

Rigidity theorem for Willmore surfaces in a sphere

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Abstract. Let M^2 be a compact Willmore surface in the $(2 + p)$ -dimensional unit sphere S^{2+p} . Denote by H and S the mean curvature and the squared length of the second fundamental form of M^2 , respectively. Set $\rho^2 = S - 2H^2$. In this note, we proved that there exists a universal positive constant C , such that if $\|\rho^2\|_2 < C$, then $\rho^2 = 0$ and M^2 is a totally umbilical sphere.

Keywords. Willmore functional; Sobolev inequality; mean curvature; totally umbilical surface.

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1. Introduction

Let M be a compact surface in the $(2 + p)$ -dimensional unit sphere S^{2+p} . Choose a local orthonormal frame field $\{e_1, e_2, \dots, e_{2+p}\}$ in S^{2+p} such that, restricted to M , the $\{e_1, e_2\}$ are tangent to M . The following convention of indices are used throughout.

$$1 \leq i, j, k \leq 2; \quad 3 \leq \alpha, \beta, \gamma \leq 2 + p.$$

Denote by H and S the mean curvature and the squared length of the second fundamental form of M , respectively. Then, we have

$$S = \sum_{\alpha, i, j} (h_{ij}^\alpha)^2, \quad \mathbf{H} = \sum_{\alpha} H^\alpha e_\alpha, \quad H^\alpha = \frac{1}{2} \sum_k h_{kk}^\alpha, \quad H = |\mathbf{H}|,$$

where h_{ij}^α is the component of the second fundamental tensor of M .

Let $\rho^2 = S - 2H^2$. In fact, if we set $\tilde{h}_{ij}^\alpha = h_{ij}^\alpha - \delta_{ij} H^\alpha$, by a direct computation, one has

$$\rho^2 = \sum_{\alpha, i, j} (\tilde{h}_{ij}^\alpha)^2.$$

So, $\rho^2 \geq 0$, and ρ vanishes at the umbilical points of M .

The Willmore functional is defined by

$$W(x) = \int_M \rho^2 dv = \int_M (S - 2H^2) dv.$$

Here the integration is with respect to the area measure of M . In [3], Chen proved that this functional is invariant under conformal transformations of S^{2+p} .

DEFINITION

$x : M \rightarrow S^{2+p}$ is called a Willmore surface if it is a critical surface of the Willmore functional $W(x)$.

It was proved by Bryant [1] and Weiner [7] that M is a Willmore surface if and only if

$$\Delta^\perp H^\alpha + \sum_{\beta, i, j} h_{ij}^\alpha h_{ij}^\beta H^\beta - 2H^2 H^\alpha = 0, \quad (1.1)$$

i.e.,

$$\Delta^\perp H^\alpha + \sum_{\beta, i, j} \tilde{h}_{ij}^\alpha \tilde{h}_{ij}^\beta H^\beta = 0,$$

where $\Delta^\perp H^\alpha = \sum_k H_{kk}^\alpha$.

From (1.1), we know that all minimal surfaces in S^{2+p} are Willmore surfaces. So, the Veronese surface must be the Willmore surface. Moreover, Pinkall [4] constructed many compact non-minimal flat Willmore surfaces in S^3 , and Castro and Urbano [2] constructed many compact non-minimal Willmore surfaces in S^4 .

In [6], Li obtained the following rigidity theorem for Willmore surfaces in a unit sphere.

Theorem A. *Let M be a compact Willmore surface in S^{2+p} . Then*

$$\int_M \rho^2 \left(2 - \frac{2}{B} \rho^2 \right) dv \leq 0,$$

where

$$B = \begin{cases} 2, & p = 1, \\ \frac{4}{3}, & p \geq 2. \end{cases}$$

In particular, if

$$\rho^2 \leq B,$$

then either $\rho^2 = 0$ and M is totally umbilical, or $\rho^2 = B$. In the latter case, $p = 2$ and M is the Veronese surface or $p = 1$ and $M = S^1 \left(\frac{1}{\sqrt{2}} \right) \times S^1 \left(\frac{1}{\sqrt{2}} \right)$.

Applying Theorem A and the Sobolev inequality, we proved the following result (see [9]).

Theorem B. *Let M be a compact Willmore surface in S^{2+p} . There exists a positive constant $\tilde{C}(H_0)$, defined by*

$$\tilde{C}(H_0) = \frac{B \left(\sqrt{9 + H_0^2} - \sqrt{1 + H_0^2} \right) \sqrt{\pi}}{48\sqrt{3}},$$

such that if

$$\|\rho^2\|_2 < \tilde{C}(H_0),$$

then M is a totally umbilical surface, where $H_0 = \max_{x \in M} H$ and B is defined in Theorem A.

We shall improve the constant of Theorem B and obtain the following global pinching theorem for compact Willmore surfaces in S^{2+p} .

Main theorem. *Let M be a compact Willmore surface in the unit sphere S^{2+p} . There exists an explicit positive constant*

$$C = \frac{(\sqrt{2} - 1)\sqrt{\pi}}{12\sqrt{3}} B,$$

such that if

$$\|\rho^2\|_2 < C,$$

None

then $\rho^2 = 0$ and M is a totally umbilical sphere, where B is defined in Theorem A.

Remark 1. By a simple calculation, we know that the pinching constant in Theorem B $\tilde{C}(H_0) \rightarrow 0$ as $H \rightarrow \infty$. But the pinching constant C in our main theorem is independent of mean curvature H . So C is superior to $\tilde{C}(H_0)$.

2. Basic lemmas

In this section, we introduce several useful lemmas

Lemma 2.1. *Let $x : M^2 \rightarrow S^{2+p}$ be a surface in a unit sphere. We have the following inequality:*

$$|\nabla \rho|^2 \leq \sum_{\alpha, i, j, k} (\tilde{h}_{ijk}^\alpha)^2. \tag{2.1}$$

Proof. We can see from $\rho^2 = \sum_{\alpha, i, j} (\tilde{h}_{ij}^\alpha)^2$ and the Cauchy–Schwarz inequality that

$$|\nabla \rho|^2 \leq \sum_{\alpha, i, j, k} (\tilde{h}_{ijk}^\alpha)^2 \tag{2.2}$$

at all points where $\rho \neq 0$ and hence by analyticity at all the points. □

Lemma 2.2 [10]. Let $x : M \rightarrow S^{2+p}$ be a surface. Then

$$\Sigma R_{\alpha\beta 12}^2 \leq \frac{2-B}{B} \rho^4, \quad (2.3)$$

where equality holds if and only if $p = 1$ or $p \geq 2$, $\sum_{\alpha} (\tilde{h}_{11}^{\alpha})^2 = \sum_{\alpha} (\tilde{h}_{12}^{\alpha})^2$ and $\sum_{\alpha} \tilde{h}_{11}^{\alpha} \tilde{h}_{12}^{\alpha} = 0$. Here B is defined in Theorem A.

Lemma 2.3. Let $x : M \rightarrow S^{2+p}$ be a surface.

$$|\nabla \tilde{h}|^2 \geq |\nabla^{\perp} \mathbf{H}|^2, \quad (2.4)$$

where $|\nabla^{\perp} \mathbf{H}|^2 = \sum_{\alpha, i} (H_i^{\alpha})^2$.

Proof. By a simple calculation, we have

$$|\nabla \tilde{h}|^2 = \sum_{\alpha, i, j, k} (\tilde{h}_{ijk}^{\alpha})^2 = \sum_{\alpha, i, j, k} (h_{ijk}^{\alpha})^2 - 2|\nabla^{\perp} \mathbf{H}|^2. \quad (2.5)$$

In [6], Li proved

$$\sum_{\alpha, i, j, k} (h_{ijk}^{\alpha})^2 \geq 3|\nabla^{\perp} \mathbf{H}|^2. \quad (2.6)$$

Substituting (2.6) into (2.5), we obtain (2.4). \square

Lemma 2.4 [10]. Let $x : M^2 \rightarrow S^{2+p}$ be a compact Willmore surface in a unit sphere. Then

$$\int_M |\nabla^{\perp} \mathbf{H}|^2 dv \leq \int_M \rho^2 H^2 dv. \quad (2.7)$$

Lemma 2.5. Let $x : M \rightarrow S^{2+p}$ be a compact surface.

$$\int_M \sum_{\alpha, i, j} \tilde{h}_{ij}^{\alpha} H_{ij}^{\alpha} dv = - \int_M |\nabla^{\perp} \mathbf{H}|^2 dv. \quad (2.8)$$

Proof. By Stoke formula, we have

$$\int_M \sum_{\alpha, i, j} \tilde{h}_{ij}^{\alpha} H_{ij}^{\alpha} dv = - \int_M \sum_{\alpha, i, j} \tilde{h}_{ijj}^{\alpha} H_i^{\alpha} dv. \quad (2.9)$$

$$\begin{aligned} \sum_{\alpha, i, j} \tilde{h}_{ijj}^{\alpha} H_i^{\alpha} &= \sum_{\alpha, i, j} (h_{ijj}^{\alpha} - \delta_{ij} H_j^{\alpha}) H_i^{\alpha} \\ &= \sum_{\alpha, i, j} h_{jji}^{\alpha} H_i^{\alpha} - \sum_{\alpha, i} H_i^{\alpha} H_i^{\alpha} \\ &= |\nabla^{\perp} \mathbf{H}|^2. \end{aligned} \quad (2.10)$$

We obtain (2.8) by putting (2.10) into (2.9). \square

Lemma 2.6. Let M be a compact 2-dimensional surface in S^{2+p} . Then for any $g \in C^1(M)$, $g \geq 0$, $t > 0$, g satisfies

$$\int_M |\nabla g|^2 dv \geq \frac{t}{A} \left(\int_M g^4 dv \right)^{\frac{1}{2}} - t^2 \int_M g^2 dv - t \int_M \left(1 + \frac{H^2}{2} \right) g^2 dv, \quad (2.11)$$

where $A = \frac{12\sqrt{3}}{\sqrt{\pi}}$.

Proof. From [5] and [8], we have

$$\left(\int_M g^2 dv \right)^{\frac{1}{2}} \leq A \int_M (|\nabla g| + \sqrt{1 + H^2}g) dv.$$

Replacing g by g^2 , we get

$$\begin{aligned} \left(\int_M g^4 dv \right)^{\frac{1}{2}} &\leq A \int_M (|\nabla g^2| + \sqrt{1 + H^2}g^2) dv \\ &= A \int_M (g|\nabla g| + \sqrt{1 + H^2}g^2) dv \\ &\leq A \left(\int_M f^2 dv \right)^{\frac{1}{2}} \left(\int_M |\nabla g|^2 dv \right)^{\frac{1}{2}} + A \int_M \left(1 + \frac{H^2}{2} \right) g^2 dv \\ &\leq At \int_M g^2 dv + \frac{A}{t} \int_M |\nabla g|^2 dv + A \int_M \left(1 + \frac{H^2}{2} \right) g^2 dv, \end{aligned}$$

where $t \in \mathbf{R}^+$. So, we have

$$\int_M |\nabla g|^2 dv \geq \frac{t}{A} \left(\int_M g^4 dv \right)^{\frac{1}{2}} - t^2 \int_M g^2 dv - t \int_M \left(1 + \frac{H^2}{2} \right) g^2 dv,$$

i.e.,

$$\|\nabla g\|_2^2 \geq \frac{t}{A} \|g^2\|_2 - (t^2 + t) \|g^2\|_1 - \frac{t}{2} \|H^2 g^2\|_1.$$

This proves Lemma 2.6. □

3. Proof of the main theorem

In this section, we give the proof of our main theorem. From Lemma 2.1 in [10] and (2.3), we have

$$\begin{aligned} \frac{1}{2} \Delta \rho^2 &= \sum_{\alpha, i, j, k} (\tilde{h}_{ijk}^\alpha)^2 + 2 \sum_{\alpha, i, j} \tilde{h}_{ij}^\alpha H_{ij}^\alpha + \rho^2 (2 - \rho^2 + 2H^2) - \sum_{\alpha, \beta} R_{\alpha\beta 12}^2 \\ &\geq \sum_{\alpha, i, j, k} (\tilde{h}_{ijk}^\alpha)^2 + 2 \sum_{\alpha, i, j} \tilde{h}_{ij}^\alpha H_{ij}^\alpha + \rho^2 (2 - \rho^2 + 2H^2) - \frac{2-B}{B} \rho^4. \end{aligned}$$

Integrating the above inequality and using Lemma 2.5, we get

$$\begin{aligned}
0 &\geq \int_M \sum_{\alpha,i,j,k} (\tilde{h}_{ijk}^\alpha)^2 dv + 2 \int_M \sum_{\alpha,i,j} \tilde{h}_{ij}^\alpha H_{ij}^\alpha dv + \int_M \rho^2 \left[2(1+H^2) - \frac{2}{B} \rho^2 \right] dv \\
&= \int_M |\nabla \tilde{h}|^2 dv - 2 \int_M |\nabla^\perp \mathbf{H}|^2 dv + \int_M \rho^2 \left[2(1+H^2) - \frac{2}{B} \rho^2 \right] dv \\
&= \eta \int_M |\nabla \tilde{h}|^2 dv + (1-\eta) \int_M |\nabla \tilde{h}|^2 dv - 2 \int_M |\nabla^\perp \mathbf{H}|^2 dv \\
&\quad + \int_M \rho^2 \left[2(1+H^2) - \frac{2}{B} \rho^2 \right] dv,
\end{aligned} \tag{3.1}$$

where $0 < \eta < 1$. From (2.1), (2.7) and (3.1), we have

$$\begin{aligned}
0 &\geq \eta \int_M |\nabla f_\varepsilon|^2 dv + (1-\eta) \int_M |\nabla^\perp \mathbf{H}|^2 dv - 2 \int_M |\nabla^\perp \mathbf{H}|^2 dv \\
&\quad + \int_M \rho^2 \left[2(1+H^2) - \frac{2}{B} \rho^2 \right] dv \\
&= \eta \int_M |\nabla f_\varepsilon|^2 dv - (1+\eta) \int_M |\nabla^\perp \mathbf{H}|^2 dv + \int_M \rho^2 \left[2(1+H^2) - \frac{2}{B} \rho^2 \right] dv \\
&\geq \eta \int_M |\nabla f_\varepsilon|^2 dv - (1+\eta) \int_M \rho^2 H^2 dv + \int_M \rho^2 \left[2(1+H^2) - \frac{2}{B} \rho^2 \right] dv \\
&= \eta \int_M |\nabla f_\varepsilon|^2 dv + (1-\eta) \int_M \rho^2 H^2 dv + 2 \int_M \rho^2 dv - \int_M \frac{2}{B} \rho^4 dv.
\end{aligned} \tag{3.2}$$

Substituting (2.11) into (3.2), we get

$$\begin{aligned}
0 &\geq \eta \frac{t}{A} \left(\int_M f_\varepsilon^4 \right)^{\frac{1}{2}} dv - \eta t^2 \int_M f_\varepsilon^2 dv - \eta t \int_M \left(1 + \frac{H^2}{2} \right) f_\varepsilon^2 dv \\
&\quad + (1-\eta) \int_M \rho^2 H^2 dv + 2 \int_M \rho^2 dv - \frac{2}{B} \int_M \rho^4 dv.
\end{aligned}$$

As $\varepsilon \rightarrow 0$, this implies

$$\begin{aligned}
0 &\geq \eta \frac{t}{A} \left(\int_M \rho^4 dv \right)^{\frac{1}{2}} - (\eta t^2 + \eta t - 2) \int_M \rho^2 dv \\
&\quad + \left(1 - \eta - \frac{\eta t}{2} \right) \int_M \rho^2 H^2 dv - \frac{2}{B} \int_M \rho^4 dv.
\end{aligned}$$

Choose $t = \frac{2(1-\eta)}{\eta}$, then $1 - \eta - \frac{\eta t}{2} = 0$. So we have

$$\begin{aligned}
0 &\geq \frac{2(1-\eta)}{A} \left(\int_M \rho^4 dv \right)^{\frac{1}{2}} - \left[\frac{4(1-\eta)^2}{\eta} + 2(1-\eta) - 2 \right] \int_M \rho^2 dv \\
&\quad - \frac{2}{B} \int_M \rho^4 dv,
\end{aligned}$$

i.e.,

$$\begin{aligned}
 0 &\geq \frac{(1-\eta)}{A} \left(\int_M \rho^4 dv \right)^{\frac{1}{2}} - \left[\frac{2(1-\eta)^2}{\eta} - \eta \right] \int_M \rho^2 dv - \frac{1}{B} \int_M \rho^4 dv \\
 &= \frac{1-\eta}{A} \|\rho^2\|_2 - \left[\frac{2(1-\eta)^2}{\eta} - \eta \right] \|\rho^2\| - \frac{1}{B} \|\rho^2\|_2^2 \\
 &= \left[\frac{1-\eta}{A} - \frac{1}{B} \|\rho^2\|_2 \right] \|\rho^2\|_2 - \left[\frac{2(1-\eta)^2}{\eta} - \eta \right] \|\rho^2\|.
 \end{aligned} \tag{3.3}$$

We take $\eta = 2 - \sqrt{2}$. This together with (3.3) yields

$$0 \geq \left\{ \frac{\sqrt{2}-1}{A} - \frac{1}{B} \|\rho^2\|_2 \right\} \|\rho^2\|_2,$$

which implies $\|\rho^2\|_2 = 0$ for

$$\|\rho^2\|_2 < C = \frac{(\sqrt{2}-1)B}{A} = \frac{(\sqrt{2}-1)\sqrt{\pi}}{12\sqrt{3}} B,$$

i.e., $S = 2H^2$ and M is a totally umbilical Willmore surface. This completes the proof of the main theorem.

As we all know, minimal surfaces must be Willmore surfaces, so we obtain the following corollary.

COROLLARY

Let M be a compact minimal surface in the unit sphere S^{2+p} . There exists a positive constant

$$C = \frac{(\sqrt{2}-1)\sqrt{\pi}}{12\sqrt{3}} B,$$

such that if

$$\|S\|_2 < C,$$

then $S = 0$ and M is a totally geodesic, where B is defined in Theorem A.

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