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# Approximation properties of fine hyperbolic graphs

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**Abstract.** In this paper, we propose a definition of approximation property which is called the metric invariant translation approximation property for a countable discrete metric space. Moreover, we use the techniques of Ozawa's to prove that a fine hyperbolic graph has the metric invariant translation approximation property.

**Keywords.** Uniform Roe algebras; fine hyperbolic graph; metric invariant translation approximation property.

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## 1. Introduction

Given a countable discrete group $G$, some nice approximation properties for the reduced $C^*$-algebras $C_r^*(G)$ can give us the approximation properties of $G$. For example, Lance [7] proved that the nuclearity of $C_r^*(G)$ is equivalent to the amenability of $G$; Haagerup [5] proved that $C_r^*(G)$ has the CBAP if and only if $G$ is weakly amenable, where $G$ is said to be weakly amenable with constant $C$ if there exists a sequence of finitely supported functions $\varphi_n$ on $G$ such that $\varphi_n \to 1$ pointwise and $\sup_n \|\varphi_n\|_{cb} \leq C$, where $\|\varphi_n\|_{cb}$ denotes the (completely bounded) norm of Schur multiplier on $B(l^2(G))$. For a metric space $X$ with bounded geometry, the approximation properties of their uniform Roe algebras are also related to some properties of the metric space. For example, $X$ has Property A if and only if $C_u^*(X)$ is nuclear, see [1, 11].

In [8], Ozawa proved that a hyperbolic group $G$ is weakly amenable. This result gives an affirmative answer to the question raised by Roe at the end of [10] that hyperbolic groups have invariant translation approximation property (ITAP, see Definition 2.2). He also pointed out that there was no serious difficulty in extending the main theorem to fine hyperbolic graphs, but he did not outline the proof. So in this paper, we first give a proof for this extension, see Theorem 1.1 below. Then we define a metric invariant translation approximation property for countable discrete metric spaces (may not have bounded geometry), see Definition 2.3 below. This definition can be viewed as a version of the ITAP for metric spaces in some sense, though the two definitions are not equivalent when we view $G$ as a countable discrete group and as a metric space. By using the extension result (Theorem 1.1) for a fine hyperbolic graph, we can prove a fine hyperbolic graph that has the metric invariant translation approximation property. We outline the main results in this paper:

**Theorem 1.1.** *Let $\Gamma$ be a fine hyperbolic graph and $d$ be the graph metric on $\Gamma$. Then there exists a constant $C > 0$ such that there exists a sequence of finitely supported functions $f_n : \mathbb{N} \to [0, 1]$ with $f_n \to 1$ pointwise and the Schur multiplier on $C_u^*(\Gamma)$ associated with the kernel*

$$K_n(x, y) = f_n(d(x, y))$$

*has completely bounded norm at most $C$ for every $n$.*

**Theorem 1.2.** *Given a fine hyperbolic graph $\Gamma$, $\Gamma$ has the metric translation invariant approximation property.*

## 2. Preliminaries of approximation properties

First, we recall the definition of the uniform Roe algebra for a countable discrete metric space. Let $X$ be a countable discrete metric space. Denote the $C^*$-algebra of all bounded linear operators on the Hilbert space $l^2(X)$ by $\mathcal{B}(l^2(X))$. Any operator $T$ in $\mathcal{B}(l^2(X))$ has a natural form of $X \times X$ matrix:

$$T = [T(x, y)]_{(x,y)\in X\times X}.$$

Let $C^*_{\text{alg}}(X)$ be the operators in $\mathcal{B}(l^2(X))$ such that

(1) $T$ has finite propagation, i.e. there exists $R \geq 0$ such that $T(x, y) = 0$ if $d(x, y) \geq R$. The propagation of $T$ is the smallest value of such $R$, denoted by Prop$(T)$;
(2) There exists $L > 0$ such that for any $y \in X$, $\sharp\{x : T(x, y) \neq 0\} \leq L$ and $\sharp\{x : T(y, x) \neq 0\} \leq L$.

DEFINITION 2.1

$C^*_{\text{alg}}(X)$ is a $*$-subalgebra of $\mathcal{B}(l^2(X))$, its norm closure is called the uniform Roe algebra, denoted by $C_u^*(X)$.

When $X$ has bounded geometry, the second condition above is automatically true. Bounded geometry means for any $r > 0$, there exists $N > 0$, such that the ball in $X$ with radius $r$ contains at most $N$ elements. The uniform Roe algebra plays an important role in both index theory [6] and exactness problems in $C^*$-algebra theory [9].

For a closed subspace $\mathscr{A}$ of $C_u^*(X)$, define $\overset{\circ}{\mathscr{A}} = \{T \in \mathscr{A} | T \text{ satisfies (1), (2) above}\}$. Now let us recall the definition of ITAP for a countable discrete group which is defined by Roe in [10]. Let $G$ be a countable discrete group and we view $G$ as a bounded geometrical metric space with a left invariant metric, let $C_u^*(|G|)$ be the uniform Roe algebra with respect to this metric. Define $C_u^*(|G|)^G$ by

$$C_u^*(|G|)^G = \{T \in C_u^*(G) | T_{gh,gh'} = T_{h,h'}, g, h, h' \in G\},$$

and let

$$C_u^*(|\overset{\circ}{G}|)^G = \{T \in C_u^*(|G|)^G | \text{Prop}(T) < +\infty\}.$$

It is clear that

$$C_u^*(|\overset{\circ}{G}|)^G = \mathbb{C}G,$$

where $\mathbb{C}G$ is the group algebra and

$$\overline{C_u^*(|\overset{\circ}{G}|)^G} = C_r^*(G).$$

DEFINITION 2.2

If $\overline{C_u^*(|\overset{\circ}{G}|)^G} = C_u^*(|G|)^G$, i.e. $C_u^*(|G|)^G = C_r^*(G)$, then we say that $G$ has the invariant translation approximation property (ITAP).

Theorem 3.2 in [12] implies the following theorem:

**Theorem 2.3.** *Weakly amenable groups have ITAP.*

In [8], Ozawa proved that hyperbolic groups are weakly amenable, hence they have ITAP.

For a discrete metric space $X$, let

$$\mathscr{A} = \{T \in C_u^*(X) | T_{x,y} = T_{x',y'},\ \text{whenever } d(x, y) = d(x', y')\},$$

then $\mathscr{A}$ is an operator system of $C_u^*(X)$ (a closed self-adjoint subspace of $C_u^*(X)$).

DEFINITION 2.4

If $\overline{\overset{\circ}{\mathscr{A}}} = \mathscr{A}$, then we say that $X$ has the metric invariant translation approximation property.

Finally, we recall the definition of Schur multiplier. For a kernel $\varphi : X \times X \to \mathbb{C}$, the Schur multiplier associated to $\varphi$ is the map $M_\varphi : C_u^*(X) \to C_u^*(X)$ by

$$M_\varphi(A) := [\varphi(x, y)A_{x,y}]_{(x,y)\in X\times X}.$$

If $M_\varphi$ is completely bounded, then we denote by $\|M_\varphi\|_{cb}$ the completely bounded norm of $M_\varphi$.

## 3. Fine hyperbolic graphs

Let $\Gamma$ be a graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. We can think of $\Gamma$ as a 1-dimensional simplex complex. A path of length $n$ connecting $x$ and $y$ is a sequence $x_0x_1 \cdots x_n$ of vertices with $x_0 = x, x_n = y$ and $x_i$ is equal to or adjacent to $x_{i+1}$ for $i = 0, 1, \ldots, n-1$. If the vertices $x_0, x_1, \ldots, x_n$ above are all distinct, we call the sequence $x_0x_1 \cdots x_n$ an arc. Now we equip $\Gamma$ with the graph distance:

$$d(x, y) = \min\{n | \text{there exists an arc of length } n \text{ connecting } x \text{ and } y\}.$$

We assume that the graph $\Gamma$ is connected so that the distance is well defined. A geodesic path $p$ is a finite sequence $p(0), p(1), \ldots, p(d)$ of vertices in $\Gamma$ such that

$d(p(m), p(n)) = |m-n|$ for $m, n \in \{0, 1, \ldots, d\}$. A geodesic ray is a map $q : \mathbb{N} \to \Gamma$ such that $d(q(m), q(n)) = |m-n|$, and we call $q(0)$ the original vertex of $q$. A $(\lambda, \varepsilon)$-quasi-geodesic path $p$ in $\Gamma$ is a finite sequence $p(0), p(1), \ldots, p(d)$ of vertices in $\Gamma$ such that

$$\frac{1}{\lambda}|m-n| - \varepsilon \le d(p(m), p(n)) \le \lambda|m-n| + \varepsilon$$

for $m, n \in \{0, 1, \ldots, d\}$. A $(\lambda, \varepsilon)$-quasi-geodesic ray is a map $q : \mathbb{N} \to \Gamma$ such that

$$\frac{1}{\lambda}|m-n| - \varepsilon \le d(q(m), q(n)) \le \lambda|m-n| + \varepsilon.$$

DEFINITION 3.1 (Chapter III. H, Definition 1.16 of [3])

Let $\Gamma$ be a graph. $\Gamma$ is said to be $\delta$-hyperbolic if the insize of every geodesic triangle $\triangle = \triangle(a, b, c)$ in $\Gamma$ is no more than $\delta$, i.e. for $i_1 \in ab, i_2 \in ac, i_3 \in bc$ with $d(a, i_1) = d(a, i_2), d(b, i_1) = d(b, i_3), d(a, i_2) = d(c, i_3)$, we have $d(i_1, i_2) \le \delta, d(i_1, i_3) \le \delta, d(i_2, i_3) \le \delta$ .

*Lemma* 3.2 (*Chapter III. H, Theorem* 1.7 *of* [3]). *Let $\Gamma$ be a $\delta$-hyperbolic geodesic space. For $\lambda \ge 1$ and $\varepsilon \ge 0$, there exists $R = R(\delta, \lambda, \varepsilon) > 0$ such that if $q$ is a $(\lambda, \varepsilon)$-quasi-geodesic segment and $p$ is a geodesic segment joining the endpoints of $q$, then $q$ is contained in the $R$-neighborhood of $p$.*

DEFINITION 3.3 (2)

We say that a graph $\Gamma$ is fine if for any $n \in \mathbb{N}$, there exists an integer $m$ such that for any $x, y \in V(\Gamma)$, the number of arcs of length $n$ connecting $x$ and $y$ is no more than $m$.

From now on, we assume that $\Gamma$ is a $\delta$-hyperbolic fine graph. We fix a geodesic ray $\mathfrak{P} : \mathbb{N} \to \Gamma$. For any $x \in \Gamma$, let $p_x$ be a geodesic ray which starts at $x$ and eventually flows into $\mathfrak{P}$. Let $\Omega$ be the set of all $(1, 2\delta)$-quasi-geodesic rays starting at $x$ and eventually flowing into $\mathfrak{P}$. Define $T(x, k)$ by

$$T(x, k) = \{\omega \in \Gamma | \ \exists q_x \in \Omega \text{ such that } \omega \in q_x \text{ with } d(\omega, x) = k-1 \text{ or } k\}.$$

PROPOSITION 3.4

*$\sharp T(x, k) \le M(\delta)$ with $M(\delta)$ is independent on $k$ and $x$.*

*Proof.* Let $\mathfrak{P} : \mathbb{N} \to \Gamma$ be an infinite geodesic path, $p_x$ be a geodesic ray which start at $x$ and eventually flows into $\mathfrak{P}$. Let $q_x$ be another $(1, 2\delta)$-quasi-geodesic ray and eventually flowing into $\mathfrak{P}$. By Lemma 3.2, there exists $K = K(\delta)$ such that $q_x \subseteq N(p_x, K)$.

Let $v$ be a point on $p_x$ such that $d(x, v) = k$. Assume $\omega \in q_x$ and $d(\omega, x) \in \{k-1, k\}$, then there exists a point $\omega'$ in $p_x$ such that $d(\omega, \omega') \le K$. So we have

$$|d(\omega', x) - d(\omega, x)| \le d(\omega', \omega) \le K$$

and

$$\begin{aligned} d(\omega, v) &\leq d(\omega, \omega') + d(v, \omega') \leq K + |d(x, \omega') - d(x, v)| \\ &\leq K + |d(x, \omega') - d(\omega, x)| + |d(x, \omega) - k| \\ &\leq 2K + 1. \end{aligned}$$

Let $o$ be the first common point such that $p_x, q_x$ flow into $\mathfrak{P}$ (notice that $o$ is related to $q_x$).

If $d(x, o) \leq 6K + 2$, we know the length of arc $\widehat{x\omega o}$ will be no more than $6K + 2 + 2\delta$ since $q_x$ is $(1, 2\delta)$-quasi-geodesic. By the uniformly fine property of $\Gamma$, we know that such an arc $\widehat{x\omega o}$ is finite, and so is $\omega$.

If $d(x, o) > 6K + 2$, we only consider for $k \geq 3K + 1$ and the case of $k < 3K + 1$ can be proved easily. We draw two circles, one with center $v$ and radius $2K + 1$, and the other with center $v$ and radius $3K + 1$. Since $d(v, \omega) \leq 2K + 1$, $q_x$ must intersect with the little circle. Choose $s, t \in q_x$ such that $d(s, v) = 3K + 1$, $d(t, v) = 3K + 1$. By Lemma 3.2, there exist $a, b \in p_x$ such that $d(s, a) \leq K$, $d(t, b) \leq K$ and $a, b$ are not contained in the little circle. So

$$\begin{aligned} l(\widehat{st}) &\leq d(s, t) + 2\delta \leq d(s, v) + d(t, v) + 2\delta \\ &\leq 6K + 2 + 2\delta \end{aligned}$$

and

$$l(\widehat{as\omega tb}) \leq 8K + 2 + 2\delta.$$

By the fine property of $\Gamma$, we know that such an arc $\widehat{as\omega tb}$ is finite, and so is $\omega$. □

PROPOSITION 3.5

*Let* $W(k, l) = \{(x, y) | T(x, k) \cap T(y, l) \neq \emptyset\}$, *then*

$$E(n) = \{(x, y) | d(x, y) \leq n\} = \bigcup_{k=0}^{n} W(k, n - k).$$

*Moreover, there exists a constant* $R$ *such that* $W(k, l) \cap W(k + j, l - j) = \emptyset$ *for all* $j > R$.

*Proof.* For any $(x, y) \in W(k, n - k)$ with $k = 0, 1, 2, \ldots, n$, there exists $\omega \in T(x, k) \cap T(y, n-k)$. We have $d(x, \omega) \leq k$ and $d(y, \omega) \leq n-k$, so $d(x, y) \leq n$, i.e. $W(k, n-k) \subseteq E(n)$ for $k = 0, 1, \ldots, n$. To prove $E(n) \subseteq \bigcup_{k=0}^{n} W(k, n - k)$, let $(x, y) \in E(n)$ with $d(x, y) \leq n$.

(1) If $x \in p_y$ or $y \in p_x$, it is easy to get the conclusion.
(2) Otherwise, let $o \in p_x \cap p_y$ such that $d(o, x) + d(o, y) \geq n$. Consider the geodesic triangle $xyo$. Choose $i_1 \in xo$, $i_2 \in yo$, $i_3 \in xy$ such that

$$d(o, i_1) = d(o, i_2), d(x, i_1) = d(x, i_3), d(y, i_2) = d(y, i_3)$$

and

$$d(i_1, i_2) \leq \delta, d(i_2, i_3) \leq \delta, d(i_3, i_1) \leq \delta.$$

Choose a geodesic segment $i_3o$ and let $\omega(m) \in i_3o$ such that $d(\omega(m), i_3) = m$. Consider the function $f(m) = d(\omega(m), x) + d(\omega(m), y)$, we know $f(0) = d(x, y) \leq n$, $f(m+1) \leq f(m) + 2$. Therefore, there is $m_0 \in \mathbb{N}$ such that $f(m_0) \in \{n-1, n\}$. We claim that $\omega := \omega(m_0) \in T(x, k) \cap T(y, n-k)$ for $k = d(\omega, x)$. We only prove that $\omega(m_0) \in T(x, k)$, $\omega(m_0) \in T(y, n-k)$ can be proved similarly.

In fact, to prove $\omega(m_0) = \omega \in T(x, k)$, it suffices to prove that the arc $\widehat{xi_3o}$ is a $(1, 2\delta)$-quasi-geodesic arc. For any $a, b$ in the arc $\widehat{xi_3o}$, without loss of generality, we assume that $a \in oi_3$, $b \in xi_3$. Consider the geodesic triangle $abi_3$, there exist $i_1' \in ab$, $i_2' \in ai_3$, $i_3' \in bi_3$ such that

$$d(a, i_1') = d(a, i_2'), d(b, i_1') = d(b, i_3'), d(i_3, i_2') = d(i_3, i_3')$$

and

$$d(i_1', i_2') \leq \delta, d(i_2', i_3') \leq \delta, d(i_3', i_1') \leq \delta.$$

We can prove that $d(i_2', i_3) \leq \delta$. Otherwise, if $d(i_2', i_3) > \delta$, then by $d(x, i_3') \leq d(x, i_3)$, we know

$$d(o, i_2') = d(o, i_3) - d(i_2', i_3) \leq d(o, i_2) + d(i_2, i_3) - d(i_2', i_3) < d(o, i_2)$$

and

$$\begin{aligned} d(o, x) &\leq d(o,a) + d(a,b) + d(b,x) = d(o,a) + d(a,i_2') + d(b,i_3') + d(b,x) \\ &= d(o, i_2') + d(x, i_3') < d(o, i_2) + d(x, i_3) = d(o, x). \end{aligned}$$

It is a contradiction. So $l(\widehat{bi_3a}) - d(a, b) = 2d(i_2', i_3) \leq 2\delta$, we get that $\widehat{xi_3o}$ is a $(1, 2\delta)$-quasi-geodesic arc and $\omega \in T(x, k)$. Similarly, we can prove $\widehat{yi_3o}$ is a $(1, 2\delta)$-quasi-geodesic arc and hence $\omega \in T(y, n-k)$. By the definition of $W(k, n-k)$, we get $(x, y) \in W(k, n-k)$ and hence

$$E(n) \subseteq \{(x, y) | d(x, y) \leq n\} = \bigcup_{k=0}^{n} W(k, n-k).$$

Moreover, suppose that $(x, y) \in W(k, l) \cap W(k + j, l - j)$ exists. We choose $v \in T(x, k) \cap T(y, l)$ and $w \in T(x, k + j) \cap T(y, l - j)$. Let $v_x, w_x$ be the points on $p_x$ such that $d(v_x, x) = k$ and $d(w_x, x) = k + j$. Then by the proof of Proposition 3.4, we know $d(v, v_x) \leq 2K + 1$ and $d(w, w_x) \leq 2K + 1$. Choose $v_y, w_y$ on $\mathfrak{P}_y$ likewise for $y$. It follows that $d(v_x, v_y) \leq 4K + 2$ and $d(w_x, w_y) \leq 4K + 2$. Choose a point $o$ on $\mathfrak{P}_x \cap \mathfrak{P}_y$, then $|d(v_x, o) - d(v_y, o)| \leq 4K + 2$ and $|d(w_x, o) - d(w_y, o)| \leq 4K + 2$. On the other hand, one has $d(v_x, o) = d(w_x, o) + j$ and $d(v_y, o) = d(w_y, o) - j$. So it follows that

$$2j = d(v_x, o) - d(w_x, o) + d(w_y, o) - d(v_y, o) \leq 8K + 4,$$

so the second assertion is proved by taking $R = 4K + 2$. □

PROPOSITION 3.6

*Let* $Z(k, l) = W(k, l) \bigcap \bigcap_{j=1}^{R} W(k + j, l - j)^c$. *Then for every* $n \in \mathbb{N}$, *one has*

$$\chi_{E(n)} = \sum_{k=0}^{n} \chi_{Z(k,n-k)}.$$

*Proof.* The proof is similar to Lemma 9 in [8]. □

Now with Propositions 3.4, 3.5 and 3.6, we can get the following theorem by using techniques similar to Ozawa's in [8] by replacing the assumption of bounded geometry by Proposition 3.4:

**Theorem 3.7.** *Let $\Gamma$ be a fine hyperbolic graph and $d$ be the graph metric on $\Gamma$. Then there exists a constant $C$ such that there exists a sequence of finite supported functions $f_n : \mathbb{N} \to [0,1]$ such that $f_n \to 1$ pointwise and the Schur multiplier on $C_u^*(\Gamma)$ associated with the kernel*

$$K_n(x,y) = f_n(d(x,y))$$

*has completely bounded norm at most $C$ for every $n$.*

By the above theorem, we can get our main theorem:

**Theorem 3.8.** *Let $\Gamma$ be a fine hyperbolic graph, then $\Gamma$ has metric invariant translation approximation property.*

*Proof.* Let $\mathscr{A} = \{A \in C_u^*(\Gamma) | A_{x,y} = A_{x',y'} \text{ if } d(x,y) = d(x',y')\}$. For any $A \in \mathscr{A}$, we choose an element $T \in C_u^*(\Gamma)$ which has finite propagation $R$ (i.e. $\mathrm{Prop}(T) = R$) such that $\|A - T\| \leq \frac{\varepsilon}{3}$ and there exists $L > 0$ such that $\sharp\{x : T_{x,y} \neq 0\} < L$, $\sharp\{x : T_{y,x} \neq 0\} < L$. Let $I : \Gamma \times \Gamma \to \{0,1\}$ with $I(x,y) = 1$ if $T(x,y) \neq 0$ and $I(x,y) = 0$ if $T(x,y) = 0$. Then by a similar proof as Lemma 3.7 in [4], we get

$$|(T\xi)_x|^2 \leq L \cdot \left( \sup_{(x,y):d(x,y)\leq R} |T_{x,y}| \right)^2 \sum_{y:\mathrm{dist}(x,y)\leq R} |I(x,y)\xi_y|^2.$$

So

$$\begin{aligned} \|T\xi\|^2 &\leq L \cdot \left( \sup_{(x,y):d(x,y)\leq R} |T_{x,y}| \right)^2 \sum_x \left( \sum_{y:\mathrm{dist}(x,y)\leq R} |I(x,y)\xi_y|^2 \right) \\ &\leq L \cdot \left( \sup_{(x,y):d(x,y)\leq R} |T_{x,y}| \right)^2 \sum_y \left( \sum_{x:\mathrm{dist}(x,y)\leq R} |I(x,y)\xi_y|^2 \right) \\ &\leq L^2 \cdot \left( \sup_{(x,y):d(x,y)\leq R} |T_{x,y}| \right)^2 \sum_y |\xi_y|^2 \end{aligned}$$

and

$$\|T\| \leq L \cdot \sup_{(x,y):d(x,y)\leq R} |T_{x,y}|.$$

By Theorem 3.7, there exists a constant $C$ such that for $\frac{\varepsilon}{3L\|T\|} > 0$, we can get a kernel $\varphi$ with finite propagation such that $|1 - \varphi(x,y)| \leq \frac{\varepsilon}{3L\|T\|}$ if $d(x,y) \leq R$ and $\|\varphi\|_{cb} \leq C$. Moreover, we know $\varphi(x,y) = \varphi(x',y')$ if $d(x,y) = d(x',y')$. So we get

$$\begin{aligned} \|T - M_\varphi(T)\| &\leq L \cdot \sup_{(x,y):d(x,y)\leq R} |(1-\varphi(x,y))T_{x,y}| \\ &\leq L\|T\| \sup_{(x,y):d(x,y)\leq R} |(1-\varphi(x,y))| \\ &\leq \frac{\varepsilon}{3} \end{aligned}$$

and

$$\begin{aligned}\|A - M_\varphi(A)\| &\leq \|A - T\| + \|T - M_\varphi(T)\| + \|M_\varphi(T) - M_\varphi(A)\| \\ &\leq \tfrac{2\varepsilon}{3} + C\tfrac{\varepsilon}{3}.\end{aligned}$$

This means $\mathscr{A} \subseteq \overline{\overset{\circ}{\mathscr{A}}}$. The reverse inclusion is obvious. □


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