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Generators for finite depth subfactor planar algebras

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Abstract. We show that a subfactor planar algebra of finite depth k is generated by a single s-box, for $s \leq \min\{k+4, 2k\}$.

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The main result of Kodiyalam and Tupurani [3] shows that a subfactor planar algebra of finite depth is singly generated with a finite presentation. If P is a subfactor planar algebra of depth k, it is shown there that a single $2k$-box generates P. It is natural to ask what the smallest s is such that a single s-box generates P. While we do not resolve this question completely, we show in this note that $s \leq \min\{k+4, 2k\}$ and that k does not suffice in general. All terminology and unexplained notation will be as in [3].

For the rest of the paper fix a subfactor planar algebra P of finite depth k. Let $2t$ be such that it is the even number of $k+3$ and $k+4$. We will show that some s-box generates P as a planar algebra, where $s = \min\{2k, 2t\}$. The main observation is the following result about involutive algebra anti-automorphisms of finite-dimensional complex semisimple algebras. We mention as a matter of terminology that we always deal with $\mathbb{C}$-algebra anti-automorphisms and automorphisms (as opposed to those that might induce a non-identity involution on the base field $\mathbb{C}$). Also, as is common in Hopf algebra literature, we will use Sa instead of $S(a)$ to demote the image of a under a map S of algebras.

Theorem 1. *Let A be a finite-dimensional complex semisimple algebra and let $S : A \to A$ be an involutive algebra anti-automorphism. Suppose that A has no 2×2 matrix summand. Then, there exists $a \in A$ such that a and Sa generate A as an algebra.*

Before beginning the proof of this theorem, we observe that the somewhat peculiar restriction on A not having an $M_2(\mathbb{C})$ summand is really necessary.

Remark 2. The map $S : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ defined by $Sa = adj(a)$ is easily verified to be an involutive algebra anti-automorphism, while there exists no $a \in M_2(\mathbb{C})$ that together with Sa generates $M_2(\mathbb{C})$ since these generate only a commutative subalgebra.

We pave the way for a proof of Theorem 1 by studying the two special cases when $A = M_n(\mathbb{C})$ and $A = M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$. In these, n is a fixed positive integer. We will need the following lemmas that specify a 'standard form' for each of these two special cases.

Lemma 3. *Let S be an involutive algebra anti-automorphism of $M_n(\mathbb{C})$. There is an algebra automorphism of $M_n(\mathbb{C})$ under which S is identified with either (i) the transpose map or (ii) the transpose map followed by conjugation by the matrix*

$$J = \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix} (= -J^T = -J^{-1}).$$

The second case may arise only when $n = 2k$ is even (and I_k denotes, of course, the identity matrix of size k).

Proof. Let T denote the transpose map on $M_n(\mathbb{C})$. The composite map TS is then an algebra automorphism of $M_n(\mathbb{C})$ and is consequently given by conjugation with an invertible matrix, say u. Thus $Sx = (uxu^{-1})^T$. Involutivity of S implies that u is either symmetric or skew-symmetric. By Takagi's factorization (see p. 204 and p. 217 of [1]), u is of the form $v^T v$ if it is symmetric and of the form $v^T J v$ if it is skew-symmetric, for some invertible v. For the algebra automorphism of $M_n(\mathbb{C})$ given by conjugation with v, S gets identified in the symmetric case with the transpose map and in the skew-symmetric case with the transpose map followed by conjugation by J. □

Lemma 4. *Let S be an involutive algebra anti-automorphism of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ that interchanges the two minimal central projections. There is an algebra automorphism of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ fixing the minimal central projections under which S is identified with the map $x \oplus y \mapsto y^T \oplus x^T$.*

Proof. The map $x \oplus y \mapsto S(y^T \oplus x^T)$ is an algebra automorphism of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ fixing the minimal central projections and is therefore given by $x \oplus y \mapsto uxu^{-1} \oplus vyv^{-1}$ for invertible u, v. Hence $S(x \oplus y) = uy^T u^{-1} \oplus vx^T v^{-1}$.

Thus, $S^2(x \oplus y) = u(v^{-1})^T xv^T u^{-1} \oplus v(u^{-1})^T yu^T v^{-1}$. Involutivity of S now implies that $v^T u^{-1}$ and $u^T v^{-1}$ are both scalar matrices, or equivalently, $v^T = \lambda u$ and $u^T = \mu v$ for non-zero scalars λ, μ. Taking transposes shows that $\lambda\mu = 1$ and finally, by replacing u by λu, we may assume that $v = u^T$.

The commutativity of the following diagram:

$$\begin{array}{ccc} M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) & \xrightarrow{x\oplus y \mapsto u^{-1}xu \oplus y} & M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \\ \downarrow S & & \downarrow x\oplus y \mapsto y^T \oplus x^T \\ M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) & \xrightarrow{x\oplus y \mapsto u^{-1}xu \oplus y} & M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \end{array}$$

now implies that under the algebra automorphism of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ given by $x \oplus y \mapsto u^{-1}xu \oplus y$, S is identified with $x \oplus y \mapsto y^T \oplus x^T$. □

The proof of Theorem 1 in the case $A = M_n(\mathbb{C})$ (for $n \neq 2$) needs some preparation. For a subset $S \subseteq M_n(\mathbb{C})$ we use the notation S', as usual, to denote its commutant in $M_n(\mathbb{C})$.

Lemma 5. *If $U \subseteq \mathbb{C}^{2N}$ is non-empty and Zariski open, then*

$$U \cap \{(z_1, \ldots, z_N, \overline{z_1}, \ldots, \overline{z_N}) : z_i \in \mathbb{C}\} \neq \emptyset.$$

Proof. It suffices to show that $S = \{(z_1, \ldots, z_N, \overline{z_1}, \ldots, \overline{z_N}) : z_i \in \mathbb{C}\}$ is Zariski dense in $\mathbb{C}^{2N}$. If a polynomial f in $2N$ variables vanishes on S, then the polynomial $p(u_1, \ldots, u_N, v_1, \ldots v_N) = f(u_1+iv_1, \ldots, u_N+iv_N, u_1-iv_1, \ldots, u_N-iv_N)$ vanishes on $\mathbb{R}^{2N}$. It is then easily seen by induction on the number of variables that p identically vanishes and then, so does f. □

PROPOSITION 6

For $n > 1$, *the set*

$$U = \left\{(P, Q) \in M_n(\mathbb{C}) \times M_n(\mathbb{C}) : P, Q \text{ invertible and } \left\{\begin{bmatrix} 0 & P \\ Q & 0 \end{bmatrix}, \begin{bmatrix} 0 & P^T \\ Q^T & 0 \end{bmatrix}\right\}' = \mathbb{C}I_{2n}\right\}.$$

is a non-empty Zariski open subset of $M_n(\mathbb{C}) \times M_n(\mathbb{C})$.

Proof. For an arbitrary matrix $\begin{bmatrix} X & Y \\ Z & W \end{bmatrix} \in M_{2n}(\mathbb{C})$, the condition that it commute with both $\begin{bmatrix} 0 & P \\ Q & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & P^T \\ Q^T & 0 \end{bmatrix}$ is given by a set of $8n^2$ homogeneous linear equations in the $4n^2$ entries of X, Y, Z, W with coefficient (linear) polynomials in the entries of P and Q.

The solution space for this system is at least one dimensional (since it certainly contains the identity matrix) and thus the coefficient matrix has rank at most $4n^2-1$. The condition that the solution space is exactly one dimensional is hence equivalent to the condition that the coefficient matrix has rank at least $4n^2 - 1$, which is clearly Zariski open condition in the entries of P and Q. It follows that U is Zariski open.

To show non-emptiness of U, choose an invertible $Q \in M_n(\mathbb{C})$ such that Q and Q^T generate $M_n(\mathbb{C})$ as an algebra. For instance, Q could be $I_n + N_n$, where N_n is the $n \times n$ nilpotent matrix with super-diagonal entries, all 1 and 0 entries elsewhere. The condition that $\begin{bmatrix} X & Y \\ Z & W \end{bmatrix} \in M_{2n}(\mathbb{C})$ commutes with both $\begin{bmatrix} 0 & I \\ Q & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & I \\ Q^T & 0 \end{bmatrix}$ is equivalent to the set of equations:

$$YQ = QY = Z = YQ^T = Q^TY,$$
$$WQ = QX,\ X = W,\ WQ^T = Q^TX.$$

Since Y commutes with Q and Q^T (which generate $M_n(\mathbb{C})$), $Y = \lambda I_n$ for a scalar $\lambda \in \mathbb{C}$. Thus $Z = \lambda Q = \lambda Q^T$. Now (and this is the crucial point where $n > 1$ is needed), since Q and Q^T generate $M_n(\mathbb{C})$ which is not commutative, they cannot be equal and so $\lambda = 0$. Since $X = W$ and hence commutes with both Q and Q^T, $X = W = \mu I$ for some scalar $\mu \in \mathbb{C}$. Thus $(I, Q) \in U$. □

PROPOSITION 7

Let S *be an involutive algebra anti-automorphism of* $M_m(\mathbb{C})$ *with* $m \neq 2$. *There exists invertible* $x \in M_m(\mathbb{C})$ *which, together with* Sx, *generates* $M_m(\mathbb{C})$ *as an algebra.*

Proof. First, we may assume by Lemma 3 that S is either (i) the transpose map or (ii) the transpose map followed by conjugation by J. In Case (i), as in the proof of Proposition 6, $x = I_m + N_m$ is invertible and such that x and Sx generate $M_m(\mathbb{C})$ as an algebra.

In Case (ii), $m = 2n$ is necessarily even. It then follows from Proposition 6 and Lemma 5 that there is an invertible $P \in M_n(\mathbb{C})$ such that

$$\left\{ \begin{bmatrix} 0 & P \\ \bar{P} & 0 \end{bmatrix}, \begin{bmatrix} 0 & P^T \\ \bar{P}^T & 0 \end{bmatrix} \right\}' = \mathbb{C} I_{2n}$$

The commutant of these two matrices is the same as that of the algebra they generate which is a $*$-subalgebra of $M_m(\mathbb{C})$ since they are adjoints of each other. By the double commutant theorem, it follows that the algebra generated by these is the whole of $M_m(\mathbb{C})$. Now take $x = \begin{bmatrix} 0 & P \\ \bar{P} & 0 \end{bmatrix}$. □

In proving Theorem 1 for $A = M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$, we will need the following lemma.

Lemma 8. *Let A and B be finite dimensional complex unital algebras and let $a \in A$ and $b \in B$ be invertible. Then, for all but finitely many $\lambda \in \mathbb{C}$, the algebra generated by $a \oplus \lambda b \in A \oplus B$ contains both a $(= a \oplus 0)$ and b $(= 0 \oplus b)$.*

Proof. We may assume that $\lambda \neq 0$ and then it suffices to see that a is expressible as a polynomial in $a \oplus \lambda b$. Note that since $a \oplus \lambda b$ is invertible and $A \oplus B$ is finite dimensional, the algebra generated by $a \oplus \lambda b$ is actually unital. In particular, it makes sense to evaluate any complex univariate polynomial on $a \oplus \lambda b$.

Let $p(X)$ and $q(X)$ be the minimal polynomials of a and b respectively. By invertibility of a and b, neither p nor q has 0 as a root. The minimal polynomial of λb is $q(\frac{X}{\lambda})$. Unless λ is the quotient of a root of p by a root of q, $p(X)$ and $q(\frac{X}{\lambda})$ will have no common roots and therefore be coprime. So there will exist a polynomial $r(X)$ that is divisible by $q(\frac{X}{\lambda})$ but is X modulo $p(X)$. Thus $r(a \oplus \lambda b) = a$, as desired. □

PROPOSITION 9

Let S be an involutive algebra anti-automorphism of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ that interchanges the two minimal central projections. There exists invertible $x \oplus y \in M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ which together with $S(x \oplus y)$ generates $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ as an algebra.

Proof. First, by Lemma 4, we may assume that S is the map $x \oplus y \mapsto y^T \oplus x^T$. Now, as in the proof of Proposition 7, there is an invertible $x \in M_n(\mathbb{C})$ such that x and x^T generate $M_n(\mathbb{C})$. By Lemma 8, for all but finitely many $\lambda \in \mathbb{C}$, the algebra generated by $x \oplus \lambda x$ contains $x \oplus 0$ and $0 \oplus x$ and similarly the algebra generated by $\lambda x^T \oplus x^T$ contains $x^T \oplus 0$ and $0 \oplus x^T$. Thus the algebra generated by $x \oplus \lambda x$ and $\lambda x^T \oplus x^T$ is the whole of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$. □

Proof of Theorem 1. Let $\hat{A}$ denote the (finite) set of all inequivalent irreducible representations of A and for $\pi \in \hat{A}$, let d_π denote its dimension. Since S is an involutive anti-automorphism, it acts as an involution on the set of minimal central projections of A. It is then easy to see that there exist subsets $\hat{A}_1$ and $\hat{A}_2$ of $\hat{A}$ and an identification

$$A \to \bigoplus_{\pi \in \hat{A}_1} M_{d_\pi}(\mathbb{C}) \oplus \bigoplus_{\pi \in \hat{A}_2} (M_{d_\pi}(\mathbb{C}) \oplus M_{d_\pi}(\mathbb{C}))$$

such that each summand is S-stable.

Now, by Propositions 7 and 9, in each summand of the above decomposition, either $M_{d_\pi}(\mathbb{C})$ or $M_{d_\pi}(\mathbb{C}) \oplus M_{d_\pi}(\mathbb{C})$, there is an invertible element which together with its image under S generates that summand.

Finally, an inductive application of Lemma 8 shows that if a is a general linear combination of these generators, then a and Sa generate A as an algebra. □

Before we prove our main result, we will need a result about connected pointed bipartite graphs. Recall that a bipartite graph has its vertex set partitioned into 'even' and 'odd' vertices and all its edges connect an even and an odd vertex. It is pointed if a certain even vertex, normally denoted by $*$, is distinguished. Its depth is the largest distance of a vertex from $*$.

PROPOSITION 10

Let Γ be a connected pointed bipartite graph of depth $k \geq 3$. For any vertex v of Γ, let t be the one of $k+3, k+4$ with the same parity as v. The number of paths of length t from $*$ to v is at least 3.

Proof. We analyse three cases depending on the distance of v from $*$.

Case I: If $v = *$, note that $t \geq 6$ is even. To show that there are at least 3 paths of length t from $*$ to $*$, it suffices to show that there are at least 3 paths of length 6 from $*$ to $*$. Since $k \geq 3$, choose any vertex at distance 2 from $*$ and a path from $*$ to the chosen vertex. It is easy to see that there are at least 3 paths of length 6 from $*$ to $*$ supported on the edges of this path.

Case II: If v is at distance 1 from $*$, then $t \geq 7$ is odd. As observed in Case I, there are at least 3 paths of length 6 from $*$ to $*$ and consequently at least 3 paths of length 7 from $*$ to v.

Case III: Suppose v is at a distance n from $*$, where $n > 1$. Observe that if n and k have the same parity, then $n \leq k$ while in the other case, $n \leq k-1$. Choose a path $\xi_1\xi_2\xi_3\cdots\xi_n$ from $*$ to v. Then $\xi_2 \neq \overline{\xi_1}$. Then we have three paths $\xi_1\overline{\xi_1}\xi_1\overline{\xi_1}\xi_1\xi_2\cdots\xi_n$, $\xi_1\xi_2\overline{\xi_2}\xi_2\overline{\xi_2}\xi_2\cdots\xi_n$, and $\xi_1\overline{\xi_1}\xi_1\xi_2\overline{\xi_2}\xi_2\cdots\xi_n$ of length $n+4$ from $*$ to v. Thus if n and k have the same parity, so that $t = k+4$, then there exist at least 3 paths of length t from $*$ to v. If n and k have opposite parity then $t = k+3$ and since $n \leq k-1$ in this case, since there exist at least 3 distinct paths of length $n+4$ from $*$ to v, there also exist 3 distinct paths of length t from $*$ to v. □

We now prove the main result.

Theorem 11. *Let P be a subfactor planar algebra of finite depth k. Let $2t$ be the even number in $\{k+3, k+4\}$. Let $s = \min\{2k, 2t\}$. Then P is generated by a single s-box.*

Proof.

Case I: If $k \leq 3$, $s = 2k$. Then by Proposition 5.1 of [3], P is generated by a single s box.

Case II: If $k > 3$, so that $s = 2t$, let Γ be the principal graph of the subfactor planar algebra P. Then from Proposition 10, the number of paths of length s from the $*$-vertex to any even vertex v in Γ is at least 3. So P_s does not have an $M_2(\mathbb{C})$ summand. Consider the t-th power, say X, of the s-rotation tangle. This tangle changes the position of $*$ on an s-box from the top left to the bottom right position. Clearly $Z_X^P : P_s \to P_s$ is an involutive algebra anti-automorphism. From Theorem 1, there exists an element $a \in P_s$ such that a and $Z_X^P(a)$ generate P_s as an algebra. Since $s \geq k$, the planar algebra generated by P_s contains P_k and thus is the whole of P. Hence the single s-box containing a generates the planar algebra P. □

We finish by showing that $k + 1$ might actually be needed.

Example 12. Let $P = P(V)$ be the tensor planar algebra (see [2]) for details) of a vector space V of dimension greater than 1. It is easy to see that depth(P) = 1. However, given any $a \in P_1 = \text{End}(V)$, if Q is the planar subalgebra of P generated by a, a little thought shows that Q_1 is just the algebra generated by a and is hence abelian while P_1 is not.

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