is an extremely strict ideal in \( Z \).

Proof. Let \( P : Z^* \to Z^* \) be the projection of norm one such that \( \ker(P) = Y^\perp \). We also have now, \( \|P(z^*)\| + \|z^* - P(z^*)\| = \|z^*\| \) for all \( z^* \in Z^* \). Let \( R : (Z \otimes_e X)^* \to (Z \otimes_e X)^* \) be a projection of norm one such that \( \ker(R) = (Y \otimes_e X)^\perp \). Since \( Y \subset Z \) is an \( M \)-ideal, it follows from Proposition VI.3.1 of [6] that \( Y \otimes_e X \) is an \( M \)-ideal in \( Z \otimes_e X \).
Now from the remarks on uniqueness of the projection, we have that \( R = Q \) which was defined in the above lemma. Thus by hypothesis, \( \partial_e Q((Z \otimes X)^*)_1 \) is a norming set for \( Z \otimes X \). This will be used in the last equality below.

We will now show that \( \partial_e P(Z^*)_1 \) is a norming set for \( Z \). It is easy to see that \( \partial_e P(Z^*)_1 \cap \partial_e Y^\perp_1 = \partial_e Z^*_1 \).

Fix \( x \in X, \|x\| = 1 \). For any \( z \in Z \),

\[
\|z\| = \|z \otimes x\| = \sup \{|P(z^*)(z)||x^*(x)| : P(z^*) \in \partial_e P(Z^*)_1, \ x^* \in \partial_e X^*_1\} \\
\leq \sup \{|P(z^*)(z)| : P(z^*) \in \partial_e P(Z^*)_1\} \leq \|z\|.
\]

In the next theorem we exhibit a situation where extremely strict ideal in the space can be used to produce extremely strict ideal in the component space. In general, the intersection of two ideals need not be an ideal. We denote by \( \partial_e M \) the weak \( \partial_e \) ideal of \( \partial_e M \otimes X \). Thus by hypothesis, \( R = \partial_e M \otimes X \). Now from the remarks on uniqueness of the projection, we have that \( \partial_e M \otimes X \) has an extremely strict ideal in the component space. In general, the intersection of two ideals need not be an ideal. We denote by \( \partial_e M \) the weak \( \partial_e \) ideal of \( \partial_e M \otimes X \). Thus in particular, \( M, N \) are \( M \)-ideals in \( X \) and have the properties of \( M \)-ideals, alluded to before. For sake of simplicity, we ignore writing in the proof below, the projection associated with an ideal.

**Theorem 14.** Let \( X \) be a Banach space such that \( X = M \oplus_\infty N \) and \( Z \) is an extremely strict ideal of \( X \) such that \( M \subset Z \). Then \( Z \cap N \) is an extremely strict ideal in \( N \).

**Proof.** Since \( M \subset Z \), we have \( Z = M \oplus_\infty (Z \cap N) \). Thus \( Z \cap N \) is an ideal in \( Z \) and as \( Z \) is an ideal in \( X \), we have by transitivity, \( Z \cap N \) is an ideal in \( X \) and again as \( Z \cap N \subset N \subset X \), we have that \( Z \cap N \) is an ideal in \( N \). To show that \( Z \cap N \) is an extremely strict ideal in \( N \), we will show that \( \partial_e N^*_1 \subset \partial_e (Z \cap N)^*_1 \). Let \( f \in \partial_e N^*_1 \). As \( X^* = M^* \oplus_1 N^* \), we have \( f \in \partial_e X^*_1 \). Since \( Z \) is an extremely strict ideal in \( X \), we have \( X^*_1 = CO(\partial_e Z^*_1) \). Now by Milman’s converse of the Krein–Milman theorem (see [3]), we get a net \( \{f_a\} \subset \partial_e Z^*_1 \) such that \( f_a \rightarrow f \) in the weak*-topology of \( X^* \). As \( X^* = M^* \oplus_1 N^* \), we have that both the summands are closed subspaces with respect to the weak*-topology of \( X^* \). As \( f \notin M^* \), we can choose a subnet, still denoted by \( \{f_a\} \), not in \( M^* \) such that \( f_a \rightarrow f \) in the weak*-topology of \( X^* \). Now as \( Z^* = M^* \oplus_1 (Z \cap N)^* \), as \( f_a \)'s are extreme points of \( Z^*_1 \), we have that this subnet is in \( \partial_e (Z \cap N)^*_1 \). We also have that \( f_a \rightarrow f \) in the weak*-topology of \( Z^* \). Thus \( f \in \partial_e (Z \cap N)^*_1 \). \( \Box \)

In view of Remark 3 above, the following corollary is easy to see.

**Corollary 15.**

Let \( X \) be a Banach space such that \( X \) has no isomorphic copy of \( \ell^1 \). If \( X = M \oplus_\infty N \) and \( Z \) is a strict ideal of \( X \) such that \( M \subset Z \), then \( Z \cap N \) is a strict ideal in \( N \).

Our next result also deals with \( M \)-ideals and this time in Banach spaces \( X \) for which \( \partial_e X^*_1 \) is weak*-dense in \( X^*_1 \). We recall that a finite dimensional space \( Y \) is said to be polyhedral if the unit ball (or equivalently the dual unit ball) has only finitely many extreme points (modulo multiplication by scalars of absolute value one).

**Proposition 16.**

Let \( X \) be a Banach space such that \( \partial_e X^*_1 = X^*_1 \). Let \( M \subset X \) be an \( M \)-ideal such that \( X|M \) is a polyhedral space. Then \( M \) is an extremely strict ideal.
Proof. Since $M$ is an $M$-ideal, we have $X^* = M^* \bigoplus_1 M^\perp$ and $\partial_e X^*_1 = \partial_e M^*_1 \cup \partial_e (M^\perp_1)$. We claim that $\partial_e M^*$ is weak*-dense in $X^*_1$. Let $f \in X^*_1$. By hypothesis, there is a net $\{f_\alpha\} \subset \partial_e X^*_1$ such that $f_\alpha \to f$ in the weak*-topology. Since $\partial_e (M^\perp_1)$ is a finite set, and as $X^*_1$ has no isolated points, we assume w.l.o.g. that this set has no intersection with the net $\{f_\alpha\}$. As the $f_\alpha$’s are extreme points of $X^*_1$, we get that $\{f_\alpha\} \subset \partial_e M^*_1$. Hence the claim. Therefore $M$ is an extremely strict ideal. □

Remark 17. Separable Banach space $X$ such that $X^*$ is isometric to a $L^1(\mu)$-space and $\overline{\partial_e X^*} = X^*_1$, is called the Gurariy spaces and is extensively studied in the literature. See [8] and the recent article [4]. If $X$ is the Gurariy space and $M \subset X$ is a $M$-ideal of finite codimension, then since $M^\perp$ is also an $L^1$-predual space, it is isometric to a finite dimensional $\ell^\infty(n)$, we have that $X|M$ is a polyhedral space. Thus this class of spaces satisfy the hypothesis of the above proposition.

Acknowledgements

The author would like to thank Professor F. Botelho and the Department of Mathematical Sciences at the University of Memphis, for the kind hospitality during his stay in January and February 2015. This work was partially supported by a travel grant from the Simons Foundation. The author is currently a Fulbright-Nehru Academic and Professional Excellence Fellow at the University of Memphis. He would also like to thank the referee for alerting him to the subtleties involved in Lemma 10.

References


Communicating Editor: B V Rajarama Bhat