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Wavelet transform of generalized functions in $K'\{M_p\}$ spaces

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Abstract. Using convolution theory in $K\{M_p\}$ space we obtain bounded results for the wavelet transform. Calderón-type reproducing formula is derived in distribution sense as an application of the same. An inversion formula for the wavelet transform of generalized functions is established.

Keywords. Continuous wavelet transform; Schwartz distributions; Gel'fand-Shilov spaces $K\{M_p\}$ and $K'\{M_p\}$.

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1. Introduction

Using translation parameter $b \in \mathbb{R}^n$ and dilation parameter $a > 0$, the wavelet $\psi_{b,a}(t)$ is defined as

$$\mathfrak{D}_a \tau_b \psi(t) = \psi_{b,a}(t) = a^{-n/2}\psi\left(\frac{t-b}{a}\right), \quad t \in \mathbb{R}^n. \tag{1}$$

The wavelet transform $W(b,a)$ of an element $f \in L^2(\mathbb{R}^n)$ with respect to the wavelet $\psi_{b,a}(t) \in L^2(\mathbb{R}^n)$ is defined by (p. 28 of [7])

$$\begin{aligned} W(b,a) &= a^{-n/2}\int_{\mathbb{R}^n} f(t)\bar{\psi}\left(\frac{t-b}{a}\right)\mathrm{d}t \\ &= \int_{\mathbb{R}^n} f(t)\overline{\psi_{b,a}(t)}\mathrm{d}t \\ &= (f * h_{a,0})(b), \end{aligned} \tag{2}$$

where $h(x) = \overline{\psi(-x)}$, provided the integral exists. In view of (2) the wavelet transform $W(b,a)$ can be regarded as the convolution of f and $h_{a,0}$. The convolution product plays a central place among the various modes of function compositions. It is a powerful mathematical tool used in symbolic calculus, approximation theory, Fourier series, and in the solution of boundary-value problems. The existence of convolution $f * g$ has been investigated by many authors [10]. The inversion formula for the wavelet transform (2) is given by (p. 130 of [7])

$$\frac{1}{C_\psi}\int_0^\infty \int_{\mathbb{R}^n} W(b,a)\psi_{b,a}(x)a^{-n-1}\mathrm{d}b\mathrm{d}a = f(x), \quad \text{in } L^2(\mathbb{R}^n), \tag{3}$$

where

$$C_\psi = \int_0^\infty |\hat{\psi}(a\xi)|^2 a^{-1}\mathrm{d}a < \infty.$$

Now, we recall of the definition and notations relevant to $K\{M_p\}$ space of Gel'fand and Shilov [4]. Let $\{M_p\}$ be a sequence of real-valued functions defined over $\mathbb{R}^n$ such that

$$1 \leqq M_1(x) \leqq M_2(x) \leqq \cdots \qquad \text{for all } x \in \mathbb{R}^n.$$

The space $K\{M_p\}$ consists of all infinitely differentiable functions ϕ on $\mathbb{R}^n$ such that

$$\|\phi\|_p = \sup\left\{M_p(x)|D^\alpha\phi(x)| : x \in \mathbb{R}^n, |\alpha| \leqq p\right\} < \infty \tag{4}$$

for all $p \geq 1$. The vector space $K\{M_p\}$ is supplied with the locally convex topology generated by the sequence of norms $\{\| \quad \|_p\}_{p=1}^\infty$ defined by (4). Under this topology, $K\{M_p\}$ is a Fréchet space (II.2.2 of [4]).

We shall consider $K\{M_p\}$ spaces where M_p satisfies some additional conditions given below [13].

(P) For each $p \in \mathbb{N}$ there is a $p' > p$ such that for every $\epsilon > 0$ there exists $T > 0$ with the property $M_p(x)M_{p'}^{-1}(x) = m_{pp'}(x) < \epsilon$ for $|x| > T$.

(M) The functions M_p are quasi-monotonic in each co-ordinate, i.e. if $|x_j'| \leqq |x_j''|$, then

$$M_p(x_1, \ldots, x_j', \ldots, x_n) \leqq C_p M_p(x_1, \ldots, x_j'', \ldots, x_n)$$

for each fixed point $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$.

(F) Each M_p is symmetric, i.e., $M_p(x) = M_p(-x)$, and for each p there is a $p' > p$ and $C_{p'} > 0$ such that

$$M_p(x+y) \leqq C_{p'}M_{p'}(x)M_{p'}(y), \quad \text{for all } x, y \in \mathbb{R}^n.$$

The dual of $K\{M_p\}$ is denoted by $K'\{M_p\}$. The Schwartz space $\mathscr{S}'$, Gel'fand–Shilov spaces $(S_{\alpha,A})'$ and $(\mathcal{W}_{M,A})'$ are special cases of $K'\{M_p\}$; and $K'\{M_p\} \subset \mathcal{D}'$.

An infinitely differentiable function ψ on $\mathbb{R}^n$ is said to be a multiplier on $K\{M_p\}$ if

(i) $\psi\phi \in K\{M_p\}$ for each $\phi \in K\{M_p\}$ and
(ii) the map $\phi \to \psi\phi$ is continuous from $K\{M_p\}$ into itself.

The vector space of all multipliers on $K\{M_p\}$ is denoted by $\Theta_M(K\{M_p\})$ (p. 597 of [12]).

For later use we recall the following fact from II.4.2 of [4]. If $\{M_p\}$ satisfies (P) and (M), the sequence of seminorms

$$\|\phi\|_p' = \sup\left\{\left|\int_{\mathbb{R}^n} M_p(t)|D^\alpha\phi(t)\mathrm{d}t\right| : |\alpha| \leqq p\right\} \quad (p = 1, 2, \ldots) \tag{5}$$

generates the same local topology as the sequence of norms $\{\|\|_p : p \geqq 1\}$ defined by (4). Various other results related to convolution in $K\{M_p\}$-spaces can be found in [5, 6, 9, 12] and [13].

In this paper, we shall study convolution in $K\{M_p\}$ space in §2. Boundedness result for the wavelet transform is obtained in §3. Calderón's formula involving distributions is established in §4. An inversion formula for the wavelet transform of generalized functions is derived in §5.

2. Convolution in $K\{M_p\}$ space

We now consider translation on $K\{M_p\}$, spaces. If $\psi \in K\{M_p\}$, $b \in \mathbb{R}^n$ and $a > 0$, the translation of ψ by b is denoted by $\tau_b\psi$, i.e. $\tau_b\psi(x) = \psi(x-b)$. If $\{M_p\}$ satisfies condition (P), (M) and (F), we have the following results.

Theorem 1. *If $T \in \mathscr{E}'(\mathbb{R}^n)$ and $\psi \in K\{M_p\}$, then the function*

$$\phi(\xi) = \langle T_\eta, \psi(\xi - \eta)\rangle$$

belongs to $K\{M_p\}$. Furthermore, if $\{\psi_j\}$ converges to zero in $K\{M_p\}$ space, then $\{\phi_j\}$ converges to zero in $K\{M_p\}$.

Proof. Since every distribution T with compact support can be written as a finite sum of derivatives of continuous functions with compact support contained in an arbitrary neighbourhood of the support of T (Theorem 2.22 of [2]), it suffices to prove the theorem when $T = \partial^\beta G$, where G is a continuous function in $\mathbb{R}^n$ such that its support is contained in an arbitrary neighbourhood of the support of T. Therefore,

$$\phi(\xi) = \langle T_\eta, \psi(\xi-\eta)\rangle = (-1)^{|\alpha|}\int_{\mathbb{R}^n} G(\eta) D^\alpha \psi(\xi-\eta)\,\mathrm{d}\eta,$$

hence using condition (F) we have

$$\begin{aligned} M_p(\xi)D^\beta\phi(\xi) &= (-1)^{|\alpha|}\int_{\mathbb{R}^n} G(\eta)M_p(\xi)D^{\alpha+\beta}\psi(\xi-\eta)\,\mathrm{d}\eta \\ &= (-1)^{|\alpha|}\int_{\mathbb{R}^n} M_p(\xi-\eta+\eta)G(\eta)D^{\alpha+\beta}\psi(\xi-\eta)\,\mathrm{d}\eta \\ &\leqq \int_{\mathbb{R}^n} |M_p(\eta)G(\eta)||M_p(\xi-\eta)D^{\alpha+\beta}\psi(\xi-\eta)|\mathrm{d}\eta. \end{aligned}$$

Then by (5),

$$\begin{aligned} \|\phi\|'_p &= \int_{\mathbb{R}^n} M_p(\xi)|D^\beta\phi(\xi)|\mathrm{d}\xi \\ &= \int_{\mathbb{R}^n} M_p(\eta)|G(\eta)|\mathrm{d}\eta \int_{\mathbb{R}^n} M_p(\xi-\eta)|D^{\alpha+\beta}\psi(\xi-\eta)|\mathrm{d}\xi \\ &\leqq C_p\|\psi\|'_{p'}, \end{aligned}$$

from which the statement of the theorem follows. □

Remark 1. By recalling that

$$(T * \psi)(\xi) = \langle T_\eta, \psi(\xi - \eta)\rangle,$$

it follows from the above theorem, that if $T \in \mathscr{E}'(\mathbb{R}^n)$ and $\psi \in K\{M_p\}$, then $T * \psi \in K\{M_p\}$.

In the following we assume that $\{M_p\}$ satisfies an additional condition:

(A) For any two subscripts p, r $(p \geq r)$ there exists $s \geq p$ such that

$$M_p(x) M_r(x) \leqq C_{pr} M_s(x) \tag{6}$$

for $x \in \mathbb{R}^n$.

Theorem 2. *If $\psi \in K\{M_p\}$, $U \in K'\{M_p\}$ and* (6) *holds, then $U * \psi \in \Theta_M(K\{M_p\})$.*

Proof. If $U \in K'\{M_p\}$, then there exists a positive integer p and bounded measurable function $f_\alpha (\alpha \in \mathbb{N}^n, |\alpha| \leqq p)$ such that (p. 596 of [12])

$$U = \sum_{|\alpha| \leqq p} D^\alpha (M_p f_\alpha). \tag{7}$$

By definition, we have

$$(U * \psi)(\xi) = \langle U_\eta, \psi(\xi - \eta)\rangle = \left\langle \sum_{|\alpha| \leqq p} D^\alpha (M_p(\eta) f_\alpha(\eta)), \psi(\xi - \eta) \right\rangle.$$

Therefore, by the technique used in the proof of the above theorem, we have

$$\begin{aligned} |D^\beta (U * \psi)(\xi)| &\leqq \sum_{|\alpha| \leqq p} \int_{\mathbb{R}^n} |(M_p(\eta) f_\alpha(\eta)) D^{\alpha+\beta} \psi(\xi - \eta)| \mathrm{d}\eta \\ &\leqq C_{p'} M_{p'}(\xi) L \sum_{|\alpha| \leqq p} \int_{\mathbb{R}^n} M_{p'}(-u) |D^{\alpha+\beta} \psi(-u)| \mathrm{d}u. \end{aligned}$$

This yields

$$\frac{1}{M_{p'}(\xi)} |D^\beta (U * \psi)(\xi)| \leqq C_{p'} L \|\psi\|'_{p'} < \infty,$$

where $p' > p$ is given by condition (F) and L is a common bound for $\{f_\alpha\}$.

Therefore, by p. 101 of [4], ψ is a multiplier in $K\{M_p\}$-space. □

Theorem 3. *If $U \in K'\{M_p\}$ and $T \in \mathscr{E}'(\mathbb{R}^n)$, then $U * T \in K'\{M_p\}$. Moreover, the bilinear map*

$$K'\{M_p\} \times \mathscr{E}'(\mathbb{R}^n) \ni (U, T) \to U * T \in K'\{M_p\}$$

is separately continuous.

Proof.

Linearity: Let $U \in K'\{M_p\}$, $T \in \mathscr{E}'(\mathbb{R}^n)$ and $\psi \in K\{M_p\}$. Then by [2, Theorem 3.2], $\langle T_\xi, \psi(\xi - \eta)\rangle$ is an infinitely differentiable function of η depending continuously on $\psi \in K\{M_p\}$. Therefore

$$\langle T_\eta, \langle U_\xi, \psi(\xi - \eta)\rangle\rangle \tag{8}$$

is well defined and depends continuously on $\psi \in K\{M_p\}$. By Theorem 1, the function

$$\phi(\xi) = \langle T_\eta, \psi(\xi - \eta)\rangle$$

belongs to $K\{M_p\}$ space and depends continuously on $\psi \in K\{M_p\}$. Since $U \in K'\{M_p\}$,

$$\langle U_\xi, \langle T_\eta, \psi(\xi - \eta)\rangle\rangle \tag{9}$$

is also defined and it depends continuously on $\psi \in K\{M_p\}$. On the other hand, since (8) and (9) do concide when $\psi \in C_c^\infty(\mathbb{R}^n)$, and $C_c^\infty(\mathbb{R}^n)$ is dense in $K\{M_p\}$ they coincide everywhere in $K\{M_p\}$ space. Then we get

$$\begin{aligned}\langle U * T, \psi\rangle &= \langle U_\xi \circledast T_\eta, \psi(\xi - \eta)\rangle \\ &= \langle U_\xi, \langle T_\eta, \psi(\xi - \eta)\rangle\rangle \\ &= \langle T_\eta, \langle U_\xi, \psi(\xi - \eta)\rangle\rangle\end{aligned}$$

for all $\psi \in K\{M_p\}$; therefore $U * T \in K'\{M_p\}$.
Continuity: For fix $T \in \mathscr{E}'(\mathbb{R}^n)$, suppose the distributions $\{U_j\}$ converge to zero strongly in $K'\{M_p\}$. We have for every $\psi \in K\{M_p\}$,

$$\begin{aligned}\left\langle U_j * T, \psi\right\rangle &= ((U_j * T) * \check{\psi})(0) \\ &= T * (U_j * \check{\psi})(0),\end{aligned} \tag{10}$$

where $\check{\psi}(x) = \psi(-x)$. Since the convolution product of a function on $K\{M_p\}$ and a distribution of $K'\{M_p\}$ is separately continuous, hence, $U_j * \check{\psi} \to 0$ in C^∞. Furthermore, it can be shown that $U_j * \check{\psi}$ converges uniformly to zero whenever ψ belongs to a bounded set in $K\{M_p\}$. Hence, it follows that $T * (U_j * \check{\psi}) \to 0$ in C^∞ uniformly whenever ψ belongs to a bounded set of $K\{M_p\}$.

Therefore $(U_j) * T \to 0$ strongly in $K'\{M_p\}$. Similarly, we can prove the continuity of $U * T$ with respect to T.

This completes the proof of the theorem. □

3. Boundedness of wavelet transform in $K\{M_p\}$ space

We now consider translation and dilation of ψ in $K\{M_p\}$ spaces. Since the injection $\mathcal{D} \subseteq K\{M_p\}$ is continuous and $\mathcal{D}$ is dense in $K\{M_p\}$, each element of $K'\{M_p\}$ can be identified with a distribution. If $\{M_p\}$ satisfies conditions (P), (M) and (F), we have the following result.

Theorem 4. *Let $\{M_p\}$ satisfy conditions* (*P*), (*M*) *and* (*F*). *For $U \in \mathcal{D}'$, the following are equivalent*:

(*a*) $U \in K'\{M_p\}$.

(*b*) *For each positive integer* $p, U = \sum_{|\alpha|\leqq p} D^{\alpha}(M_p f_{\alpha})$, *where each* $M_p f_{\alpha}$ *is a continuous, bounded function on* $\mathbb{R}^n$.

(*c*) *There is a positive integer* k *such that for any* $\psi \in \mathcal{D}$, $\frac{(1+1/a)^{-(p+\frac{n}{2})}(U*\mathfrak{D}_a\psi)}{M_k(a)M_k(b)}$ *is bounded on* $\mathbb{R}^n$ *for* $b \in \mathbb{R}^n$ *and* $a > 0$.

(*d*) *There is a positive integer* k *such that* $\left\{\frac{(1+1/a)^{-(p+\frac{n}{2})}}{M_k(a)M_k(b)}\mathfrak{D}_a\tau_{-b}U : b \in \mathbb{R}^n, a > 0\right\}$ *is bounded in* $\mathcal{D}'$,

Proof. That (a) and (b) are equivalent is established in pp. 111–113 of [4]. First we show that (b) implies (c): Since $K\{M_p\}$ has a differentiable translation (p. 261 of [13]), so for each $\psi \in K\{M_p\}$ the function $\psi : (b, a) \to \langle f, \mathfrak{D}_a\tau_b\psi\rangle$ is in $C^{\infty}(\mathbb{R}^n)$ (pp. 139–141 of [4]). Then using representation of U as given by (7) we have

$$
\begin{aligned}
|U * h_{a,0}(b)| &= |U * \mathfrak{D}_a\psi(t)| \leqq \sum_{|\alpha|\leqq p}\int_{\mathbb{R}^n} M_p(t)|f_{\alpha}(t)a^{-n/2}D^{\alpha}\mathfrak{D}_a\tau_b\psi(t)|\mathrm{d}t \\
&\leqq \sum_{|\alpha|\leqq p}\int_{\mathbb{R}^n} M_p(au+b)|f_{\alpha}(au+b)(1/a)^{|\alpha|+n/2}D^{\alpha}\psi(u)|\mathrm{d}u \\
&\leqq (1+1/a)^{(p+\frac{n}{2})}C'_pM_{p'}(a)M_{p'}(b)L\sum_{|\alpha|\leqq p}\int_{\mathbb{R}^n} M_{p'}(u) \\
&\quad \times |D^{\alpha}\psi(u)|\mathrm{d}u \\
&\leqq (1+1/a)^{(p+\frac{n}{2})}C'_pM_{p'}(a)M_{p'}(b)L\|\psi\|'_{p'}
\end{aligned}
$$

where $p' > p$ and L is a common bound for $\{f_{\alpha}\}$. Thus if we set $p' = k$, (c) is established.

Now we show (c)$\Rightarrow$(d). Let $\psi \in \mathcal{D}$. By (c) there exists $B = B_{\psi} > 0$ such that $|(U * \mathfrak{D}_a)\psi(b)| \leqq B(1+1/a)^{(p+\frac{n}{2})}M_k(a)M_k(b)$ for $b \in \mathbb{R}^n$ and $a > 0$. That is,

$$
\sup\left\{\left|\left\langle\frac{(1+1/a)^{-(p+\frac{n}{2})}}{M_k(a)M_k(b)}\mathfrak{D}_a\tau_{-b}U, \psi\right\rangle\right| : b \in \mathbb{R}^n, a > 0\right\} \leqq B
$$

so that $\left\{\left|\frac{(1+1/a)^{-(p+\frac{n}{2})}}{M_k(a)M_k(b)}\mathfrak{D}_a\tau_{-b}U\right| : b \in \mathbb{R}^n, a > 0\right\}$ is weakly bounded in $\mathcal{D}'$ (and also strongly bounded [11]).

For (d)$\Rightarrow$(b), $\left\{\left|\frac{(1+1/a)^{-(p+\frac{n}{2})}}{M_k(a)M_k(b)}\mathfrak{D}_a\tau_{-b}U\right| : b \in \mathbb{R}^n, a > 0\right\}$ bounded in $\mathcal{D}'$ implies there is a compact neighbourhood K of 0 in $\mathbb{R}^n$ and a positive integer m such that if $\psi \in \mathcal{D}^m_K$, the family of continuous functions

$$
\left\{\left|\frac{(1+1/a)^{-(p+\frac{n}{2})}}{M_k(a)M_k(b)}\mathfrak{D}_a\tau_{-b}U * \psi\right| : b \in \mathbb{R}^n, a > 0\right\} \tag{11}
$$

is bounded on K [11].

The elementary solution E of Δ^n is m-times continuously differentiable for large N so if $\gamma \in \mathcal{D}_K$ is such that $\gamma(t) = 1$ for t in some neighbourhood of 0, then $\gamma E \in \mathcal{D}^m_K$ and $\delta = \Delta^N(\gamma E) - \phi$ where $\phi \in \mathcal{D}$, therefore

$$
U = U * \delta = \Delta^N(T * \gamma E) - T * \phi. \tag{12}
$$

Now $U*\psi \in \mathscr{E}$ since $U \in \mathcal{D}'$ and $\psi \in \mathcal{D}$, and (11) gives $\sup\{\frac{(1+1/a)^{-(p+\frac{n}{2})}}{M_k(a)M_k(b)}\langle U, \mathfrak{D}_a\tau_b\psi\rangle| : b \in \mathbb{R}^n, a > 0\} = \sup\{\frac{(1+1/a)^{-(p+\frac{n}{2})}}{M_k(a)M_k(b)}\langle U*\mathfrak{D}_a\psi(b)\rangle| : b \in \mathbb{R}^n, a > 0\} < \infty$. Since $0 \in K$, also $\gamma E \in \mathcal{D}_K^m$ implies $\sup\{\frac{(1+1/a)^{-(p+\frac{n}{2})}}{M_k(a)M_k(b)}\langle U*(\gamma E)(b)\rangle| : b \in \mathbb{R}^n, a > 0\} < \infty$ by (11) since $0 \in K$. Thus (12) yields (b). □

Theorem 5. *If $T \in \mathscr{E}'(\mathbb{R}^n)$ and $\psi \in K\{M_p\}$, then the function*

$$\phi_a(\xi) = \langle T_\eta, a^{-n/2}\psi_a(\xi-\eta)\rangle$$

belongs to $K\{M_p\}$. Furthermore, if ψ converges to zero in $K\{M_p\}$ space, then ϕ_a converges to zero in $K\{M_p\}$.

Proof. Following the technique of proof of Theorem 1, we have

$$\int_{\mathbb{R}^n} M_p(\xi)|D^\beta\phi_a(\xi)|\mathrm{d}\xi \leqq \int_{\mathbb{R}^n}|G(\eta)|M_p(\eta)\mathrm{d}\eta\int_{\mathbb{R}^n} a^{-n/2}M_p(\xi-\eta) \times D^{\alpha+\beta}\psi_a(\xi-\eta)\,\mathrm{d}\xi$$

hence, taking into account that G is a continuous function with compact support and by condition (F), we get the inequality

$$\int_{\mathbb{R}^n} M_p(\xi)|D^\beta\phi_a(\xi)|\mathrm{d}\xi \leqq C_p(1+1/a)^{p+|\beta|+n/2}M_{p'}(a)\int_{\mathbb{R}^n} M_{p'}(u)) \times D^{\alpha+\beta}\psi(u)\,\mathrm{d}u.$$

Thus, we get

$$\|\phi_a\|'_{p'} \leqq C_p(1+1/a)^{p+|\beta|+n/2}M_{p'}(a)\|\psi\|'_{p'}.$$

This completes the proof of the theorem. □

DEFINITION 6

The generalized wavelet transform of $T \in \mathscr{E}'(\mathbb{R}^n)$ is defined by $(WT)(b,a) = (U*h_{a,0})(b) = \langle U_t, a^{-n/2}\psi_a(t-b)\rangle$. The space $W_a\{M_p\}$ is defined to be the space of all C^∞-functions such that

$$\sup_{t\in\mathbb{R}^n, a\in\mathbb{R}_+}\left|(1+1/a)^{-(p+|\beta|+n/2)}\frac{D^\beta[U*h_{a,0}](b)}{M_{p'}(a)M_{p'}(b)}\right| < \infty, \tag{13}$$

for $p' > p$.

Theorem 7. *If $U \in K'\{M_p\}$ and $\psi \in K\{M_p\}$, then $WU \in W_a\{M_p\}$.*

Proof. By Definition 6 and representation (7) we have

$$\begin{aligned}(WU)(b) = (U*h_{a,0})(b) &= \langle U_t, a^{-n/2}\psi_a(t-b)\rangle \\ &= \left\langle \sum_{|\alpha|\leqq p} D^\alpha(M_p(t)f_\alpha(t)), a^{-n/2}\psi_a(t-b)\right\rangle.\end{aligned} \tag{14}$$

Moreover,

$$\begin{aligned}
&|D^{\beta}[U * h_{a,0}](x)| \\
&\quad = \left| \sum_{|\alpha| \leqq p} \int (M_p(t) f_\alpha(t)) a^{-n/2} D^{\alpha+\beta} \psi_a\,(t-b) \right| \mathrm{d}t \\
&\quad \leqq \left| \sum_{|\alpha| \leqq p} (1+1/a)^{p+|\beta|+n/2} \int_{\mathbb{R}^n} (M_p(b+au) f_\alpha(b+au)) D^{\alpha+\beta} \psi\,(u) \right| \mathrm{d}u \\
&\quad \leqq C_{p'} (1+1/a)^{p+|\beta|+n/2} M_{p'}(b) M_{p'}(a) L \sum_{|\alpha| \leqq p} \int_{\mathbb{R}^n} M_{p'}(u) |D^{\alpha+\beta} \psi\,(u)| \mathrm{d}u
\end{aligned}$$

$$\sup_{t \in \mathbb{R}^n, a \in \mathbb{R}_+} \left| (1+1/a)^{-(p+|\beta|+n/2)} \frac{D^{\beta}[U * h_{a,0}](b)}{M_{p'}(a) M_{p'}(b)} \right| \leqq C_{p'} L \|\psi\|'_{p'} < \infty,$$

where $p' > p$ with p given by condition (F) and L is a common bound for $\{f_\alpha\}$.

This completes the proof of the theorem. □

Theorem 8. *If $U \in K'\{M_p\}$ and $T \in \mathscr{E}'(\mathbb{R}^n)$, then $W(T * U) \in (W_a\{M_p\})'$.*

Proof.

(1) Let $U \in K'\{M_p\}$, $T \in \mathscr{E}'(\mathbb{R}^n)$ and $\psi \in K\{M_p\}$. Then we can show that $\langle T_\xi, a^{-n/2} \psi_a\,(\xi - \eta)\rangle$ is an infinitely differentiable function of η depending continuously on $\psi \in K\{M_p\}$ (Theorem 4.10, p. 134 of [2]). Therefore

$$\langle T_\eta, \langle U_\xi, a^{-n/2} \psi_a\,(\xi - \eta)\rangle\rangle \tag{15}$$

is well defined and depends continuously on $\psi \in K\{M_p\}$. By Theorem 5, the function

$$\phi_a(\xi) = \langle T_\eta, a^{-n/2} \psi_a\,(\xi - \eta)\rangle$$

belongs to $K\{M_p\}$ space and depends continuously on $\psi \in K\{M_p\}$. Since $U \in K'\{M_p\}$,

$$\langle U_\xi, \langle T_\eta, a^{-n/2} \psi_a\,(\xi - \eta)\rangle\rangle \tag{16}$$

is also defined and it depends continuously on $\psi \in K\{M_p\}$. On the other hand, since (15) and (16) do concide when $\psi \in \mathcal{D}$ and $\mathcal{D}$ is dense in $K\{M_p\}$, they coincide everywhere in $K\{M_p\}$ space. Then we get

$$\begin{aligned}
\langle U * T, h_{a,0}\rangle &= \langle U_\xi \circledast T_\eta, a^{-n/2} \psi_a\,(\xi - \eta)\rangle \\
&= \langle U_\xi, \langle T_\eta, a^{-n/2} \psi_a\,(\xi - \eta)\rangle\rangle \\
&= \langle T_\eta, \langle U_\xi, a^{-n/2} \psi_a\,(\xi - \eta)\rangle\rangle
\end{aligned}$$

for all $\psi \in K\{M_p\}$; therefore $U * T \in K'\{M_p\}$.

(2) For fix $T \in \mathscr{E}'(\mathbb{R}^n)$ and suppose that the distribution $\{U_j\}$ converges to zero strongly in $K'\{M_p\}$. We have for every $\psi \in K\{M_p\}$,

$$\begin{aligned}
\langle U_j * T, h_{a,0}\rangle &= ((U_j * T) * \check{h}_{a,0})(0) \\
&= T * (U_j * \check{h}_{a,0})(0).
\end{aligned} \tag{17}$$

As mentioned above, the convolution product of a function on $K\{M_p\}$ and a distribution of $K'\{M_p\}$ is separately continuous, hence, $U_j * \check{h}_{a,0} \to 0$ in C^∞. Furthermore, it can be shown that $U_j * \check{h}_{a,0}$ converges uniformly to zero whenever ψ belongs to a bounded set in $K\{M_p\}$. Hence, it follows that $T * (U_j * \check{h}_{a,0}) \to 0$ in C^∞ uniformly whenever ψ belongs to a bounded set of $K\{M_p\}$.

This implies taking into account (15), that $(U_j) * T \to 0$ strongly in $K'\{M_p\}$. Similarly, we can prove the continuity of $U * T$ with respect to T. □

4. Calderón's formula

In this section, we obtain Calderón's reproducing identity in $K'\{M_p\}$ space using the following theorem (p. 8 of [3]). In this section we use the following structure formula for $K'\{M_p\}$ (p. 261 of [12]): For each positive integer k,

$$U = \sum_{|\alpha| \le \eta_k} D^\alpha f_\alpha, \tag{18}$$

where each $M_k f_\alpha$ is a continuous, bounded function on $\mathbb{R}$.

Theorem 9. *Suppose $\psi \in L^1(\mathbb{R}^n)$ is real valued radial and satisfies $\int_0^\infty [\hat{\psi}(t\xi)]^2 \mathrm{d}t/t = 1$ if $\xi \in \mathbb{R}^n \backslash \{0\}$. If $f \in L^2(\mathbb{R}^n)$ and*

$$f_{\varepsilon,\delta}(x) = \int_\varepsilon^\delta (f * \psi_a * \psi_a)(x) \frac{\mathrm{d}a}{a}, \tag{19}$$

then $\|f - f_{\varepsilon,\delta}\|_{L^2(\mathbb{R}^n)} \to 0$ as $\varepsilon \to 0$ and $\delta \to \infty$.

Theorem 10. *Let $\psi \in K\{M_p\}$ and satisfy conditions of Theorem 9. Then the following Calderón's reproducing identity holds*:

$$U(x) = \int_0^\infty (U * \psi_a * \psi_a)(x) \frac{\mathrm{d}a}{a}, \quad \forall U \in K'\{M_p\}.$$

Proof. Let $\phi \in K\{M_p\}$. Then using structure formula (18) we have

$$\begin{aligned}\int_\varepsilon^R \left\langle (U * \psi_a * \psi_a) \frac{\mathrm{d}a}{a}, \phi \right\rangle &= \sum_{|\alpha| \leqq p} \int_\varepsilon^R \left\langle (D^\alpha(f_\alpha) * \psi_a * \psi_a) \frac{\mathrm{d}a}{a}, \phi \right\rangle \\ &= \sum_{|\alpha| \leqq p} \left\langle \int_\varepsilon^R (f_\alpha * \psi_a * \psi_a) \frac{\mathrm{d}a}{a}, (-1)^{|\alpha|} D^\alpha \phi \right\rangle.\end{aligned}$$

Since $M_p(x) f_\alpha(x)$ is bounded, $f_\alpha(x) \in L^2(\mathbb{R}^n) \subset K'\{M_p\}$ and converges in $L^2(\mathbb{R}^n)$ implies converges in $K'\{M_p\}$, by Theorem 9 we have

$$\begin{aligned}\left\langle \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \int_\varepsilon^R (f_\alpha * \psi_a * \psi_a) \frac{\mathrm{d}a}{a}, (-1)^{|\alpha|} D^\alpha \phi \right\rangle &= \langle f_\alpha, (-1)^{|\alpha|} D^\alpha \phi(t) \rangle \\ &= \langle D^\alpha(f_\alpha), \phi \rangle.\end{aligned}$$

Thus

$$\left\langle \int_0^\infty (U * \psi_a * \psi_a)(x) \frac{\mathrm{d}a}{a}, \phi \right\rangle = \left\langle \sum_{|\alpha| \leqq p} D^\alpha (f_\alpha), \phi \right\rangle$$
$$= \langle U, \phi \rangle .$$

This completes the proof of the theorem. □

5. Inversion of the wavelet transform of generalized functions

In this section in order to simplify analysis we consider the wavelet transform defined on $\mathbb{R}$. We also assume that the space $K\{M_p\}$ satisfies an additional condition (p. 87(4) of [6]): for arbitrary $p \in \mathbb{N}$ and $k \in \mathbb{N}_0$ there are an index $q := q(p,k) \in \mathbb{N}, q \geq p$, and a constant $C_{p,k} > 0$ such that

$$|M_p^{(k)}(t)| \leq C_{p,k} M_q(t), \quad t \in \mathbb{R}. \tag{20}$$

In order to derive inversion formula for the distributional wavelet transform, we construct a structure formula for the distribution $f \in K'\{M_p\}$ (pp. 272–274 of [14]) and follow the technique used in [8].

If $U \in K'\{M_p\}$ and $\phi \in K\{M_p\}$, then by boundedness property of distributions, there exists a $C > 0$ and a non-negative integer m satisfying

$$|\langle U, \phi \rangle| \leq C \max_{0 \leq k \leq m} \sup_{t \in \mathbb{R}^n} |M_p(t) D_t^k \phi(t)|, \quad \forall t > 0. \tag{21}$$

Let m be the least possible value of the non-negative integer. Then

$$\begin{aligned}
|\langle U, \phi \rangle| &\leq C \max_{0 \leq k \leq m} \sup_{t \in \mathbb{R}^n} \left| \int_{-\infty}^t \left| \frac{\mathrm{d}}{\mathrm{d}t} [M_p(t) D_t^k \phi(t)] \mathrm{d}t \right| \right| \\
&\leq C \max_{0 \leq k \leq m} \sup_{t \in \mathbb{R}^n} \left| \int_{-\infty}^t [|M_p(t) D_t^{k+1} \phi(t)| + |D_t M_p(t)||D_t^k \phi(t)|] \mathrm{d}t \right| \\
&\leq C \int_{-\infty}^\infty \left[\frac{M_p(t)}{M_{p'}(t)} |M_{p'}(t) D_t^{k+1} \phi(t)| + C_{p,1} \frac{M_q(t)}{M_{q'}(t)} |M_{q'}(t) D_t^k \phi(t)| \right] \mathrm{d}t, \\
&\quad \times (p' > p, q' > q) \\
&\leq C \|m_{pp'}\|_2 \|M_{p'}(t)) D_t^{k+1} \phi(t)\|_2 + C_{p,1} \|m_{qq'}\|_2 \|M_{q'}(t)) D_t^k \phi(t)\|_2.
\end{aligned}$$

Now, using Hahn–Banach theorem and Riesz representation theorem we get g_k belonging to the space $L^2(\mathbb{R})$ satisfying

$$\begin{aligned}
|\langle U, \phi \rangle| &= \langle g_1(t), M_{p'}(t) D_t^{k+1} \phi(t) \rangle + \langle g_2(t), M_{q'}(t) D_t^k \phi(t) \rangle \\
&= \langle (-1)^{k+1} D_t^{k+1} \{M_{p'}(t) g_1\} + (-1)^k D_t^k \{g_2 M_{q'}(t)\}, \ \phi(t) \rangle.
\end{aligned}$$

Therefore our structure formula is

$$U = (-1)^{k+1} D_t^{k+1} [M_{p'}(t) g_1] + (-1)^k D_t^k [g_2 M_{q'}(t)], \tag{22}$$

where $k = 0, 1, 2, 3, \ldots$.

Now we define functions $g_{1\nu}$ and $g_{2\nu}$ as follows [1]:

$$g_{1\nu} = \begin{cases} g_1(t), & -\nu \le t \le \nu \\ 0, & \text{elsewhere}, \end{cases}$$

$$g_{2\nu} = \begin{cases} g_2(t), & -\nu \le t \le \nu \\ 0, & \text{elsewhere}. \end{cases}$$

Also define $U_\nu \in K'\{M_p\}$ by

$$\langle U_\nu,\ \phi\rangle = \langle g_{1\nu}(t), M_{p'}(t)D_t^{k+1}\phi(t)\rangle + \langle g_{2\nu}(t), M_{q'}(t)D_t^k\phi(t)\rangle, \tag{23}$$

$g_{1\nu}$ and $g_{2\nu} \to g_1$ and g_2 respectively in $L^2(\mathbb{R})$ as $\nu \to \infty$ therefore, $\langle U_\nu, \phi\rangle \to \langle U, \phi\rangle$ as $\nu \to \infty$.

We now extend the inversion formula for the wavelet transform defined by (3) to distributions in $K'\{M_p\}$.

Theorem 11. *Assume that the wavelet transform $W(b, a)$ of $U \in K'\{M_p\}$ is given by* (14). *Then*

$$\lim_{\substack{N \to \infty \\ R \to \infty}} \left\langle \frac{1}{C_\psi}\int_0^R \int_{-N}^N W(b, a)\psi_{b,a}(x)\frac{\mathrm{d}b\mathrm{d}a}{a^2},\ \phi(x)\right\rangle = \langle U, \phi\rangle\,, \tag{24}$$

for each $\phi \in \mathcal{D}$, $a > 0$ and $b \in \mathbb{R}$, where $\psi_{b,a}(x)$ is defined by (1).

Proof. Using the structure formula for f as given in (22) we have

$$W_{U_\nu}(b, a) = \int_{-\infty}^{\infty} g_{1\nu}(t)M_{p'}(t)D_t^{k+1}\psi_{b,a}(t)\mathrm{d}t + \int_{-\infty}^{\infty} g_{2\nu}(t)M_{q'}(t)D_t^k\psi_{b,a}(t)\mathrm{d}t. \tag{25}$$

We wish to derive the inversion formula

$$J \equiv \frac{1}{C_\psi}\int_0^\infty \int_{-\infty}^\infty W_{U_\nu}(b, a)\psi_{b,a}(x)\frac{\mathrm{d}b\mathrm{d}a}{a^2} = U_\nu$$

interpreting convergence in the weak topology of $\mathcal{D}'$, i.e.

$$J \equiv \left\langle \frac{1}{C_\psi}\int_0^\infty \int_{-\infty}^\infty W_{U_\nu}(b, a)\psi_{b,a}(x)\frac{\mathrm{d}b\mathrm{d}a}{a^2},\ \phi(x)\right\rangle = \langle U_\nu, \phi\rangle\,,\ \forall\phi \in \mathcal{D}.$$

Using structure formula, we have

$$\begin{aligned} J &= \left\langle \frac{1}{C_\psi}\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty g_{1\nu}(t)M_{p'}(t)D_t^{k+1}\overline{\psi_{b,a}(t)}\psi_{b,a}(x)\frac{\mathrm{d}t\mathrm{d}b\mathrm{d}a}{a^2} \right.\\ &\quad \left. + \frac{1}{C_\psi}\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty g_{2\nu}(t)M_{q'}(t)D_t^k\overline{\psi_{b,a}(t)}\psi_{b,a}(x)\frac{\mathrm{d}t\mathrm{d}b\mathrm{d}a}{a^2},\ \phi(x)\right\rangle, \\ &= \left\langle \frac{1}{C_\psi}\int_0^\infty \left[\int_{-\infty}^\infty \left\{\int_{-\infty}^\infty g_{1\nu}(t)M_{p'}(t)(-1)^{k+1}D_b^{k+1}\overline{\psi_{b,a}(t)}\right\}\psi_{b,a}(x)\mathrm{d}t\right]\right. \end{aligned}$$

$$\times \frac{\mathrm{d}b\mathrm{d}a}{a^2},\ \phi(x)\Big\rangle$$
$$+\Big\langle \frac{1}{C_\psi}\int_0^\infty \Big[\int_{-\infty}^\infty \Big\{\int_{-\infty}^\infty g_{2\nu}(t)M_{q'}(t)(-1)^k D_b^k\overline{\psi_{b,a}(t)}\Big\}\psi_{b,a}(x)\mathrm{d}t\Big]$$
$$\times\ \frac{\mathrm{d}b\mathrm{d}a}{a^2},\ \phi(x)\Big\rangle$$

as $D_t\psi_{b,a}(t) = -D_b\psi_{b,a}(t)$. Therefore,

$$J = \Big\langle \frac{1}{C_\psi}\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty g_{1\nu}(t)M_{p'}(t)\overline{\psi_{b,a}(t)}D_b^{k+1}\psi_{b,a}(x)\frac{\mathrm{d}t\mathrm{d}b\mathrm{d}a}{a^2},\ \phi(x)\Big\rangle$$
$$+\Big\langle \frac{1}{C_\psi}\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty g_{2\nu}(t)M_{q'}(t)\overline{\psi_{b,a}(t)}D_b^{k}\psi_{b,a}(x)\frac{\mathrm{d}t\mathrm{d}b\mathrm{d}a}{a^2},\ \phi(x)\Big\rangle$$

(by integration by parts with respect to b)

$$= \Big\langle \frac{1}{C_\psi}\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty g_{1\nu}(t)M_{p'}(t)\overline{\psi_{b,a}(t)}(-1)^{k+1}D_x^{k+1}\psi_{b,a}(x)$$
$$\times\frac{\mathrm{d}t\mathrm{d}b\mathrm{d}a}{a^2},\ \phi(x)\Big\rangle$$
$$+\Big\langle \frac{1}{C_\psi}\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty g_{2\nu}(t)M_{q'}(t)\overline{\psi_{b,a}(t)}(-1)^{k}D_x^{k}\psi_{b,a}(x)$$
$$\times\frac{\mathrm{d}t\mathrm{d}b\mathrm{d}a}{a^2},\ \phi(x)\Big\rangle$$

as $D_b\psi_{b,a}(x) = -D_x\psi_{b,a}(x)$. Hence, by distributional differentiation,

$$J = \Big\langle \frac{1}{C_\psi}\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty g_{1\nu}(t)M_{p'}(t)\overline{\psi_{b,a}(t)}\psi_{b,a}(x)\frac{\mathrm{d}t\mathrm{d}b\mathrm{d}a}{a^2},\ D_x^{k+1}\phi(x)\Big\rangle$$
$$+\Big\langle \frac{1}{C_\psi}\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty g_{2\nu}(t)M_{q'}(t)\overline{\psi_{b,a}(t)}\psi_{b,a}(x)\frac{\mathrm{d}t\mathrm{d}b\mathrm{d}a}{a^2},\ D_x^{k}\phi(x)\Big\rangle. \tag{26}$$

The integrands

$$D_x^{k+1}\phi(x)\psi_{b,a}(x)\overline{\psi_{b,a}(t)}g_{1\nu}(t)\frac{M_{p'}(t)}{a^2}$$

and

$$D_x^{k}\phi(x)\psi_{b,a}(x)\overline{\psi_{b,a}(t)}g_{2\nu}(t)\frac{M_{q'}(t)}{a^2}$$

are absolutely integrable with respect to x and t in the x, t-plane and so Fubini's theorem is applicable with respect to integration by x and t. Therefore (26) yields

$$J = \frac{1}{C_\psi}\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty D_x^{k+1}\phi(x)\psi_{b,a}(x)\overline{\psi_{b,a}(t)}g_{1\nu}(t)M_{p'}(t)\frac{\mathrm{d}x\mathrm{d}t\mathrm{d}b\mathrm{d}a}{a^2}$$
$$+\frac{1}{C_\psi}\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty D_x^{k}\phi(x)\psi_{b,a}(x)\overline{\psi_{b,a}(t)}g_{2\nu}(t)M_{q'}(t)\frac{\mathrm{d}x\mathrm{d}t\mathrm{d}b\mathrm{d}a}{a^2}$$

$$
\begin{aligned}
&= \frac{1}{C_\psi}\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty\left[\overline{\overline{W_\psi\left\{D_x^{k+1}\phi(x)\right\}}(b,a)\psi_{b,a}(t)\frac{\mathrm{d}b\mathrm{d}a}{a^2}}\right]g_{1\nu}(t)M_{p'}(t)\mathrm{d}t \\
&\quad +\frac{1}{C_\psi}\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty\left[\overline{\overline{W_\psi\left\{D_x^{k}\phi(x)\right\}}(b,a)\psi_{b,a}(t)\frac{\mathrm{d}b\mathrm{d}a}{a^2}}\right]g_{2\nu}(t)M_{q'}(t)\mathrm{d}t \\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(invoking Fubini's theorem)} \\
&= \int_{-\infty}^\infty\overline{D_t^{k+1}\phi(t)}g_{1\nu}(t)M_{p'}(t)\mathrm{d}t+\int_{-\infty}^\infty\overline{D_t^{k}\phi(t)}g_{2\nu}(t)M_{q'}(t)\mathrm{d}t \\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(by inversion formula (3))} \\
&= \left\langle g_{1\nu}(t),\ M_{p'}(t)D_t^{k+1}\phi(t)\right\rangle+\left\langle g_{2\nu}(t),\ M_{q'}(t)D_t^{k}\phi(t)\right\rangle\ \text{(using duality)} \\
&= \langle U_\nu,\ \phi\rangle\qquad\text{(by structure formula (22))} \\
&\to \langle U,\ \phi\rangle\ \text{as}\ \nu\to\infty.
\end{aligned}
$$

This completes the proof of the theorem. □

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