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Structures of generalized 3-circular projections for symmetric norms

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Abstract. Recently several authors investigated structures of *generalized bi-circular projections* in spaces where the descriptions of the group of surjective isometries are known. Following the same idea in this paper we give complete descriptions of generalized 3-circular projections for symmetric norms on $\mathbb{C}^n$ and $\mathbb{M}_{m\times n}(\mathbb{C})$.

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1. Introduction

Let X be a complex Banach space and T a surjective isometry of X such that $T^n = I$, for some $n \geq 2$, where I denotes the identity operator on X. Then $P = \frac{I+T+\cdots+T^{n-1}}{n}$ is a projection on X. Such a projection is called generalized n-circular projection, see [4]. Let $\lambda_0 = 1, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be the n distinct roots of unity. For $i = 1, 2, \ldots, n-1$, we define $P_i = \frac{I+\overline{\lambda_i}T+\overline{\lambda_i}^2T^2+\cdots+\overline{\lambda_i}^{n-1}T^{n-1}}{n}$. Then each P_i is a projection, $P_0 \oplus P_1 \oplus P_2 \oplus \cdots \oplus P_{n-1} = I$ and $P_0 + \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_{n-1} P_{n-1} = T$. Motivated from [4], we have the following definitions. We denote the unit circle of the complex plane by $\mathbb{T}$.

DEFINITION 1.1

Let X be a complex Banach space. A projection P_0 on X is said to be *n-circular projection*, $n \geq 2$, if there exist non trivial projections $P_1, P_2, \ldots, P_{n-1}$ on X such that

(a) $P_0 \oplus P_1 \oplus \cdots \oplus P_{n-1} = I$
(b) $P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1}$ is a surjective isometry for all $\lambda_i \in \mathbb{T}$, $i = 1, 2, \ldots, n-1$.

DEFINITION 1.2

Let X be a complex Banach space. A projection P_0 on X is said to be a *generalized n-circular projection* (GnP, for short) $n \geq 2$, if there exist $\lambda_1, \lambda_2, \ldots, \lambda_{n-1} \in \mathbb{T} \setminus \{1\}$,

$\lambda_i,\ i = 1, 2, \ldots, n-1$ of finite order and non trivial projections $P_1, P_2, \ldots, P_{n-1}$ on X such that

(a) $\lambda_i \neq \lambda_j$ for $i \neq j$
(b) $P_0 \oplus P_1 \oplus \cdots \oplus P_{n-1} = I$
(c) $P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1}$ is a surjective isometry.

In the case $n = 2$, a projection satisfying conditions in Definition 1.1 (respectively, Definition 1.2) is referred to as *bi-circular projection* (respectively, *generalized bi-circular projection* – henceforth, GBP).

Remark 1.3. Definition 1.2 seems to be a more natural one to start with compared to the definition given in [4], if we want to put the definition of GBP in this general set up. As we will see later in Theorem 3.2, not every G3P is of the form $\frac{I+T+T^2}{3}$, where T is a surjective isometry such that $T^3 = I$.

Generalized bi-circular projection has been studied by many authors (see the subsequent paragraph and references at the end of this paper). In particular, Botelho [4] and Botelho and Jamison [5–8] extensively investigated the structures of GBPs for different Banach spaces whose isometry group has concrete description.

In [8], it was shown that a GBP on spaces of continuous functions on a compact, connected and Hausdorff space, $C(\Omega)$ and $C(\Omega, X)$, is equal to the average of the identity with an isometric reflection. The same result was proved in [1] for $C_0(\Omega, X)$, with Ω a locally compact Hausdorff space (not necessarily connected) and X a Banach space with trivial centralizer. Similar characterization also holds for GBPs on L_p-spaces ($1 < p < \infty, p \neq 2$) [14], minimal norm ideal of operators [5], spaces of Lipschitz functions [6], JB^*-triples [9] and certain Hardy spaces [12].

We note that if $P + \lambda(I - P)$ is a surjective isometry and $\lambda \in \mathbb{T} \setminus \{1\}$ is of infinite order then P is a bi-circular projection (see [14]). Such projections were called trivial in [10, 14].

It is easy to observe that any GBP is a bi-contractive projection (see [14]). In general, any GnP is a contractive projection.

Recently in [2] the authors gave complete description of G3Ps for the space $C(\Omega)$. It was also shown in the same paper that if the convex combination of three surjective isometries on $C(\Omega)$ is a projection P, then P is either a GBP or a G3P.

We recall the definition of symmetric norm on $\mathbb{C}^n$ and $\mathbb{M}_{m\times n}(\mathbb{C})$.

DEFINITION 1.4

A norm $\|\cdot\|$ on $\mathbb{C}^n$ is called a symmetric norm if for every permutation matrix Π we have $\|\Pi x\| = \|x\|$ for all $x \in \mathbb{C}^n$.

A norm on$\mathbb{M}_{m\times n}(\mathbb{C})$ is called a symmetric norm if for every $A \in \mathbb{M}_{m\times n}(\mathbb{C})$, $\|UAV\| = \|A\|$, for all $m \times m$ unitary matrix U and for all $n \times n$ unitary matrix V.

A symmetric norm on $\mathbb{M}_{m\times n}(\mathbb{C})$ is also referred as an unitarily invariant norm (see [3]). In [11] the authors described GBPs on $\mathbb{C}^n$ with symmetric norm and on spaces of matrices with symmetric norms and unitary congruence invariant norms. Much in the spirit of their work, in this note we describe G3P on $\mathbb{C}^n$ and $\mathbb{M}_{m\times n}(\mathbb{C})$ where these spaces are equipped with symmetric norm.

For our purpose we strongly use the structure of the isometry groups on the above spaces for symmetric norms. Fortunately for us, such descriptions are well known. For a symmetric norm on $\mathbb{C}^n$, which is not a multiple of inner product norm, any isometry T is given by $T = DR$ where D is a diagonal matrix with diagonal entries from the unit circle $\mathbb{T}$ and R is a permutation matrix (see Theorem 2.5 of [13]). For a symmetric norm on $\mathbb{M}_{m\times n}(\mathbb{C})$, $m \neq n$, which is not a multiple of the Frobenius norm (that is the Hilbert–Schmidt norm) any isometry T is given by $T(A) = UAV$ where U is a unitary in $\mathbb{M}_m(\mathbb{C})$ and V is a unitary in $\mathbb{M}_n(\mathbb{C})$. If $m = n$ then an isometry T on $\mathbb{M}_n(\mathbb{C})$ has the form either $T(A) = UAV$ or $T(A) = UA^tV$ where U, V are unitary matrices in $\mathbb{M}_n(\mathbb{C})$ and A^t denotes the transpose of a matrix A (see Theorem 2.4 of [13]).

Remark 1.5. It follows from [11] that if a norm on $\mathbb{C}^n$ is multiple of inner product norm, then any GBP or G3P is a bi-circular projection and it is precisely an orthogonal projection.

It is also interesting to note here that the techniques used to describe G3Ps in the spaces mentioned above can be tried to describe GnPs as well, for $n > 3$. However, as it is evident from the proofs in the next sections, the number of cases to be considered become increasingly large and larger with greater values of n.

In the sequel, whenever we mention that P_0 is a G3P and write $P_0 + \lambda_1 P_1 + \lambda_2 P_2 = T$, we will always mean T, λ_i and P_i, $i = 1, 2$ are as in Definition 1.2.

2. Structure of G3P for symmetric norm on $\mathbb{C}^n$

As described in the Introduction, for a symmetric norm on $\mathbb{C}^n$, any isometry T is given by $T = DR$, where D is a diagonal matrix with diagonal entries in $\mathbb{T}$ and R is a permutation matrix.

We note the following simple fact. Let R be a permutation matrix such that the permutation associated with R fixes m elements, $m \geq 0$, and has k disjoint cycles of lengths $n_1, n_2, \ldots, n_k$, where $n = m + n_1 + \cdots + n_k$. Let π_j be the cycle $(1\ 2\ \ldots\ j-1\ j)$ and R_j the permutation matrix for the cycle π_{n_j}, $j = 1, 2, \ldots, k$. Then R is permutationally similar to $R_1 \oplus R_2 \oplus \cdots \oplus R_k \oplus I_m$.

We will identify the structure of G3P for a symmetric norm which is not a multiple of inner product norm on $\mathbb{C}^n$ up to permutation similarity.

We will need the following result proved in [11]. Since, we are only concerned with complex field, we state it according to our need.

Theorem 2.1 (Proposition 3.1 of [11]). *Let $\|\cdot\|$ be a symmetric norm on $\mathbb{C}^n$ which is not a multiple of the norm induced by the inner product, and P a generalized bi-circular projection. Then one and only one of the following holds*:

(*a*) *P is a bi-circular projection.*
(*b*) *There exist $m = n - 2k$, $k \geq 1$ such that P is permutationally similar to $P_1 \oplus P_2 \oplus \cdots \oplus P_k \oplus \operatorname{diag}(p_1, p_2, \ldots, p_m)$ with $p_j \in \{0, 1\}$ for all $j = 1, 2, \ldots, m$; and*

$$P_i = \frac{1}{2}\begin{pmatrix} 1 & d_1 \\ d_2 & 1 \end{pmatrix}$$

with $d_{i1}d_{i2} = 1$, $i = 1, \ldots, k$.

The following simple lemma will be used to describe the structure of G3P on $\mathbb{C}^n$ and later on $M_{m\times n}(\mathbb{C})$ for a symmetric norm. We omit the proof.

Lemma 2.2. *Let P_0 be a G3P on a Banach space X such that $P_0+\lambda_1 P_1+\lambda_2 P_2=T$. If P_1 or P_2 is a bi-circular projection, then so is P_0.*

Remark 2.3. Let P_0 be a G3P on a Banach space X such that $P_0+\lambda_1 P_1+\lambda_2 P_2=T$. Then

$$P_0=\frac{(T-\lambda_1 I)(T-\lambda_2 I)}{(1-\lambda_1)(1-\lambda_2)},\quad P_1=\frac{(T-I)(T-\lambda_2 I)}{(\lambda_1-1)(\lambda_1-\lambda_2)}$$

and

$$P_2=\frac{(T-I)(T-\lambda_1 I)}{(\lambda_2-1)(\lambda_2-\lambda_1)}.$$

Theorem 2.4. *Let $\|\cdot\|$ be a symmetric norm on $\mathbb{C}^n$ which is not a multiple of the norm induced by the inner product, and P_0 a generalized 3-circular projection. Then one of the following assertions holds*:

(*a*) *P_0 is a bi-circular projection.*
(*b*) *P_0 is a generalized bi-circular projection.*
(*c*) *There exist $m\geq 0$, $k\geq 1$, projections $P_{0,i}$, $i=0,\ldots,k$ such that P_0 is permutationally similar to $P_{0,1}\oplus P_{0,2}\oplus\cdots\oplus P_{0,k}\oplus P_{0,0}$ where $P_{0,0}=\operatorname{diag}(p_1,p_2,\ldots,p_m)$ with $p_j\in\{0,1\}$ for all $j=1,2,\ldots,m$; and*

$$P_{0,i}=\frac{1}{2}\begin{pmatrix}1 & d_{i1}\\ d_{i2} & 1\end{pmatrix}$$

with $d_{i1}d_{i2}=1$ or

$$P_{0,i}=\frac{1}{3}\begin{pmatrix}1 & d_{i1} & d_{i1}d_{i2}\\ d_{i2}d_{i3} & 1 & d_{i2}\\ d_{i3} & d_{i1}d_{i3} & 1\end{pmatrix}$$

with $d_{i1}d_{i2}d_{i3}=1$.

Proof. Let $P_0+\lambda_1 P_1+\lambda_2 P_2=T$ and $T=DR$. Let R be permutationally similar to $R_1\oplus R_2\oplus\cdots\oplus R_k\oplus I_m$. We write $D=D_1\oplus D_2\oplus\cdots\oplus D_k\oplus D_0$ accordingly. Then T will be permutationally similar to

$$\begin{pmatrix}D_1R_1 & 0 & \cdots & 0\\ 0 & D_2R_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & D_kR_k & 0\\ 0 & 0 & \cdots & D_0\end{pmatrix} \tag{1}$$

We note that D_iR_i is a matrix of order n_i. By Remark 2.3,

$$P_0=\frac{(T-\lambda_1 I)(T-\lambda_2 I)}{(1-\lambda_1)(1-\lambda_2)}.$$

Hence, equation (1) implies that P_0 is permutationally similar to $P_{0,1} \oplus P_{0,2} \oplus \cdots \oplus P_{0,k} \oplus P_{0,0}$, where

$$P_{0,i} = \frac{(D_i R_i - \lambda_1 I)(D_i R_i - \lambda_2 I)}{(1-\lambda_1)(1-\lambda_2)},$$

$i = 1, 2, \ldots, k$ and

$$P_{0,0} = \frac{(D_0 - \lambda_1 I)(D_0 - \lambda_2 I)}{(1-\lambda_1)(1-\lambda_2)}.$$

We first consider the case $k = 0$. Then $T = D_0$ and entries of the diagonal matrix D_0 are 1, λ_1 and λ_2.

Hence, P_0 is a diagonal matrix whose elements are 0 or 1. This implies that for any $\lambda \in \mathbb{T} \setminus \{1\}$, $P_0 + \lambda(I - P_0)$ is a diagonal matrix with entries 1 or λ and hence an isometry. Thus, P_0 is a bi-circular projection.

We now consider $k > 0$. Since the eigenvalues of T are $\{1, \lambda_1, \lambda_2\}$, we conclude that the eigenvalues of $D_i R_i$, for each i, and of D_0 is a subset of $\{1, \lambda_1, \lambda_2\}$.

Claim. If R is the permutation matrix associated with the cycle $(1\ 2\ \ldots\ m-1\ m)$ and $D = \operatorname{diag}(d_1, \ldots, d_m)$, then all the eigenvalues of DR are distinct.

To see the Claim, we observe that the characteristic polynomial of DR is $\lambda^m - (d_1 \ldots d_m) = 0$. Thus, DR has m distinct eigenvalues.

From the Claim, we conclude that each $D_i R_i$ has n_i distinct eigenvalues. Therefore, $n_i = 2$ or $n_i = 3$.

We consider both the cases. Let $n_i = 2$ for some $i = 1, \ldots, k$. Then eigenvalues of $D_i R_i$ can be $\{1, \lambda_1\}$, $\{1, \lambda_2\}$ or $\{\lambda_1, \lambda_2\}$. Suppose $D_i = \operatorname{diag}(d_{i1}, d_{i2})$.

(a) Suppose the eigenvalues of $D_i R_i$ is $\{1, \lambda_1\}$. Then

$$\begin{aligned} D_i R_i &= \begin{pmatrix} d_{i1} & 0 \\ 0 & d_{i2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & d_{i1} \\ d_{i2} & 0 \end{pmatrix}. \end{aligned}$$

Since $\operatorname{tr}(D_i R_i) = 0$, we have $1 + \lambda_1 = 0$ or $\lambda_1 = -1$. Further, $\det(D_i R_i) = -d_{i1} d_{i2} = \lambda_1$ or $d_{i1} d_{i2} = 1$.

We also note $P_0 - P_1 + \lambda_2 P_2 = T$. Thus, $P_0 + P_1 + \lambda_2^2 P_2 = (I - P_2) + \lambda_2^2 P_2 = T^2$ is an isometry. Therefore, by Theorem 2.1, we have P_2 is either a bi-circular projection or $\lambda_2^2 = -1$.

If P_2 is a bi-circular projection, then by Lemma 2.2, P_0 is also a bi-circular projection. If $\lambda_2^2 = -1$, then $\lambda_2 = \pm i$. Further,

$$\begin{aligned} P_{0,i} &= \frac{(D_i R_i - \lambda_1 I)(D_i R_i - \lambda_2 I)}{(1-\lambda_1)(1-\lambda_2)} \\ &= \frac{(D_i R_i + I)(D_i R_i \mp i I)}{2(1 \mp i)} \\ &= \frac{1}{2} \begin{pmatrix} 1 & d_{i1} \\ d_{i2} & 1 \end{pmatrix} \end{aligned}$$

and

$$P_{0,0} = \frac{(D_0 + I)(D_0 \mp iI)}{2(1 \mp i)} = \mathrm{diag}(p_1, p_2, \ldots, p_m).$$

Here, $p_j \in \{0, 1\}$ for all $j = 1, 2, \ldots, m$. The case when $\{1, \lambda_2\}$ is the eigenvalues of $D_i R_i$ is similar.

(b) Suppose the eigenvalues of $D_i R_i$ is $\{\lambda_1, \lambda_2\}$. Then $\mathrm{tr}(D_i R_i) = 0 = \lambda_1 + \lambda_2$. Hence, $\lambda_1 = -\lambda_2$. Moreover, $\det(D_i R_i) = -d_{i1} d_{i2} = \lambda_1 \lambda_2$ or $d_{i1} d_{i2} = \lambda_1^2$. Now, $P_0 + \lambda_1(P_1 - P_2) = T$. This implies that $P_0 + \lambda_1^2(P_1 + P_2) = T^2$. By Theorem 2.1, P_0 is either a bi-circular projection or a GBP.

Let $n_i = 3$ for some $i = 1, \ldots, k$. Then the eigenvalues of $D_i R_i$ are $\{1, \lambda_1, \lambda_2\}$. Suppose $D_i = \mathrm{diag}(d_{i1}, d_{i2}, d_{i3})$. Then

$$D_i R_i = \begin{pmatrix} d_{i1} & 0 & 0 \\ 0 & d_{i2} & 0 \\ 0 & 0 & d_{i3} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & d_{i1} & 0 \\ 0 & 0 & d_{i2} \\ d_{i3} & 0 & 0 \end{pmatrix}$$

Since $\mathrm{tr}(D_i R_i) = 0$, we have $1 + \lambda_1 + \lambda_2 = 0$. Thus, λ_1 and λ_2 are the cube roots of identity. This implies that

$$P_{0,i} = \frac{I + D_i R_i + (D_i R_i)^2}{3},$$

$i = 1, 2, \ldots, k$ and

$$P_{0,0} = \frac{I + D_0 + D_0^2}{3}.$$

We conclude that

$$P_{0,i} = \frac{1}{3} \begin{pmatrix} 1 & d_{i1} & d_{i1} d_{i2} \\ d_{i2} d_{i3} & 1 & d_{i2} \\ d_{i3} & d_{i1} d_{i3} & 1 \end{pmatrix}$$

and

$$P_{0,0} = \mathrm{diag}(p_1, p_2, \ldots, p_m).$$

Here, $p_j \in \{0, 1\}$ for all $j = 1, 2, \ldots, m$. Moreover, $\det(D_i R_i) = d_{i,1} d_{i,2} d_{i,3} = 1$. The proof is complete. □

Remark 2.5.

(1) In the case $n_i = 2$ and eigenvalues of $D_i R_i$ are $\{1, \lambda_1\}$ (or $\{1, \lambda_2\}$), $P_{0,i}$ appearing in the proof is a GBP. In the case $n_i = 3$, $P_{0,i}$ is a G3P.
(2) For a G3P P_0 it may be that $n_i = 2$ for all $i = 1, \ldots, k$. In this case, P_0 will be a GBP and in our proof we recover the structure of GBP described in Theorem 2.1.

3. Structure of G3P for symmetric norms on $\mathbb{M}_{m\times n}(\mathbb{C})$

For a symmetric norm on $\mathbb{M}_{m\times n}(\mathbb{C})$, $m \neq n$, any isometry is given by $T(A) = UAV$, where $U \in \mathbb{M}_m(\mathbb{C})$ and $V \in \mathbb{M}_n(\mathbb{C})$ are unitary matrices. If $m = n$, then any isometry is given by either $T(A) = UAV$ or $T(A) = UA^tV$ where $U, V \in \mathbb{M}_n(\mathbb{C})$ are unitary matrices. For convenience, we separate out the cases where isometries are of the form $T(A) = UAV$ and $T(A) = UA^tV$.

Remark 3.1. Let us assume that U has eigenvalues $u_1, \ldots, u_m$ and V has eigenvalues $v_1, \ldots, v_n$. If $T(A) = UAV$, then identifying $\mathbb{M}_{m\times n}(\mathbb{C})$ as $\mathbb{R}^m \otimes \mathbb{R}^n$ we see that $T(x \otimes y^t) = (Ux) \otimes (y^tV)$. Thus, T has eigenvalues u_iv_j, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$. Without loss of generality, we may assume that $u_1 = v_1 = 1$. Therefore, the spectrum of U and V is a subset of $\{1, \lambda_1, \lambda_2\}$. (This will determine P_0 up to a multiple of u_1v_1).

Theorem 3.2. *Let $\|\cdot\|$ be a symmetric norm on $\mathbb{M}_{m\times n}(\mathbb{C})$ and P_0 a generalized 3-circular projection such that the isometry associated with it is of the form $A \longmapsto UAV$ for some $U \in M_m(\mathbb{C})$ and $V \in M_n(\mathbb{C})$, U, V are unitary matrices. Then one and only one of the following assertions holds*:

(*a*) *P_0 is a bi-circular projection. In this case, there exist $R \in \mathbb{M}_n(\mathbb{C})$ with $R = R^* = R^2$ such that $P_0(A) = AR$ or there exist $S \in \mathbb{M}_m(\mathbb{C})$ with $S = S^* = S^2$ such that $P_0(A) = SA$.*
(*b*) *Either P_1 or P_2 is a generalized bi-circular projection. If P_i is a a generalized bi-circular projection, $i = 1, 2$, then*

$$P_0 = \frac{\lambda_i I}{2(\lambda_i - 1)} + \frac{T}{1 - \lambda_i^2} + \frac{\lambda_i T^q}{2(1 + \lambda_i)},$$

where q is the order of λ_j, $j = 1, 2$ and $j \neq i$.
(*c*) *There exist $R_i = R_i^* = R_i^2$ in $\mathbb{M}_m(\mathbb{C})$ and $S_i = S_i^* = S_i^2$ in $\mathbb{M}_n(\mathbb{C})$ such that*

$$P_0(A) = \sum_{i=0}^{p-1} R_i A S_i,$$

where

(*i*) $i = 0, 1, \ldots, p - 1$ *and p is an odd integer* ≥ 3,
(*ii*) $R_iR_j = 0$, $S_iS_j = 0$ *for $i \neq j$*,
(*iii*) $\sum_{i=0}^{p-1} R_i = I$ *and* $\sum_{i=0}^{p-1} S_i = I$.

Proof. Suppose that P_0 is a G3P and $P_0 + \lambda_1 P_1 + \lambda_2 P_2 = T$. Since we have assumed that $u_1 = v_1 = 1$ (see Remark 3.1), we get that spectra of U and V are any one of the following sets:

$$\{1\},\ \{1, \lambda_1\},\ \{1, \lambda_2\} \text{ or } \{1, \lambda_1, \lambda_2\}.$$

So we have following three exclusive cases:

Case I. Suppose the spectrum of U is $\{1\}$. Therefore, the spectrum of V will be $\{1, \lambda_1, \lambda_2\}$. In this case, $U = I$ and $T(A) = AV$. From Remark 2.3, we have

$$P_0 A = \frac{(T - \lambda_1 I)(T - \lambda_2 I)A}{(1 - \lambda_1)(1 - \lambda_2)} = AR,$$

where

$$R = \frac{(V - \lambda_1 I)(V - \lambda_2 I)}{(1 - \lambda_1)(1 - \lambda_2)}.$$

It is routine to verify that $R = R^*$. Since P_0 is a projection it also follows that $R^2 = R$.

We now show that P_0 is indeed a bi-circular projection. Let $\mu \in \mathbb{T} \setminus \{1\}$. We have

$$\begin{aligned} [P_0 + \mu(I - P_0)]A &= P_0 A + \mu(A - P_0 A) \\ &= AR + \mu(A - AR) \\ &= A[R + \mu(I - R)] \\ &= AW, \end{aligned}$$

where $W = R + \mu(I - R)$. It is now easy to verify that $W^* W = W W^* = I$. Therefore, P_0 is a bi-circular projection.

Similarly, if the spectrum of V is $\{1\}$ we will get $P_0(A) = SA$ for some $S \in \mathbb{M}_m(\mathbb{C})$ with $S = S^* = S^2$. Hence, assertion (a) is proved

Case II. Suppose the spectrum of U is $\{1, \lambda_1\}$. So, the choices of spectrum of V are $\{1, \lambda_1\}$, $\{1, \lambda_2\}$ or $\{1, \lambda_1, \lambda_2\}$. We consider following three possible subcases here.

(A) If the spectrum of V is $\{1, \lambda_1\}$, then T will have spectrum $\{1, \lambda_1, \lambda_1^2\}$. This implies that $\lambda_1^2 = \lambda_2$.

Let p and q be the order of λ_1 and λ_2 respectively. Then we have $\lambda_1^{2q} = \lambda_2^q = 1$ and $\lambda_2^p = \lambda_1^{2p} = 1$. This implies that p divides $2q$ and q divides p. Thus, $2q = k_1 p$ and $p = k_2 q$ for some positive integers k_1 and k_2. Hence, we have $k_1 k_2 = 2$. So, either $k_1 = 1, k_2 = 2$ or $k_1 = 2, k_2 = 1$. If $k_1 = 1$ and $k_2 = 2$, we get $p = 2q$. If $k_1 = 2$ and $k_2 = 1$, we get $p = q$.

(1) Suppose $p = 2q$. Then we have

$$\begin{aligned} T^q &= P_0 + \lambda_1^q P_1 + \lambda_2^q P_2 \\ &= P_0 + \lambda_1^q P_1 + P_2. \end{aligned}$$

This implies that P_1 is a GBP. By Proposition 4.1 of [11] we get that P_1 is either a bi-circular projection or P_1 is a proper GBP (not bi-circular) with $\lambda_1^q = -1$.

If P_1 is a bi-circular projection, then by Lemma 2.2 we conclude that P_0 is also a bi-circular projection. Thus, we get assertion (a) back.

If P_1 is not a bi-circular projection, then $\lambda_1^q = -1$. Since $\lambda_1^2 = \lambda_2$, we consider the following two cases:

Suppose $\lambda_1 = \sqrt{\lambda_2}$, then we get $(\lambda_2)^{q/2} = -1$.
Suppose $\lambda_1 = -\sqrt{\lambda_2}$, then we get $(-1)^q (\lambda_2)^{q/2} = -1$. This shows that $(\lambda_2)^{q/2} = -1$, otherwise if $(\lambda_2)^{q/2} = 1$ then we will get $\lambda_1^q = 1$, which is a contradiction.

So, in both cases we have $(\lambda_2)^{q/2} = -1$. Since $q = p/2$ we also have $\lambda_1^{p/2} = -1$. For the form of P_0, we consider the following three equations:

$$\begin{aligned} P_0 - P_1 + P_2 &= T^q, \\ P_0 + \lambda_1 P_1 + \lambda_2 P_2 &= T, \\ P_0 + P_1 + P_2 &= I. \end{aligned}$$

Eliminating P_1 and P_2, we get

$$P_0 = \frac{\lambda_1 I}{2(\lambda_1 - 1)} + \frac{T}{1 - \lambda_1^2} + \frac{\lambda_1 T^q}{2(1 + \lambda_1)}.$$

Hence, assertion (b) is proved.

(2) Suppose $p = q$. Since $\lambda_1^2 = \lambda_2$, we have $\lambda_1 = \pm\sqrt{\lambda_2}$. We first claim that $\lambda_1 \neq -\sqrt{\lambda_2}$. To see this, if $\lambda_1 = -\sqrt{\lambda_2}$ then we have $\lambda_1^p = (-\sqrt{\lambda_2})^p = 1$ or $(-1)^p(\lambda_2)^{p/2} = 1$. This shows that p is odd, otherwise $(\lambda_2)^{p/2} = 1$, a contradiction because the order of λ_2 is p. Hence, we get $(\lambda_2)^{p/2} = -1$. It follows that $\lambda_1^p = -1$, a contradiction since the order of λ_1 is p.

Thus we must have $\lambda_1 = \sqrt{\lambda_2}$. Hence, $\lambda_1^p = (\sqrt{\lambda_2})^p = (\lambda_2)^{p/2} = 1$. This implies that p is odd. As the order of λ_1 is p, we have $U^p = I$ and $V^p = I$. Further, for $i = 0, 1, \ldots, p-1$, we have

$$P_0 + \lambda_1^i P_1 + \lambda_2^i P_2 = T^i.$$

Adding these equations, we get

$$pP_0 + \left(\sum_{i=0}^{p-1} \lambda_1^i\right) P_1 + \left(\sum_{i=0}^{p-1} \lambda_2^i\right) P_2 = I + T + T^2 + \cdots + T^{p-1}.$$

Since $\sum_{i=0}^{p-1} \lambda_1^i = \sum_{i=0}^{p-1} \lambda_2^i = 0$, we obtain

$$P_0 = \frac{I + T + T^2 + \cdots + T^{p-1}}{p}.$$

We now define

$$R_i = \frac{1}{p}\sum_{j=0}^{p-1} \lambda_1^{ij} U^j \quad \text{and} \quad S_i = \frac{1}{p}\sum_{j=0}^{p-1} \overline{\lambda_1}^{ij} V^j,$$

where $i = 0, 1, \ldots, p-1$. It is straightforward to verify that $R_i = R_i^* = R_i^2$, $S_i = S_i^* = S_i^2$, for $i \neq j$; $R_i R_j = 0$, $S_i S_j = 0$ and $\sum_{i=0}^{p-1} R_i = I$, $\sum_{i=0}^{p-1} S_i = I$.

Therefore, P_0 will be of the form

$$P_0(A) = \sum_{i=0}^{p-1} R_i A S_i$$

and assertion (c) is proved.

We can also get the form of P_1 and P_2. We first observe that P_j, $j = 1, 2$, will have the form

$$P_j = \frac{I + \overline{\lambda_j} T + \overline{\lambda_j}^2 T^2 + \cdots + \overline{\lambda_j}^{p-1} T^{p-1}}{p}.$$

But $\overline{\lambda_j} = \lambda_j^{p-1}$ and $\lambda_1^2 = \lambda_2$, so we get

$$P_1(A) = \sum_{i=0}^{p-1} R_i A S_{(i+1)(\text{mod } p)}.$$

Similarly,

$$P_2(A) = \sum_{i=0}^{p-1} R_i A S_{(i+2)(\text{mod } p)}.$$

(B) If the spectrum of V is $\{1, \lambda_2\}$, then T will have spectrum $\{1, \lambda_1, \lambda_2, \lambda_1\lambda_2\}$. This implies that $\lambda_1\lambda_2 = 1$ and hence λ_1 and λ_2 are of the same order. Now,

$$T = P_0 + \lambda_1 P_1 + \overline{\lambda_1} P_2 \implies \lambda_1 T = P_2 + \lambda_1 P_0 + \lambda_1^2 P_1.$$

Because $\lambda_1 T$ is again an isometry, we are reduced to Case II, Part (A)(2). So, P_2 will be of the form $P_2(A) = \sum_{i=0}^{p-1} R_i A S_i$, where R_i and S_i are as in assertion (c).

Proceeding in the same manner as above, we can easily obtain the form of P_0. Therefore, we get back assertion (c).

(C) If the spectrum of V is $\{1, \lambda_1, \lambda_2\}$, then T will have spectrum $\{1, \lambda_1, \lambda_2, \lambda_1\lambda_2, \lambda_1^2\}$. This implies that $\lambda_1\lambda_2 = 1$ and $\lambda_1^2 = \lambda_2$. Therefore, we have $\lambda_1^3 = \lambda_2^3 = 1$. Here, we get assertion (c) with $p = 3$.

Case III. The spectrum of U is $\{1, \lambda_2\}$. This case is symmetric to Case II.

Case IV. Suppose that the spectrum of U is $\{1, \lambda_1, \lambda_2\}$.

(1) If the spectrum of V is $\{1\}$, then $V = I$. We proceed in the same way as in Case I to get $S \in \mathbb{M}_m(\mathbb{C})$ such that $S = S^* = S^2$ and $P_0 A = SA$. Thus, P_0 is a bi-circular projection.
(2) If the spectrum of V is $\{1, \lambda_1\}$ or $\{1, \lambda_2\}$, then we proceed exactly as in Case II above.
(3) If the spectrum of V is $\{1, \lambda_1, \lambda_2\}$, then the spectrum of T will be $\{1, \lambda_1, \lambda_2, \lambda_1\lambda_2, \lambda_1^2, \lambda_2^2\}$. Thus, we have $\lambda_1\lambda_2 = 1$, $\lambda_1^2 = \lambda_2$ and $\lambda_2^2 = \lambda_1$. Hence, $1 = \lambda_1\lambda_2 = \lambda_1\lambda_1^2 = \lambda_1^3$. Similarly, we have $\lambda_2^3 = 1$. Thus, we get assertion (c) for $p = 3$. This completes the proof of the theorem. □

Remark 3.3.

(1) In case (b) of Theorem 3.2, we do not know if P_0 is itself a GBP. However as the proof shows, in this case we do have $\lambda_j^{\frac{q}{2}} = -1$ and $\lambda_i^q = -1$.

(2) Condition (c) in Theorem 3.2 is sufficient for $p = 3$. To see this, define

(i) $P_1(A) = R_0AS_1 + R_1AS_2 + R_2AS_0$,
(ii) $P_2(A) = R_0AS_2 + R_1AS_0 + R_2AS_1$,
(iii) $U = R_0 + \omega R_1 + \omega^2 R_2$ and
(iv) $V = S_0 + \omega^2 S_1 + \omega S_2$, where ω is cube root of unity.

It can be easily verified that P_1 and P_2 are projections. Further, for $i \neq j$; $P_i P_j = 0$ and $P_0 + P_1 + P_2 = I$. Also, U and V are unitary matrices such that $(P_0 + \omega^2 P_1 + \omega P_2)A = UAV$. This implies that P_0 is a generalized 3-circular projection.

We now consider the case $m = n$ and the associated isometry T is of the form $T(A) = UA^tV$ for some unitary matrices U and V in $\mathbb{M}_n(\mathbb{C})$. Let $P_0 + \lambda_1 P_1 + \lambda_2 P_2 = T$ and $T(A) = UA^tV$. Then we have

$$T^2(A) = P_0(A) + \lambda_1^2 P_1(A) + \lambda_2^2 P_2(A) = UV^tAU^tV.$$

Let $X = UV^t$ and $Y = U^tV$. So, $T^2(A) = XAY$, where X and Y are unitary matrices. Following the same idea of the proof of Theorem 3.2 above, we get the following result.

Theorem 3.4. *Let $\|\cdot\|$ be a symmetric norm on $\mathbb{M}_n(\mathbb{C})$ and P_0 a generalized 3-circular projection such that the isometry associated with it is of the form $A \longmapsto UA^tV$ for some unitary matrices U and V in $\mathbb{M}_n(\mathbb{C})$. Then one and only one of the following holds*:

(*a*) *P_0 is a bi-circular projection.*
(*b*) *P_1 or P_2 is a generalized bi-circular projection. If P_i, $i = 1, 2$, is a generalized bi-circular projection*, *then P_0 is of the form*

$$A \longmapsto \frac{\lambda_i^2 A}{2(\lambda_i^2 - 1)} + \frac{UV^tAU^tV}{1 - \lambda_i^4} + \frac{\lambda_i^2 (UV^t)^q A (U^tV)^q}{2(1 + \lambda_i^2)},$$

where q is the order of λ_j^2, $j = 1, 2$ and $j \neq i$.
(*c*) *There exist $R_i = R_i^* = R_i^2$ and $S_i = S_i^* = S_i^2$ in $\mathbb{M}_n(\mathbb{C})$ such that*

$$P_0(A) = \sum_{i=0}^{p-1} R_i A S_i,$$

where

(*i*) $i = 0, 1, \ldots, p - 1$ *and p is an odd integer*,
(*ii*) $R_i R_j = 0$, $S_i S_j = 0$ *for $i \neq j$*,
(*iii*) $\sum_{i=0}^{p-1} R_i = I$ *and* $\sum_{i=0}^{p-1} S_i = I$

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